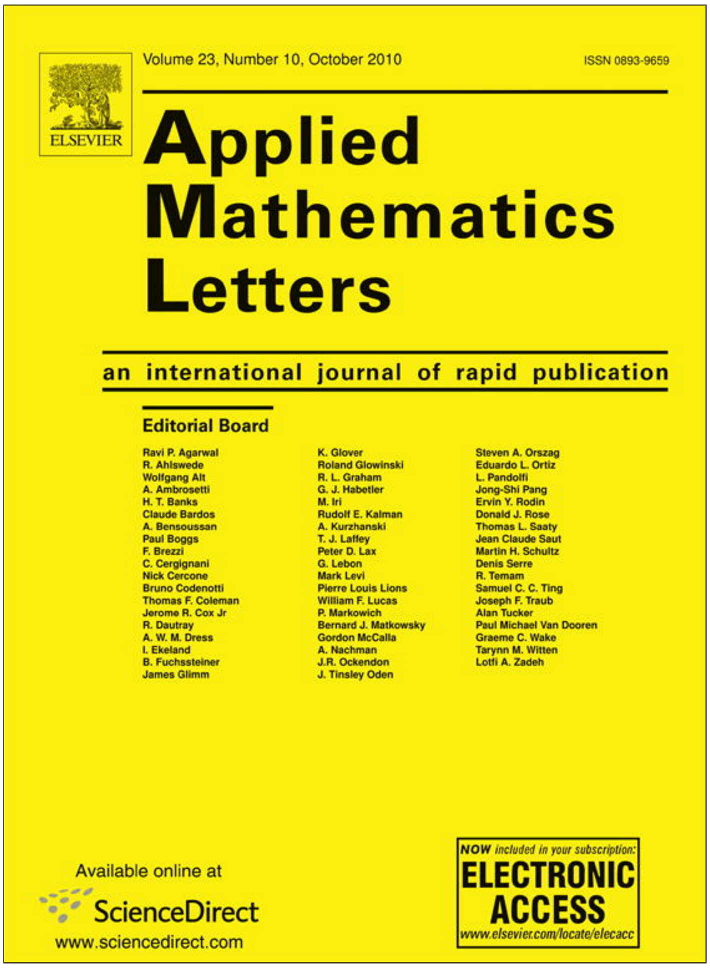
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On the von Neumann–Sion minimax theorem in KKM spaces

Sehie Park *

The National Academy of Sciences, Seoul 137–044, Republic of Korea
Department of Mathematical Sciences, Seoul National University, Seoul 151–747, Republic of Korea

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ABSTRACT

In an abstract convex space $(E, D; \Gamma)$, we show that the partial KKM principle is equivalent to a Fan–Browder type fixed point theorem and that this theorem implies generalized forms of the von Neumann–Sion minimax theorem.

1. Introduction

The von Neumann–Sion minimax theorem is fundamental in convex analysis and in game theory. von Neumann [1] proved his theorem for simplexes by reducing the problem to the one-dimensional cases. Sion's generalization [2] was proved by the aid of Helly's theorem and the KKM theorem due to Knaster et al. [3]. In a recent paper, Kindler [4] proved Sion's theorem by applying the one-dimensional KKM theorem (i.e., every interval in $\mathbb{R}$ is connected), the one-dimensional Helly theorem (i.e., any family of pairwise intersecting compact intervals in the real line $\mathbb{R}$ has a nonempty intersection), and Zorn's lemma (or other method).

In a recent work of the author [5], for convex subsets X of a topological vector space E, he showed that a KKM principle implies a Fan–Browder type fixed point theorem and that this theorem implies a generalized form of the Sion minimax theorem.

In the present paper, the procedure in [5] can be generalized and applied to abstract convex spaces recently due to the author. In fact, in an abstract convex space $(E, D; \Gamma)$, he shows that the partial KKM principle is equivalent to a Fan–Browder type fixed point theorem and that this theorem implies generalized forms of the von Neumann–Sion minimax theorem.

2. Abstract convex spaces

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Recall the following in [6–10]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the *Γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_N \mid N \in \langle D'\rangle\} \subset E.$$

* Corresponding address: The National Academy of Sciences, 137-044 Seoul, Republic of Korea. Tel.: +82 2 565 3120.
E-mail addresses: shpark@math.snu.ac.kr, parkcha38@hanmail.net.

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A subset X of E is called a *Γ-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D'\rangle})$ is called a *Γ-convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such a case, a subset X of E is said to be *Γ-convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example 2.1. The following are known examples of abstract convex spaces; see [6–10].

