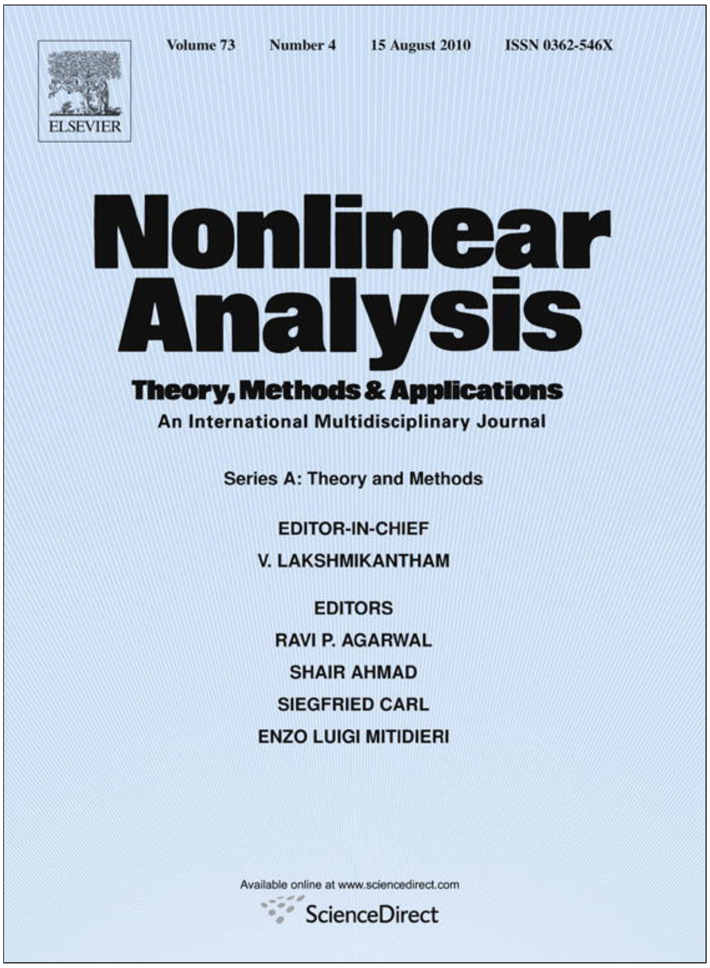
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The KKM principle in abstract convex spaces: Equivalent formulations and applications

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ABSTRACT

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In this paper, we clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptiness of the intersection of a certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or other solutions of various equilibrium problems. One of the remarkable results on the nonempty intersection is the celebrated Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM theorem) in 1929 [1], which is concerned with certain types of multimaps called the KKM maps later. This is the origin of the KKM theory, first named by the author [2], as the study of applications of equivalent formulations or generalizations of the KKM theorem.

The KKM theorem was based on the celebrated Sperner combinatorial lemma and first applied in [1] to a simple proof of the Brouwer fixed point theorem. Later it was known that these three theorems are mutually equivalent and regarded as

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a sort of mathematical trinity; see [3]. All are extremely important and have many applications. The first application of the KKM theorem after [1] was Sion's 1958 generalization [4] of the von Neumann minimax theorem.

From 1961, Ky Fan showed that the KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. He extended the KKM theorem to arbitrary topological vector spaces and applied it to various problems; see [3,5]. Fan's works were expanded systematically by Granas [5] to topological methods in convex analysis mainly on convex subsets of topological vector spaces. Later, it had been extended to convex spaces by Lassonde [6], and to C-spaces (or H-spaces) by Horvath [7,8], and many others. In the last decade of the 20th century, the KKM theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the author; for details, see [3,9] and references therein.

Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify known results in the theory on convex spaces, H-spaces, G-convex spaces, and others; see [10–14]. Moreover, in 2007, we found that most of variations of G-convex spaces can be subsumed in the concept of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ and made into G-convex spaces; see [11,15]. Such contents of the KKM theory have numerous applications on various fields, especially, on fixed point theory [16] and equilibrium theory [17].

The partial KKM principle for an abstract convex space is an abstract form of the KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In our recent works [12–14], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are closely related to KKM spaces or spaces satisfying the partial KKM principle. Moreover, a number of such results are equivalent to each other.

