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THE RISE AND DECLINE OF GENERALIZED CONVEX SPACES

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Abstract. In the KKM theory, various types of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ due to other authors are simply G-convex spaces. Various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. In 2006-09, G-convex spaces are extended to KKM spaces. In the present paper, we review the recent transition from G-convex spaces to KKM spaces, and introduce a basic KKM theorem on abstract convex spaces satisfying the partial KKM principle.

1. Introduction

In 1929, Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) obtained the "closed" version of the following; see [30-32].

The KKM theorem. *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map* (*that is,* $\operatorname{co} A \subset G(A)$ *for each* $A \subset D$) *with closed* [*resp., open*] *values. Then* $\bigcap_{z\in D} G(z) \neq \emptyset$.

The KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. The KKM theory, first called by the author, is the study on applications of equivalent formulations or generalizations of the KKM theorem.

Since 1993, the author has initiated the study of the KKM theory on generalized convex spaces (or G-convex spaces) $(X, D; \Gamma)$ as a common generalization of various general convexities without linear structures due to other authors. We have established within such a frame the foundations of the KKM theory, as well as fixed point theorems and many other equilibrium results for multimaps. This direction of study has been followed by a number of other authors.

In the last decade, there have appeared authors who introduced spaces of the form $(X, \{\varphi_A\})$ having a family $\{\varphi_A\}$ of continuous functions defined on simplexes

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and claimed that such spaces generalize G-convex spaces without giving any justifications or proper examples. In fact, a number of modifications or imitations of G-convex spaces have followed; for example, see [1, 3, 6-20, 25, 26, 28, 50-53], and many others. Some authors also introduced various types of generalized KKM maps and tried to generalize the KKM theorem for their own settings; for example, see [2-10, 12-20, 25, 26, 29, 50-53], and some others. Most of such generalizations are disguised forms of known ones.

In order to destroy such inadequate concepts and to upgrade the KKM theory, in 2006-09, we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [33-36, 38, 39, 42-48]. Moreover, we noticed that all spaces of the form $(X, \{\varphi_A\})$ can be unified to ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ or spaces having a family $\{\phi_A\}_{A\in\langle D\rangle}$ of singular simplexes.

In [34-37, 40-42] we showed that various types of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ are simply G-convex spaces, and various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. As such examples, we obtained KKM type theorems and very general fixed point theorems on ϕ_A-spaces.

In the present paper, we review the recent transition from G-convex spaces to KKM spaces in the KKM theory, and introduce a basic KKM theorem on abstract convex spaces satisfying the partial KKM principle.

2. Imitations of generalized convex spaces

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. The following is well-known:

Definition. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

For details on G-convex spaces, see [30-32, 49] and the references therein, where basic theory was indicated and lots of examples of G-convex spaces were given.

Recently, there have appeared authors in [1, 3, 6-20, 25, 26, 28, 50-53] and others who introduced spaces of the form $(X, \{\varphi_A\})$. Some of them tried to rewrite certain results on G-convex spaces by simply replacing $\Gamma(A)$ by $\varphi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples.

In this section, we give some examples of spaces of the form $(X, \{\varphi_A\})$ given by other authors:

(I) In 1998, Ben-El-Mechaiekh et al. [1] defined an L-space (E, Γ), which is a particular form of our G-convex space $(X, D; \Gamma)$ for the case $E = X = D$. Some authors incorrectly claimed that the class of L-spaces contains our class of G-convex spaces; for example, [11, 12], where a number of particular results (with certain defects) of known ones.

(II) Since 2001, many authors studied L-spaces; for example, [6, 11, 12, 19, 50] and others. Some incautious authors adopt the term L-convex spaces instead of L-spaces in [1].

(III) [25] A topological space Y is said to have property (H) if, for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$.

(IV) [13, 14] $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. This definition appears in a large number of papers of the same author and his followers; see [37]. Note that for each N, there should be infinitely many φ_N's.

The author of [13, 14] stated in more than a dozen papers that: "It is easy to see that the class of FC-spaces includes the classes of convex sets in topological vector spaces, C-spaces (or H-spaces) [21, 22], G-convex spaces, L-convex spaces [1], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces." Here he failed to give any justification or any proper example of his space which is not G-convex. One wonders how could a pair $(Y, \{\varphi_N\})$ generalize a triple $(X, D; \Gamma)$!