(1) The original KKM theorem is for the triple $(\Delta_n, V; \mathrm{co})$, where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and co: $\langle V\rangle \multimap \Delta_n$ the convex hull operation.
(2) Fan's celebrated KKM lemma is for $(E, D; \mathrm{co})$, where D is a nonempty subset of a topological vector space E.
(3) A *convex space* $(X; \Gamma)$ due to Lassonde.
(4) A C-space $(X; \Gamma)$ due to Horvath.
(5) Hyperconvex metric spaces due to Aronszajn and Panitchpakdi.
(6) Hyperbolic spaces due to Reich and Shafrir.
(7) Any topological semilattice $(X, \leq)$ with path-connected interval introduced by Horvath and Llinares.
(8) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park.
(9) A *ϕ_A-space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ due to Park.
(10) A space $(H, X; \Gamma)$ due to Kirk and Panyanak, where X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X\rangle$, $\Gamma_A := \mathrm{conv}_H(A)$.
(11) Horvath's convexity spaces.
(12) A $\mathbb{B}$-space due to Briec and Horvath.

Note that each of (2)–(12) has a large number of concrete examples.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \quad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map*.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

In our recent works [6,8,10], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Example 2.2. We give known examples of KKM spaces:

(1) Every G-convex space is a KKM space.
(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space.
(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space.
(4) For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [\![A]\!] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X\rangle$.
(5) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space.

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{aligned}\text{Simplex} &\Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space}\\ &\Longrightarrow H\text{-space} \Longrightarrow G\text{-convex space} \Longleftrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space}\\ &\Longrightarrow \text{A space satisfying the partial KKM principle}\\ &\Longrightarrow \text{Abstract convex space.}\end{aligned}$$

3. From the KKM principle to the minimax theorem

For an abstract convex space $(E, D; \Gamma)$, let us consider the following:

Definition. A multimap $T : E \multimap E$ is called a *Fan–Browder map* provided that there exists a companion map $S : E \multimap D$ such that

(a) for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and
(b) $E = \bigcup_{z\in N} \mathrm{Int}\, S^-(z)$ for some finite subset N of D.

Here, Int denotes the interior with respect to E and, for each $z \in D$, $S^-(z) := \{x \in E \mid z \in S(x)\}$.

Let us consider the following statements:

(A) *The partial KKM principle. For any closed-valued KKM map $G : D \multimap E$, the family $\{G(z)\}_{z\in D}$ has the finite intersection property.*

(B) *The Fan–Browder fixed point theorem. Any Fan–Browder map $T : E \multimap E$ has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

Recall that (A) originates from the Knaster–Kuratowski–Mazurkiewicz theorem [3] and (B) from Fan [11] and Browder [12].

Theorem 3.1. *The statements* (A) *and* (B) *are equivalent for an abstract convex space* $(E, D; \Gamma)$.

Proof. (A) $\Longrightarrow$ (B). Suppose $(E, D; \Gamma)$ satisfies the partial KKM principle (A). Define a map $G : D \multimap E$ by $G(z) := E \setminus \operatorname{Int} S^-(z)$ for each $z \in D$. Then each $G(z)$ is closed, and

$$\bigcap_{z\in N} G(z) = E \setminus \bigcup_{z\in N} \operatorname{Int} S^-(z) = E \setminus E = \emptyset$$

by (b). Therefore, the family $\{G(z)\}_{z\in D}$ does not have the finite intersection property, and hence, G is not a KKM map by (A). Thus, there exists an $M \in \langle D\rangle$ such that $\Gamma_M \not\subset G(M) = \bigcup\{E \setminus \operatorname{Int} S^-(z) \mid z \in M\}$. Hence, there exists an $x_0 \in \Gamma_M$ such that $x_0 \in S^-(z)$ for all $z \in M$; that is, $M \subset S(x_0)$. Therefore, $x_0 \in \Gamma_M \subset \operatorname{co}_\Gamma S(x_0) \subset T(x_0)$ by (a).