On the other hand, some authors studied particular types of KKM spaces and deduced certain important results like the Nash type equilibrium existence theorem from the partial KKM principle. Therefore, in order to avoid unnecessary repetitions for each particular type of KKM spaces, it would be necessary to state clearly them for KKM spaces; see [16]. Moreover, we noticed that the partial KKM principle implies many useful results in any abstract convex spaces. This was stated as a metatheorem in our previous work [18] without proof.

In this paper, we clearly show that a sequence of a dozen statements characterize the KKM spaces and are equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of the Fan minimax inequality, variational inequalities, the von Neumann minimax theorem, the von Neumann intersection lemma, and the Nash equilibrium theorem for any spaces satisfying partial KKM principle. Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

This paper is a refinement of [12–14], where more preliminaries can be found.

2. Abstract convex spaces and the KKM spaces

A multimap $F : X \multimap Y$ is a function $F : X \to 2^Y$ to the power set of Y and $F^- : Y \multimap X$ is defined by $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. Multimaps are also called simply maps. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [10–14]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the *Γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a *Γ-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be *Γ-convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. The following are known examples of abstract convex spaces:

(1) The original KKM theorem [1] is for the triple $(\Delta_n, V; \mathrm{co})$, where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and $\mathrm{co} : \langle V\rangle \multimap \Delta_n$ the convex hull operation.
(2) A triple $(X \supset D; \Gamma)$, where X is a subset of a t.v.s. E such that $\mathrm{co}\, D \subset X$ and $\Gamma := \mathrm{co}$. Fan's celebrated KKM lemma [19] is for $(E, D; \mathrm{co})$, where D is a nonempty subset of E.
(3) A *convex space* $(X \supset D; \Gamma)$ is a triple where X is a subset of a vector space such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde [6] for $X = D$. However he obtained several KKM type theorems w.r.t. $(X, D; \Gamma)$.
(4) A triple $(X \supset D; \Gamma)$ is called an *H-space* if X is a topological space and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D\rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D\rangle$. If $D = X$, $(X; \Gamma) := (X, X; \Gamma)$ is called a C-space by Horvath [7,8].
(5) Hyperconvex metric spaces due to Aronszajn and Panitchpakdi are particular cases of C-spaces; see [8].

(6) Hyperbolic spaces due to Reich and Shafrir [20] are particular cases of C-spaces. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic.
(7) Any topological semilattice $(X, \leq)$ with path-connected interval introduced by Horvath and Llinares [21]. See also [22].
(8) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.
Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$.
For $X = D$, G-convex spaces reduce to L-spaces due to Ben-El-Mechaiekh et al. Recall that all examples (1)–(7) are G-convex spaces. For details, see references of [5,9,15,22].
(9) A *ϕ_A-space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [11,15]. When $X = D$, a ϕ_A-space is called an FC-space [15] or a simplicial space [23]. Later, a ϕ_A-space is called GFC-space by Khanh et al. [24].
(10) Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_H(A)$, where $conv_H(A)$ is the intersection of all closed convex subsets of H that contain A; see Kirk and Panyanak [25]. Then $(H \supset X; \Gamma)$ is an abstract convex space.
(11) According to Horvath [26], a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from the power set $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.
(12) A $\mathbb{B}$-space due to Briec and Horvath [27] is an abstract convex space.

Note that each of the above has a large number of concrete examples.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a map $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \quad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

Example. We give known examples of KKM spaces:

(1) Every G-convex space is a KKM space [9].
(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space [13].
(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space; see [13].
(4) For a closed convex subset X of a complete $\mathbb{R}$-tree H, and $\Gamma_A := conv_H(A)$ for each $A \in \langle X \rangle$, the triple $(H \supset X; \Gamma)$ satisfies the partial KKM principle; see Kirk and Panyanak [25]. Later we found that $(H \supset X; \Gamma)$ is a KKM space [18].
(5) For Horvath's convex space $(X; \Gamma)$ with the weak Van de Vel property is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$; see [26,18].
(6) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space [27].

In our recent work [12–14], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle
$\Longrightarrow$ Abstract convex space.

It is not known yet whether there is an abstract convex space satisfying the partial KKM principle that is not a KKM space.