(V) In [52], a pair $(Y, \mathcal{C})$ is introduced, where Y is a topological space and $\mathcal{C}$ is a family of subsets of Y such that $(Y, \mathcal{C})$ is similar to the convexity space in the classical sense.

A pair $(X, \mathcal{C})$ is said to have the selection property with respect to a topological space S if every multimap $F : S \multimap X$ admits a single-valued continuous selection whenever F is lower semicontinuous and nonempty closed convex valued.

A pair $(Y, \mathcal{C})$ is said to satisfy H-condition if $\mathcal{C}$ has the following property:

(H) For each finite subset $\{y_0, \ldots, y_n\} \subset Y$, there exists a continuous mapping $f : \Delta_N \to \overline{conv}\{y_0, \ldots, y_n\}$, where Δ_N is the standard n-simplex, such that $f(\Delta_J) \subset \overline{conv}\{y_j : j \in J\}$ for each nonempty subset $J \subset N = \{0, 1, \ldots, n\}$, where $\overline{conv}$ denotes the closed convex hull.

For these definitions, we note the following remarks:

(i) A pair $(Y, \mathcal{C})$ is a particular form of our abstract convex space $(Y; \Gamma)$ with $\Gamma_A := \mathrm{conv}(A) = \bigcap\{B \in \mathcal{C} \mid A \subset B\}$ for $A \in \langle Y \rangle$.

(ii) A pair $(Y, \mathcal{C})$ satisfying the H-condition is a particular form of our G-convex space $(X, D; \Gamma)$ with $Y = X = D$ such that Γ is closed-valued.

(VI) Cain and González [3] considered relationship among some subclasses of the class of G-convex spaces and introduced a subclass of L-spaces. In [3, Theorems 3.2 and 3.4], it was shown that L-spaces are G-convex spaces.

González et al. [19] repeated to show that G-convex spaces and L-spaces satisfy the partial KKM principle. They added that L-spaces satisfy the properties of the Fan type minimax inequality, Fan-Browder type fixed point, and the Nash type equilibrium. All of such results are already known for more general G-convex spaces.

(VII) In 2008 [28], Kulpa and Szymanski defined: A collection $\mathcal{S}$ of singular simplexes in a space X is called a simplicial structure on X if: (a) For any finite subset $\{a_0, a_1, \ldots, a_n\}$ of (not necessarily distinct) points of X, there exists $\sigma \in \mathcal{S}$ such that $\sigma : [e_0, e_1, \ldots, e_n] \to X$ and $\sigma(e_i) = a_i$ for each $i = 0, 1, \ldots, n$; (b) If $\sigma \in \mathcal{S}$, then the restriction of σ to any face of the domain of σ belongs to $\mathcal{S}$.

A simplicial space $(X, \mathcal{S})$ is a topological space X together with a simplicial structure $\mathcal{S}$. This extends the simplicial convexity due to Bielawski.

3. ϕ_A-spaces

Motivated by the examples in the preceding section, we are concerned with a reformulation of the class of G-convex spaces as in [34-37, 40-42]:

Definition. A *ϕ_A-space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Any G-convex space is a ϕ_A-space. The converse also holds. The following are given in [37, 40-42]:

Theorem 1. *A ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ can be made into a G-convex space $(X, D; \Gamma)$.*

Definition. For a ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is called a *KKM map*.

Theorem 2. (1) *A KKM map $G : D \multimap X$ on a G-convex space $(X, D; \Gamma)$ is a KKM map on the corresponding ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$.*

(2) *A KKM map $T : D \multimap X$ on a ϕ_A-space $(X, D; \{\phi_A\})$ is a KKM map on a new G-convex space $(X, D; \Gamma)$.*

The following is a KKM theorem for ϕ_A-spaces. The proof is just a simple modification of the corresponding one in [31-33]:

Theorem 3. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$, let $G : D \multimap X$ be a KKM map with closed* [*resp., open*] *values. Then $\{G(z)\}_{z \in D}$ has the finite intersection property.* (*More precisely, for each $N \in \langle D \rangle$ with $|N| = n + 1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z \in N} G(z) \neq \emptyset$.*)

Further, if

(3) *$\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Remarks. (1) We may assume that, for each $a \in D$ and $N \in \langle D \rangle$, $G(a) \cap \phi_N(\Delta_n)$ is closed [resp., open] in $\phi_N(\Delta_n)$. This is said by some authors that G has finitely closed [resp., open] values. However, by replacing the topology of X by its finitely generated extension, we can eliminate "finitely"; see [32].

