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COMMENTS ON HOU JICHENG'S "ON SOME KKM TYPE THEOREMS"

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Abstract. In a paper by Hou Jicheng, *On some KKM type theorems*, Advaces in Mathematics **36** (2007), no. 1, 86–88, the author claimed that some previous KKM type theorems are false by giving a counterexample. In the present paper, we show that the counterexample does not work and, consequently, the results are correct. Moreover, we claim that the artificial concept like transfer compactly closed-valued maps can be destroyed. Finally, we introduce a theorem generalizing the main target of Hou.

1. Introduction

In a recent paper [2], Hou Jicheng claimed that some previous KKM type theorems of Park and Kim [7] and Kalmoun and Rihai [3] are false by giving a counterexample. In the present paper, we show that the counterexample does not work and hence Hou's claim is false. Consequently, the results are correct.

Moreover, in this paper, we claim that the artificial concept like *transfer compactly closed-valued maps* can be destroyed. In fact, any KKM type theorems on such maps can be easily deduced from the corresponding theorems on closed-valued maps.

Finally, we introduce a recent theorem generalizing the main target of Hou.

2. Some KKM type theorems

In [1], Ding introduced the following notions: Let X and Y be two topological spaces. A multimap $G : X \multimap Y$ is said to be *transfer compactly closed-valued* [resp., *transfer compactly open-valued*] on X if for every $x \in X$ and each nonempty compact subset K of Y, $y \notin Gx \cap K$ [resp., $y \in Gx \cap K$] implies that there exists a point $x' \in X$ such that $y \notin \mathrm{cl}_K(Gx' \cap K)$ [resp., $y \in \mathrm{int}_K(Gx' \cap K)$], where $\mathrm{cl}_K(Gx' \cap K)$ and $\mathrm{int}_K(Gx' \cap K)$ is the closure and the interior of $Gx' \cap K$ in K, resp. Note that $\mathrm{cl}_K(Gx' \cap K) = \overline{Gx' \cap K}$.

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The following in [7, Theorem 3] is one of the targets of Hou [2]:

Theorem A. *Let* $(X, D; \Gamma)$ *be a* G*-convex space,* Y *a topological space, and* $F \in \mathfrak{A}_c^\kappa(X, Y)$ *an admissible map. Let* $G : D \multimap Y$ *be a map such that*

(1) *for each* $x \in D$*,* Gx *is transfer compactly closed in* Y*;*

(2) *for any* $N \in \langle D\rangle$*,* $F(\Gamma_N) \subset \overline{G}(N)$*; and*

(3) *there exist a nonempty compact subset* K *of* Y *and, for each* $N \in \langle D\rangle$*, a compact* G*-convex subset* L_N *of* X *containing* N *such that* $F(L_N) \cap \bigcap\{\overline{G}x : x \in L_N \cap D\} \subset K$*.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$*.*

Here $\langle D\rangle$ denotes the set of nonempty finite subsets of a set D.

Hou [2] claimed that the proofs of Theorem A and three of its variants due to Kalmoun and Rihai [7] are not correct. However, he failed to indicate which part of the proofs are incorrect.

3. Defects of Hou's example

Instead of indicating incorrect parts of the proofs of the pending theorems, Hou [2] tried to make a counterexample showing the above theorems for the case F is a constant map which belongs trivially to the admissible class $\mathfrak{A}_c^\kappa(X, X)$.

In order to show that Hou's example does not work, we note the following well-known fact:

Lemma B. *Let* X *and* Y *be two topological spaces. A multimap* $G : X \multimap Y$ *is transfer compactly closed-valued if and only if for any compact subset* K *of* Y*, we have*

$$K \cap \bigcap_{x\in X} G(x) = K \cap \bigcap_{x\in X} \overline{G(x)}.$$

We show that Hou's counterexample does not work. In his example, X is a convex subset of a t.v.s. E containing the origin θ, $K = \{\theta\}$, $F : X \multimap X$ is a constant map $F(x) = \{\theta\}$, and a map $G : X \multimap X$ is defined. He claimed G is transfer compactly closed-valued. He also showed $\bigcap_{x\in X} G(x) = \emptyset$, the empty set, and $\bigcap_{x\in X} \overline{G(x)} = \{\theta\}$. Hence we have

$$\emptyset = \overline{F(X)} \cap K \cap \bigcap_{x\in X} G(x) \neq \overline{F(X)} \cap K \cap \bigcap_{x\in X} \overline{G(x)} = \{\theta\},$$

which shows that G is not transfer compactly closed-valued by Lemma B since $\overline{F(X)} \cap K$ is compact. This is a contradiction.

Note that, from the beginning, G did not satisfy condition (1) of Theorem A.

4. Generalizations

Since the appearance of [7], there are certain progresses of the KKM theory. Especially, we notice the following:

(1) In the definition of transfer compactly closed-valuedness of a map $G : X \multimap Y$, if we replace the topology of Y by its compactly generated extension, then G has simply transfer closedness. Therefore "compactly" can be easily eliminated and does not generalize anything.

(2) There are large number of papers treating equivalent conditions of the transfer compactly closedness or some similar concepts. If we choose Lemma B instead of the original definition of Ding [1], then we have more clear situation and could more easily destroy such useless papers.

(3) More recently in [4, 7], we showed that any "transfer" closed-valued version of KKM type theorems are equivalent to usual closed-valued version of them. Therefore, *now is the proper time to discard the "transfer" cases from the KKM theory.*

In view of these remarks, we introduce new KKM type theorems related to Theorem A. We follow mainly [5, 7] and references therein.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

A *KKM space* $(E, D; \Gamma)$ is an abstract convex space satisfying the *KKM principle* $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$. Every KKM space satisfies the *partial KKM principle* $1_E \in \mathfrak{KC}(E, D, E)$.

From the partial KKM principle we can deduce a whole intersection property of the Fan type. In fact, we have the following equivalent form of [5, Theorem 1], which was stated for "transfer" closed-valued maps:

Theorem C. *Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, K a nonempty compact subset of X, and $G : D \multimap X$ a map such that*

(1) *G is closed-valued (that is, for each $x \in D$, $G(x)$ is closed in X);*

(2) *G is a KKM map; and*

(3) *either*

(i) *$\bigcap\{G(z) \mid z \in M\} \subset K$ for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \cap \bigcap\{G(z) \mid z \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

More generally, we obtained the following generalization of Theorem A:

Theorem D. *Let $(X, D; \Gamma)$ be an abstract convex space, Y a topological space, and $F \in \mathfrak{KC}(X, D, Y)$. Let $G : D \multimap Y$ be a map such that*

(1) *G is closed-valued;*

(2) *G is a KKM map with respect to F (that is, for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$); and*

(3) *there exist a nonempty compact subset K of Y and, for each $N \in \langle D \rangle$, a Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$, $F(L_N)$ is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap\{G(z) \mid z \in D'\}.$$

Then $\overline{F(X)} \cap K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

This is given in [6, Theorem 2.12]. Note that Theorem D subsumes a very large number of particular forms of KKM type theorems in the literature and can be reformulated to the equivalent forms of coincidence theorems, matching theorems, analytic alternatives, minimax inequalities, geometric and section properties.

Final Remark. The original version (except the main part of Section 4) of this paper was submitted to *Advances in Mathematics* in October 2008 and its receipt was acknowledged by the editor. In May and June 2009, the author asked the editor three times by e-mail for the current status of the submission. No replies!

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