3. Equivalents of the KKM principle

For an abstract convex space $(E, D; \Gamma)$, let us consider the following statements:

(0) **The KKM principle.** *For any closed-valued [resp., open-valued] KKM map $G : D \multimap E$, the family $\{G(z)\}_{z\in D}$ has the finite intersection property.*

(I) **The Fan matching property.** *Let* $S : D \multimap E$ *be a map satisfying*

(1.1) $S(z)$ *is open [resp., closed] for each* $z \in D$; *and*
(1.2) $E = \bigcup_{z\in M} S(z)$ *for some* $M \in \langle D\rangle$.

Then there exists an $N \in \langle M\rangle$ *such that*

$$\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset.$$

(II) **Another finite intersection property.** *Let* $S : D \multimap E,\ T : E \multimap E$ *be maps satisfying*

(2.1) S *has closed [resp., open] values;*
(2.2) *for each* $y \in E$, $\mathrm{co}_\Gamma(D \setminus S^-(y)) \subset E \setminus T^-(y)$; *and*
(2.3) $x \in T(x)$ *for each* $x \in E$.

Then $\{S(z)\}_{z\in D}$ *has the finite intersection* property.
(III) **The geometric property or the section property.** *Let* $A \subset D \times E,\ B \subset E \times E$ *maps satisfying*

(3.1) $\{y \in E \mid (z, y) \in A\}$ *is closed [resp., open] for each* $z \in D$;
(3.2) *for each* $y \in E$, $\mathrm{co}_\Gamma\{z \in D \mid (z, y) \notin A\} \subset \{x \in E \mid (x, y) \notin B\}$; *and*
(3.3) $(x, x) \in B$ *for each* $x \in E$.

Then, for each $N \in \langle D\rangle$, *there exists an* $x_N \in E$ *such that* $N \times \{x_N\} \subset A$.
(IV) **Another geometric property.** *For any sets* $A \subset D \times E,\ B \subset E \times E$ *satisfying*

(4.1) $\{y \in E \mid (z, y) \in A\}$ *is open [resp., closed] for each* $z \in D$;
(4.2) *for each* $y \in E$, $\mathrm{co}_\Gamma\{z \in D \mid (z, y) \in A\} \subset \{x \in E \mid (x, y) \in B\}$; *and*
(4.3) *there exists an* $M \in \langle D\rangle$ *such that for any* $y \in E$, $(z, y) \in A$ *for some* $z \in M$.

Then there exists an $x_0 \in E$ *such that* $(x_0, x_0) \in B$.
(V) **The Fan–Browder fixed point property.** *Let* $S : E \multimap D,\ T : E \multimap E$ *be maps satisfying*

(5.1) *for each* $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$;
(5.2) $S^-(z)$ *is open [resp., closed] for each* $z \in D$; *and*
(5.3) $E = \bigcup_{z\in M} S(z)$ *for some* $M \in \langle D\rangle$.

Then T *has a fixed point* $x_0 \in E$; *that is,* $x_0 \in T(x_0)$.

Definition. For any topological space X and an abstract convex space $(E, D; \Gamma)$, a map $T : X \multimap E$ is called a *Φ-map* or a *Fan–Browder map* whenever there is a *companion map* $S : X \multimap D$ satisfying

(1) for each $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and
(2) $S^-(z)$ is open for each $z \in D$.

(VI) **Existence of maximal elements.** *Let* $S : E \multimap D,\ T : E \multimap E$ *be maps satisfying*

(6.1) $S^-(z)$ *is open [resp., closed] for each* $z \in D$;
(6.2) *for each* $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; *and*
(6.3) *for each* $x \in E$, $x \notin T(x)$.

Then there exists an $x \in E$ *such that* $S(x) = \emptyset$.
(VII) **Analytic formulation.** *Let* $A \subset C$ *be sets and* $\phi : D \times E \to C$ *be a function such that*

(7.1) *for each* $N \in \langle D\rangle$ *and* $y \in \Gamma_N$, *there exists a* $z \in N$ *satisfying* $\phi(z, y) \in A$, *and*
(7.2) *for each* $z \in D$, *the set* $\{y \in E \mid \phi(z, y) \in A\}$ *is open [resp., closed].*

Then, for each $N \in \langle D\rangle$, *there exists a* $y_N \in E$ *such that*

$$\phi(z, y_N) \in A \quad \text{for all } z \in N.$$

(VIII) **Minimax inequality.** *Let* $\phi : D \times E \to \overline{\mathbb{R}}$ *be an extended real-valued function and* $\gamma \in \overline{\mathbb{R}}$ *such that*