(2) For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \text{co}$, the convex hull, Theorem 3 reduces to the original KKM theorem and its open version; see [30, 37].

(3) If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 3 extends Fan's KKM lemma; see [30, 37].

(4) Note that any KKM theorem on spaces of the form $(X, \{\varphi_A\})$ can not generalize the original KKM principle nor Fan's KKM lemma; see [37].

(5) Recently, ϕ_A-spaces are called generalized finitely continuous spaces or GFC-spaces by Khanh et al. [26].

Here we give a new example of ϕ_A-spaces:

Example. In [24], its author gave a necessary and sufficient condition for the existence of a pure-strategy Nash equilibrium for non-cooperative games in topological spaces. He adopted the following concept:

Let X be a topological space, and $D, Y \subset X$. A real function $f : X \times Y \to \mathbb{R}$ is said to be $\mathcal{C}$-*quasiconcave* on D if, for any $N := \{x^0, x^1, \dots, x^n\} \in \langle D \rangle$, there exists a continuous map $\phi_N : \Delta_n \to Y$ such that

$$f(\phi_N(\lambda), \phi_N(\lambda)) \geq \min\{f(x^i, \phi_N(\lambda)) \mid i \in J\}$$

for all $\lambda := (\lambda_0, \lambda_1, \dots, \lambda_n) \in \Delta_n$, where $J := \{i \in \{0, 1, \dots, n\} \mid \lambda_i \neq 0\}$.

Note that $(Y, D; \{\phi_A\}_{A \in \langle D \rangle})$ is a ϕ_A-space.

4. Various KKM maps

There have also appeared a large number of the so-called generalized KKM maps in the literature. In fact, a number of authors tried to generalize the concept of KKM maps on particular forms of ϕ_A-spaces. In this section, we show that all of them are particular forms of our KKM maps.

(I) In [17], its author introduced variants of several concepts in the KKM theory on generalized convex spaces as follows:

For a G-convex space $(X, D; \Gamma)$, a multimap$F : D \to 2^X$ is called a Φ-*KKM map* if, for each $A \in \langle D \rangle$, we have

$$\phi_A(\Delta_{|A|-1}) \subset F(A).$$

Similarly, for a nonempty set Y, a *generalized* Φ-*KKM map* $F : Y \to 2^X$ is defined.

Moreover, F is called *finitely* G-*closed* (*resp., open*) if for every finite subset B of D and $y \in Y$, $F(y) \cap G\text{-co}\, B$ is closed (resp., open) in

$$G\text{-co}\, B := \bigcap\{A \subset X : A \text{ is a } \Gamma\text{-convex subset of } X \text{ containing}\, B\}.$$

Further, F is called *finitely* Φ-*closed* (*resp., open*) if for every finite subset B of D with $|B| = n + 1$ and $y \in Y$, $F(y) \cap \phi_B(\Delta_n)$ is closed (resp., open) in $\phi_B(\Delta_n)$.

The author of [17] claims that these notions generalize that F has closed (resp., open) values. In [17], its author continues as follows:

Let a G-convex space $(X, D; \Gamma)$, a Hausdorff space Y, and a multimap $F : X \to 2^Y$ be given. A multimap $G : X \to 2^Y$ is called a *$F\Phi$-KKM map* if for any finite subset A of D, we have $F\phi_A(\Delta_{|A|-1}) \subseteq G(A)$. A multimap $G : D \to 2^Y$ is called a *generalized $F\Phi$-KKM map* if for any subset $A = \{x_0, \ldots, x_n\}$ of D, there exists a finite subset $B = \{y_0, \ldots, y_n\}$ of D, not necessarily all different, such that: for each $\{i_0, \ldots, i_j\}$ we have $F\phi_B(\Delta_J) \subseteq \bigcup_{k=0}^{j} G(x_{i_k})$. G is called *finitely $F\phi$-closed (resp., open) valued* if for any finite subset $A = \{x_0, \ldots, x_n\} \subseteq D$ and each $x \in D$, the set $G(x) \cap F\phi_A(\Delta_n)$ is closed (resp., open) in $F\phi_A(\Delta_n)$