(B) $\Longrightarrow$ (A). Let $G : D \multimap E$ be a KKM map with closed values. Suppose the family $\{G(z)\}_{z\in D}$ does not have the finite intersection property; that is, there exists an $M \in \langle D\rangle$ such that $\bigcap_{z\in M} G(z) = \emptyset$. Define a map $S : E \multimap D$ by $S^-(z) := E \setminus G(z)$ for $z \in D$ and a map $T : E \multimap E$ by $T(x) := \operatorname{co}_\Gamma S(x)$ for each $x \in E$. Note that $E = \bigcup_{z\in M}(E \setminus G(z)) = \bigcup_{z\in M} S^-(z)$. Then the requirements (a) and (b) are satisfied. Hence there exists an $x_0 \in E$ such that $x_0 \in T(x_0)$. Then $x_0 \in T(x_0) = \operatorname{co}_\Gamma S(x_0)$ and hence, there exists an $N \in \langle S(x_0)\rangle$ such that $x_0 \in \Gamma_N \subset \operatorname{co}_\Gamma S(x_0)$. Therefore, for each $z \in N$, we have $x_0 \in S^-(z)$ or $x_0 \notin G(z)$; that is, $\Gamma_N \not\subset G(N)$. Hence G is not a KKM map, a contradiction. Therefore $(E, D; \Gamma)$ satisfies the partial KKM principle (A). □

Remark. There are several more statements equivalent to the partial KKM principle for abstract convex spaces; see [8,10].

Recall that an extended real-valued function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is open for each $r \in \overline{\mathbb{R}}$.

The following is known:

Lemma 3.1. *Let $\{(X_i, D_i; \Gamma_i)\}_{i\in I}$ be any family of abstract convex spaces. Let $X := \prod_{i\in I} X_i$ be equipped with the product topology and $D = \prod_{i\in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D\rangle$, define $\Gamma(A) := \prod_{i\in I} \Gamma_i(\pi_i(A))$. Then $(X, D; \Gamma)$ is an abstract convex space.*

Let $\{(X_i, D_i; \Gamma_i)\}_{i\in I}$ be a family of G-convex spaces. Then $(X, D; \Gamma)$ is a G-convex space.

For an abstract convex space $(E \supset D; \Gamma)$, a real-valued function $f : E \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in E \mid f(x) > r\}$ [resp., $\{x \in E \mid f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbb{R}}$.

From now on, for simplicity, we are mainly concerned with abstract convex spaces $(E; \Gamma)$ satisfying the partial KKM principle.

From (B), we have the following:

Theorem 3.2. *Let $(X; \Gamma_1)$ and $(Y; \Gamma_2)$ be abstract convex spaces, $(E; \Gamma) := (X \times Y; \Gamma_{X\times Y})$ the product abstract convex space defined as in Lemma 3.1, and $f, s, t, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be four functions,*

$$\mu := \inf_{y\in Y} \sup_{x\in X} f(x, y) \quad \text{and} \quad \nu := \sup_{x\in X} \inf_{y\in Y} g(x, y).$$

Suppose that

(2.1) *$f(x, y) \le s(x, y) \le t(x, y) \le g(x, y)$ for each $(x, y) \in X \times Y$;*

(2.2) *for each $r < \mu$ and $y \in Y$, $\{x \in X \mid s(x, y) > r\}$ is Γ_1-convex; for each $r > \nu$ and $x \in X$, $\{y \in Y \mid t(x, y) < r\}$ is Γ_2-convex;*

(2.3) *for each $r > \nu$, there exists a finite subset $\{x_i\}_{i=1}^m$ of X such that*

$$Y = \bigcup_{i=1}^{m} \operatorname{Int}\{y \in Y \mid f(x_i, y) > r\}; \quad \text{and}$$

(2.4) *for each $r < \mu$, there exists a finite subset $\{y_j\}_{j=1}^n$ of Y such that*

$$X = \bigcup_{j=1}^{n} \operatorname{Int}\{x \in X \mid g(x, y_j) < r\}.$$

If $(E; \Gamma)$ *satisfies the partial KKM principle, then we have* $\mu \le \nu$*, that is,*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \le \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

Proof. Suppose that there exists a real c such that

$$\nu := \sup_x \inf_y g(x, y) < c < \inf_y \sup_x f(x, y) =: \mu.$$

Define a map $T : X \times Y \multimap X \times Y$ by

$$T(x, y) := \{\bar{x} \in X \mid s(\bar{x}, y) > c\} \times \{\bar{y} \in Y \mid t(x, \bar{y}) < c\},$$

for each $(x, y) \in X \times Y$. Then each $T(x, y)$ is Γ-convex by (2.2). Moreover, for each $(\bar{x}, \bar{y}) \in X \times Y$, we have

$$\begin{aligned} T^-(\bar{x}, \bar{y}) &= \{x \in X \mid s(x, \bar{y}) > c\} \times \{y \in Y \mid t(\bar{x}, y) < c\} \\ &\supset \{x \in X \mid f(x, \bar{y}) > c\} \times \{y \in Y \mid g(\bar{x}, y) < c\} \\ &\supset \mathrm{Int}\{x \in X \mid f(x, \bar{y}) > c\} \times \mathrm{Int}\{y \in Y \mid g(\bar{x}, y) < c\}. \end{aligned}$$