(8.1) *for each* $N \in \langle D\rangle$ *and* $y \in \Gamma_N$, $\min\{\phi(z, y) \mid z \in N\} \le \gamma$, *and*
(8.2) *for each* $z \in D$, *the set* $\{y \in E \mid \phi(z, y) \le \gamma\}$ *is open [resp., closed].*

Then for each $N \in \langle D\rangle$,

(i) *there exists a* $y_N \in E$ *such that*

$$\phi(z, y_N) \le \gamma \quad \text{for all } z \in N; \text{ and}$$

(ii) *if* $E = D$ *and* $\gamma = \sup\{\phi(x, x) \mid x \in E\}$, *then we have the minimax inequality:*

$$\inf_{y\in E} \max_{z\in N} \phi(z, y) \le \sup_{x\in E} \phi(x, x).$$

(IX) **Analytic alternative.** *Let $A, B \subset C$ sets and $f : D \times E \to C$, $g : E \times E \to C$ functions. Suppose that*

(9.1) *for each $y \in E$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) \in A\} \subset \{x \in E \mid g(x, y) \in B\}$, and*
(9.2) *for each $z \in D$, the set $\{y \in E \mid f(z, y) \in A\}$ is open [resp., closed].*

Then either

(a) *for each $N \in \langle D\rangle$, there exists a $y_N \in E$ such that $f(z, y_N) \notin A$ for all $z \in N$; or*
(b) *there exists an $\hat{x} \in E$ such that $g(\hat{x}, \hat{x}) \in B$.*

(X) **Analytic alternative.** *Let $\alpha, \beta \in \mathbb{R}$, and $f : D \times E \to \overline{\mathbb{R}}$, $g : E \times E \to \overline{\mathbb{R}}$ extended real-valued functions. Suppose that*

(10.1) *for each $y \in E$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \alpha\} \subset \{x \in E \mid g(x, y) > \beta\}$, and*
(10.2) *for each $z \in D$, the set $\{y \in E \mid f(z, y) > \alpha\}$ is open [resp., closed].*

Then either

(a) *for each $N \in \langle D\rangle$, there exists a $y_N \in E$ such that $f(z, y_N) \le \alpha$ for all $z \in N$; or*
(b) *there exists an $\hat{x} \in E$ such that $g(\hat{x}, \hat{x}) > \beta$.*

(XI) **Minimax inequality.** *Under the hypothesis of* (X), *if*

$$\alpha = \beta = \sup\{g(x, x) \mid x \in E\},$$

then for each $N \in \langle D\rangle$,

(c) *there exists a $y_N \in E$ such that*

$$f(z, y_N) \le \sup_{x\in E} g(x, x) \quad \text{for all } z \in N; \text{ and}$$

(d) *we have the following minimax inequality*

$$\inf_{y\in E} \max_{z\in N} f(z, y) \le \sup_{x\in E} g(x, x).$$

Now we show that (0)–(XI) are mutually equivalent:

Theorem 1 (*Characterizations of the KKM Spaces*). *For a KKM space $(E, D; \Gamma)$, the following mutually equivalent statements* (0)–(XI) *hold.*

Proof. (0) $\Longrightarrow$ (I). Let $G : D \multimap E$ be a map given by $G(z) := E \setminus S(z)$ for $z \in D$. Then G has closed [resp., open] values. Suppose, on the contrary to the conclusion, that for any $N \in \langle M\rangle$, we have $\Gamma_N \cap \bigcap_{z\in N} S(z) = \emptyset$; that is, $\Gamma_N \subset E \setminus \bigcap_{z\in N} S(z) = \bigcup_{z\in N}(E \setminus S(z)) = G(N)$. Then $G|_M : M \multimap E$ is a KKM map. It is easily checked that $(E, M; \Gamma|_{\langle M\rangle})$ also satisfies the KKM principle (0) [12, Lemma 2]; and hence there exists a $\widehat{y} \in \bigcap_{z\in N} G(z) = \bigcap_{z\in N}(E \setminus S(z))$. Hence $\widehat{y} \notin S(z)$ for all $z \in N$. Since N is arbitrary, this violates condition (1.2).
(I) $\Longrightarrow$ (II). Suppose there exist an $M \in \langle D\rangle$ such that $\bigcap_{z\in M} S(z) = \emptyset$, that is, $E = \bigcup_{z\in M}(E \setminus S(z))$. Then by (I), there exist an $N \in \langle M\rangle$ and a