Let Y be a topological space. A map $F : X \to 2^Y$ is said to have *the generalized Φ-KKM property* if, for any map $G : D \to 2^Y$ with compactly closed values which is a generalized $F\Phi$-KKM map the class $\{G(x) \mid x \in D\}$ has the finite intersection property. We denote

$$\mathfrak{K}(X, Y) := \{F : X \to 2^Y \mid F \text{ has the generalized } \Phi\text{-KKM property}\}.$$

In fact, the main idea in [17] was *to rewrite some results on G-convex spaces by simply replacing $\Gamma(A)$ by $\phi_A(\Delta_n)$ everywhere* and the author claimed to *obtain generalizations without giving any justifications or proper examples.*

(II) In 2003 [50, Definition 2], for an L-space (X, Γ) and a topological space Y, a correspondence $G : Y \multimap X$ is called a *generalized KKM-correspondence*, if for all $A = \{y_0, y_1, \ldots, y_n\} \in \langle Y \rangle$, there exists a subset $B = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, such that for all $J \subseteq \{0, 1, \ldots, n\}$, it is satisfied that $\phi_B(\Delta_J) \subseteq \bigcup_{j \in J} G(y_j)$.

Note that *a generalized KKM-correspondence becomes simply our KKM map on a ϕ_A-space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ by putting $D := Y$ and, for any $A \in \langle D \rangle$, by defining $\phi_A(\Delta_{|A|-1}) := \phi_B(\Delta_{|B|-1})$ for $B \in \langle X \rangle$ corresponding to A.

(III) In 2003 [8, Definition 2.1], for a nonempty set X and a topological space Y, $T : X \to 2^Y$ is said to be generalized relatively KKM (R-KKM) mapping if for any $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, there exists a continuous mapping $\phi_N : \Delta_n \to Y$ such that, for each $e_{i_0}, e_{i_1}, \ldots, e_{i_k}$,

$$\phi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

where Δ_k is a standard k-simplex of Δ_n with vertices $e_{i_0}, e_{i_1}, \ldots, e_{i_k}$.

For a ϕ_A-space $(Y, X; \{\phi_N\}_{N \in \langle X \rangle})$, $T : X \to 2^Y$ is simply a KKM map.

(IV) Let X be a nonempty set and Y be a topological space with property (H). In 2005 [25], $T : X \to 2^Y$ is said to be a generalized R-KKM mapping if for each $\{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ such that

$$\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

for all $\{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$.

Similarly to (III), *a generalized R-KKM map $T : X \to 2^Y$ is simply a KKM map for the ϕ_A-space $(Y, X; \{\phi_A\}_{A \in \langle X \rangle})$.*

The author of [15] claimed as follows: "The above class of generalized *R-KKM* mappings includes those classes of *KKM* mappings, *H-KKM* mappings, *G-KKM* mappings, generalized *G-KKM* mappings, generalized *S-KKM* mappings, *GLKKM* mappings and *GMKKM* mappings defined in topological vector spaces, H-spaces, G-convex spaces, G-H-spaces, L-convex spaces and hyperconvex metric spaces, respectively, as true subclasses." This is partially incorrect.

However, in view of this claim and Theorem 2, *so many variants of KKM type theorems in* [8-20, 50, 52] *and a large number of other papers can be reduced to the ones in our G-convex space theory.* We should recognize that, in the KKM theory on G-convex spaces, every argument is related to the finite intersection property of functional values of KKM maps having closed [resp., open] values, in other words, related to some $N \in \langle D \rangle$ in $(X, D; \Gamma)$.

(V) In [6], its author introduced a generalized *R-KKM* mapping and a new class R_{ψ_N}-$KKM(X,Y)$, and claimed some fixed point theorems, matching theorems, coincidence theorems, and minimax inequalities on particular G-convex spaces.

Note that its author's generalized *R-KKM* mappings are different from those of other authors. Because of this and the incorrect application of the KKM theorem, Theorems 1-3 of [6] cannot be considered as proved. Moreover, in the definition of the class R_{ψ_N}-$KKM(X,Y)$, the role of G-convex spaces (its author's L-convex spaces) is not clear. This remark also applies to all of Theorems 4-23 of [6].