Therefore, by (2.3) and (2.4), $X \times Y$ is covered by

$$\{\mathrm{Int}\, T^-(x_i, y_j) \mid 1 \le i \le m, 1 \le j \le n\}.$$

Hence, T is a Fan–Browder map. Since $(E; \Gamma) = (X \times Y; \Gamma_{X \times Y})$ satisfies the partial KKM principle, (A) and (B) hold. Therefore, by (B), we have an $(x_0, y_0) \in X \times Y$ such that $(x_0, y_0) \in T(x_0, y_0)$. Therefore, $t(x_0, y_0) < c < s(x_0, y_0)$, a contradiction. □

An abstract convex space $(E, D; \Gamma)$ is said to be *compact* if E is a compact topological space. For example, any compact G-convex space, any compact H-space, or any compact convex space is such a space.

Theorem 3.3. *Let* $(X; \Gamma_1)$ *and* $(Y; \Gamma_2)$ *be compact abstract convex spaces,* $(E; \Gamma) := (X \times Y; \Gamma_{X \times Y})$ *the product abstract convex space defined as in Lemma 3.1, and* $f, s, t, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ *be functions satisfying*

(3.1) $f(x, y) \le s(x, y) \le t(x, y) \le g(x, y)$ *for each* $(x, y) \in X \times Y$;
(3.2) *for each* $x \in X$, $f(x, \cdot)$ *is l.s.c. and* $t(x, \cdot)$ *is quasiconvex on* Y*; and*
(3.3) *for each* $y \in Y$, $s(\cdot, y)$ *is quasiconcave and* $g(\cdot, y)$ *is u.s.c. on* X.

If $(E; \Gamma)$ *satisfies the partial KKM principle, then we have*

$$\min_{y \in Y} \sup_{x \in X} f(x, y) \le \max_{x \in X} \inf_{y \in Y} g(x, y).$$

Proof. Note that $y \mapsto \sup_{x \in X} f(x, y)$ is l.s.c. on Y and $x \mapsto \inf_{y \in Y} g(x, y)$ is u.s.c. on X. Therefore, both sides of the inequality exist. Then all the requirements of Theorem 3.2 are satisfied. □

For $f = s = t = g$ in Theorem 3.3, we have the following generalization of the Sion minimax theorem [2]:

Theorem 3.4. *Let* $(X; \Gamma_1)$ *and* $(Y; \Gamma_2)$ *be compact abstract convex spaces and* $f : X \times Y \to \mathbb{R}$ *a real function such that*

(4.1) *for each* $x \in X$, $f(x, \cdot)$ *is l.s.c. and quasiconvex on* Y*; and*
(4.2) *for each* $y \in Y$, $f(\cdot, y)$ *is u.s.c. and quasiconcave on* X.

If $(E; \Gamma) = (X \times Y; \Gamma_{X \times Y})$ *satisfies the partial KKM principle, then*

(i) f *has a saddle point* $(x_0, y_0) \in X \times Y$*; and*
(ii) *we have*

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

Proof. It is well known and easy to see that the minima and the maxima in Theorem 3.4 exist under our topological assumptions. Hence, there exists an $(x_0, y_0) \in X \times Y$ such that

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} f(x, y_0) \ge f(x_0, y_0) \ge \min_{y \in Y} f(x_0, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

Moreover, all the requirements of Theorem 3.3 with $f = g$ are satisfied. Therefore, the $\ge$s in the above should be $=$ and we have the conclusion. □

Remark. 1. von Neumann [1] obtained Theorem 3.4 when X and Y are subsets of Euclidean spaces and f is continuous.
2. For KKM spaces, (A) also holds for open-valued KKM maps, and so does (B) when T^- has closed values. In this case, (A) and (B) are also equivalent.
3. For another simple proof of the Sion minimax theorem, see [4].
4. Theorem 3.2 is motivated from [13, Theorem 8], which is for $f = s = t = g$.
5. For the history of the KKM theory, see [14].
6. Theorems 3.2–3.4 hold for G-convex spaces and hence, for H-spaces, for convex spaces, and for convex subsets of topological vector spaces (see [5]). Therefore they contain numerous particular cases in the literature.

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