$$y_0 \in \Gamma_N \cap \bigcap_{z\in N}(E \setminus S(z)) \neq \emptyset.$$

Since $y_0 \in E \setminus S(z) \Leftrightarrow y_0 \notin S(z) \Leftrightarrow z \notin S^-(y_0)$ for all $z \in N$, we have $N \subset D \setminus S^-(y_0)$. Since

$$y_0 \in \Gamma_N \subset \mathrm{co}_\Gamma(D \setminus S^-(y_0)) \subset E \setminus T^-(y_0)$$

by (2.2), we have $y_0 \notin T^-(y_0)$ or $y_0 \notin T(y_0)$, which violates (2.3).
(II) $\Longrightarrow$ (III). For each $z \in D$, let $S(z) := \{y \in E \mid (z, y) \in A\}$. Then (3.1) $\Rightarrow$ (2.1). Moreover, for each $x \in E$, let $T(x) := \{y \in E \mid (x, y) \in B\}$. Then (3.2) $\Rightarrow$ (2.2). Further (3.3) $\Rightarrow$ (2.3). Therefore, by (II), for each $N \in \langle D\rangle$, we have

$$\bigcap_{z\in N} S(z) = \bigcap_{z\in N}\{y \in E \mid (z, y) \in A\} \neq \emptyset.$$

Hence there exists an $x_N \in E$ such that $(z, x_N) \in A$ for all $z \in N$; that is, $N \times \{x_N\} \subset A$.
(III) $\Longrightarrow$ (IV). Consider (III) replacing (A, B) by their respective complements (A^c, B^c). Then (3.1) and (3.2) are satisfied by (4.1) and (4.2). Since (4.3) is the negation of the conclusion of (III), we should have the negation of (3.3). Therefore, the conclusion follows.
(IV) $\Longrightarrow$ (V). Let A and B be the graphs of S^- and T^-, resp. Then (5.1)–(5.3) imply (4.1)–(4.3). Therefore, by (IV), there exists an $x_0 \in E$ such that $(x_0, x_0) \in B$, that is, T has a fixed point $x_0 \in E$.
(V) $\Longrightarrow$ (VI). Suppose that E is covered by a finite number of $S^-(z)$'s, $z \in D$. Then all of the requirements of (V) are satisfied. Therefore, there exists an $x_0 \in E$ such that $x_0 \in T(x_0)$. But this violates (6.3).
(VI) $\Longrightarrow$ (0). Let $G : D \multimap E$ be a KKM map with closed [resp., open] values. Suppose the family $\{G(z)\}_{z\in D}$ does not have the finite intersection property; that is, there exists an $M \in \langle D\rangle$ such that $\bigcap_{z\in M} G(z) = \emptyset$. Define a map $S : E \multimap D$ by

$S^-(z) = G^c(z) := E \setminus G(z)$ for $z \in D$ and a map $T : E \multimap E$ by $\mathrm{co}_\Gamma S(x) = T(x)$ for each $x \in E$. Then the requirements (6.1) and (6.2) hold. Moreover, $\bigcap_{z\in M} G(z) = \emptyset$ implies that $E = \bigcup_{z\in M} G^c(z)$, which violates the conclusion of (VI). Hence (6.3) does not hold, that is, there exists an $x_0 \in E$ such that $x_0 \in T(x_0)$= $\mathrm{co}_\Gamma S(x_0)$. So, there exists an $N \in \langle S(x_0)\rangle$ such that $x_0 \in \Gamma_N \subset \mathrm{co}_\Gamma S(x_0)$. Therefore, for each $z \in N$, we have $x_0 \in S^-(z)$ or $x_0 \notin G(z)$; that is, $\Gamma_N \not\subset G(N)$. Hence G is not a KKM map, a contradiction. Therefore $(E, D; \Gamma)$ satisfies the KKM principle (0).
(0) $\Longrightarrow$ (VII). Let $G(z) := \{y \in E \mid \phi(z, y) \in A\}$ for $z \in D$. We show that (7.1) implies G is a KKM map. Suppose that there exists an $N \in \langle D\rangle$ such that $\Gamma_N \not\subset G(N)$. Choose a $y \in \Gamma_N$ such that $y \notin G(N)$, whence $\phi(z, y) \notin A$ for all $z \in N$. This contradicts (7.1). From (7.2), by (0), for each $N \in \langle D\rangle$, there exists a $y_N \in E$ such that $y_N \in G(z)$ for all $z \in N$; that is, $\phi(z, y_N) \in A$ for all $z \in N$. This completes the proof.
(VII) $\Longrightarrow$ (VIII). Put $A := [-\infty, \gamma]$ and $C := \overline{\mathbb{R}}$. Then (i) follows from (VII) and it is clear that (ii) follows from (i).
(VIII) $\Longrightarrow$ (0). Define $\phi : D \times E \to \overline{\mathbb{R}}$ by