Its author's aim in [6] seems to generalize the well-known results of the KKM theory in his own setting, but he failed to give any proper examples or any suggested applications.

(VI) Motivated by a large number of recent works on generalized KKM maps, we introduced the following definition in [49]: Let $(X, D; \Gamma)$ be a G-convex space and I a nonempty set. A map $F : I \multimap X$ is called a *generalized KKM map* provided that for each $N \in \langle I \rangle$, there exists a function $\sigma : N \to D$ such that $\Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle N \rangle$.

In [49], a unified account on results for such maps was given; for example, the KKM type theorem, characterizations of such maps, an equilibrium theorem implying minimax inequalities, variational inequalities, and so on. A little later than [49], similar results appeared in [11, 12], which have trivial defects in certain aspects.

In fact, we can show that *a generalized KKM map $F : I \multimap X$ is a KKM map on the ϕ_A-space* $(X, I; \{\phi_N\}_{N \in \langle I \rangle})$, where ϕ_N is the trivial composition

$$\Delta_{|N|-1} \to \Delta_{|\sigma(N)|-1} \to \Gamma_{\sigma(N)}.$$

(VII) For particular forms of G-convex spaces, some authors obtained KKM type theorems or equivalents which can not be applicable even to the original KKM theorem for $(\Delta_n \supset V; \mathrm{co})$ or to the Fan lemma for $(X \supset D; \mathrm{co})$, where X is a topological vector space; see [36].

In our previous paper [41], we gave two KKM type theorems which improve corresponding ones in [8, 50].

5. Abstract convex spaces and KKM spaces

From 2006, we began to study abstract convex spaces and KKM spaces instead of G-convex spaces as in [33-36, 38, 39, 42-48].

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Examples of abstract convex spaces were given in [38-48].

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A map $F : E \multimap Z$ is said to have the *KKM property* and called *a* $\mathfrak{K}$*-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, a $\mathfrak{KC}$*-map* is defined for closed-valued maps G, and a $\mathfrak{KO}$*-map* for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)]$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

In our recent works [38, 39, 45-48], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Example. We give known examples of KKM spaces:

(1) Every G-convex space is a KKM space [31].

(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space; see [45].

(3) The extended long line L^* is a KKM space $(L^* \supset D; \Gamma)$ with the ordinal space $D := [0, \Omega]$; see [45]. But L^* is not a G-convex space.

(4) Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree M. Then $(M, X; conv_M)$ satisfies the partial KKM principle; see Kirk and Panyanak [27]. Recently, we showed that this is a KKM space [46].

(5) For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ for each $A \in \langle X \rangle$; see [23].

(6) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space [2, Corollary 2.2].

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space

$\Longrightarrow$ Space satisfying the partial KKM principle

$\Longrightarrow$ Abstract convex space.

It is not known yet whether there is a space satisfying the partial KKM principle which is not a KKM space.

In the KKM theory, there are many equivalent reformulation of the KKM principle that can be used to characterize KKM spaces. For example, the Fan-Browder type fixed point theorem is used for the following [45]:

Theorem 4. *An abstract convex space $(X, D; \Gamma)$ is a KKM space iff for any maps $S : D \multimap X$, $T : X \multimap X$ satisfying*
(1) *$S(z)$ is open [resp., closed] for each $z \in D$;*
(2) *for each $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*
(3) *$X = \bigcup_{z \in M} S(z)$ for some $M \in \langle D \rangle$,*
T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.

Moreover, from the partial KKM principle we have a whole intersection property of the Fan type. From this, we can deduce the following [47]:

Theorem 5. *Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, K a nonempty compact subset of X, and $G : D \multimap X$ a map such that*
(1) *$\bigcap_{z \in D} G(z) = \bigcap_{z \in D} \overline{G(z)}$ [that is, G is transfer closed-valued];*
(2) *$\overline{G}$ is a KKM map; and*
(3) *either*
(i) *$\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ for some $M \in \langle D \rangle$; or*
(ii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \cap \bigcap\{\overline{G(z)} \mid z \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Theorem 5 might have more than one hundred particular forms appeared in the literature.

Remark. Recently in [48], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [48] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces. For the more historical backgrounds or details on the results in this paper, see the references of [38, 39, 45-48] and the references therein.

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