$$\phi(z, y) := \begin{cases} 0 & \text{if} \quad y \in G(z); \\ 1 & \text{otherwise} \end{cases}$$

for $(z, y) \in D \times E$. Put $\gamma = 0$ in (VIII). From the fact that G is a KKM map, we have (8.1). In fact, suppose that there exist an $N \in \langle D\rangle$ and a $y \in \Gamma_N$ such that $\min\{\phi(z, y) \mid z \in N\} > 0$. Then $y \notin G(z)$ for all $z \in N$; that is, $\Gamma_N \not\subset G(N)$, a contradiction. Therefore, all of the requirements of (VIII) are satisfied. Hence, for each $N \in \langle D\rangle$, there exists a $y_N \in E$ such that $\phi(z, y_N) = 0$ for all $z \in N$; that is, $y_N \in \bigcap\{G(z) \mid z \in N\}$. Therefore (0) holds.
(VI) $\Longrightarrow$ (IX). Define $S : E \multimap D$, $T : E \multimap E$ by

$$S(y) := \{z \in D \mid f(z, y) \in A\}, \quad T(y) := \{x \in E \mid g(x, y) \in B\}$$

for each $y \in E$. Then (9.1) $\Longrightarrow$ (6.2) and (9.2) $\Longrightarrow$ (6.1). Suppose that (b) does not hold, that is, $g(x, x) \notin B \Longleftrightarrow x \notin T(x)$ for all $x \in E$. Hence (6.3) holds. Then, by (VI), E cannot be covered by a finite number of $S^-(z)$'s, $z \in D$. Hence the conclusion (a) holds.
(IX) $\Longrightarrow$ (X). Put $C := \overline{\mathbb{R}}$, $A := (\alpha, \infty]$, and $B := (\beta, \infty]$ in (IX).
(X) $\Longrightarrow$ (XI). Clear.
(XI) $\Longrightarrow$ (II). Define real-valued functions $f : D \times E \to \mathbb{R}$ and $g : E \times E \to \mathbb{R}$ by

$$f(z, y) := \begin{cases} 0 & \text{if} \quad y \in S(z); \\ 1 & \text{otherwise} \end{cases}$$

for $(z, y) \in D \times E$ and

$$g(x, y) := \begin{cases} 0 & \text{if} \quad y \in T(x); \\ 1 & \text{otherwise} \end{cases}$$

for $(x, y) \in E \times E$. Put $\alpha = \beta = 0$. Then (2.2) implies (10.1). In fact, $M \in \langle\{z \in D \mid f(z, y) > 0\}\rangle = \langle\{z \in D \mid f(z, y) = 1\}\rangle = \langle D \setminus S^-(y)\rangle$ implies $\Gamma_M \subset E \setminus T^-(y) = \{x \in E \mid g(x, y) = 1\} = \{x \in E \mid g(x, y) > 0\}$. Since $\sup\{g(x, x) \mid x \in E\} \le \sup\{g(x, y) \mid (x, y) \in T\} = 0$ by (2.2) and the definition of g, $\sup\{g(x, x) \mid x \in E\} = 0$. Therefore, by (XI), for each $N \in \langle D\rangle$, there exists a $y_N \in E$ such that

$$f(z, y_N) \le \sup_{x\in E} g(x, x) = 0 \quad \text{for all } z \in N.$$

Hence $f(z, y_N) = 0$ for all $z \in N$; that is, $y_N \in S(z)$ for all $z \in N$. Therefore,

$$\bigcap\{S(z) \mid z \in N\} \neq \emptyset.$$

This completes our proof of Theorem 1. □

4. Equivalents of the generalized partial KKM principle

For an abstract convex space $(E, D; \Gamma)$, let us consider the following partial condition of (0):
(0)′ **The partial KKM principle.** *For any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property.*
(I)′– (XI)′ the corresponding partial conditions of (I)–(XI), resp.
From Theorem 1, we have the following:

Theorem 2 (*Characterizations of Spaces Satisfying the Partial KKM Principle*)**.** *For an abstract convex space $(E, D; \Gamma)$ satisfying the partial KKM principle, the equivalent statements* (0)′–(XI)′ *hold.*

In the present section, we consider further properties of abstract convex spaces satisfying the partial KKM principle. The following equivalent form of [14, Theorem 8.2] is basic:

Theorem 3 (*Generalized Partial KKM Principle*)**.** *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and $G : D \multimap E$ a map such that*
(1) *G is closed-valued;*

(2) *G is a KKM map (that is, $\Gamma_A \subset G(A)$ for all $A \in \langle D\rangle$); and*
(3) *there exists a nonempty compact subset K of E such that one of the following holds:*
 (i) *$K = E$;*
 (ii) *$K = \bigcap\{G(z) \mid z \in M\}$ for some $M \in \langle D\rangle$; or*
 (iii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \cap \bigcap_{z\in D'} G(z) \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. Case (i): In this case every $G(y)$ is compact. Hence Case (i) reduces to (ii).

Case (ii): Since $\{G(z) \mid z \in D\}$ has the finite intersection property, so does $\{K \cap G(z) \mid z \in D\}$ in the compact set K. Hence it has the whole intersection property.

Case (iii): Suppose that $K \cap \bigcap\{G(z) \mid z \in D\} = \emptyset$; that is, $K \subset \bigcup\{X \setminus G(z) \mid z \in N\}$ for some $N \in \langle D\rangle$. Let L_N be the compact Γ-convex subset of E in (iii). Define $G' : D' \multimap L_N$ by $G'(z) := G(z) \cap L_N$ for $z \in D'$. Then $A \in \langle D'\rangle$ implies $\Gamma'_A := \Gamma_A \cap L_N \subset G(A) \cap L_N = G'(A)$ by (2); and hence $G' : D' \multimap L_N$ is a KKM map on $(L_N, D'; \Gamma')$ with closed values. Since $(X, D; \Gamma)$ satisfies the partial KKM principle, so does $(L_N, D'; \Gamma')$; see [12, Lemma 2]. Hence, $\{G'(z) \mid z \in D'\}$ has the finite intersection property. Since L_N is compact, and $\bigcap\{G'(z) \mid z \in D'\} \neq \emptyset$ by Case (i). For any $y \in \bigcap\{G'(z) \mid z \in D'\}$, we have $y \in K$ by (ii). However, since $y \in K \subset \bigcup\{X \setminus G(z) \mid z \in N\}$, we have $y \notin G(z)$ for some $z \in N \subset D'$. This is a contradiction.

Therefore, we must have $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$. □

Recall that conditions (i)–(iii) in Theorem 3 are usually called the *compactness conditions* or the *coercivity conditions*. More formally we define as follows:

Definition. For an abstract convex space $(E, D; \Gamma)$ and a closed-valued map $G : D \multimap E$, a *coercivity condition* for G is the one guaranteeing the whole intersection property of the family $\{G(y)\}_{y\in D}$ whenever it has the finite intersection property.

Example. (1) Theorem 3 shows that each of (i)–(iii) is a coercivity condition for any $(E, D; \Gamma)$ and any G. There appeared several hundred particular cases of the condition (iii).

For particular spaces and particular maps, we may have another coercivity conditions; for example, we have the following:

(2) An abstract convex space $(H, D; \Gamma)$ is called a *hyperconvex metric space* if (H, d) is a hyperconvex metric space in the sense of Aronszajn and Panitchpakdi, D is a nonempty set, and $\Gamma : \langle D\rangle \to \mathcal{A}(H)$ is a map having admissible values (that is, $\mathcal{A}(H)$ is the class of intersections of closed balls) such that

$$A, B \in \langle D\rangle,\ A \subset B \text{ implies } \Gamma_A \subset \Gamma_B.$$

Since any family of closed balls in a hyperconvex metric space has nonempty intersection whenever each two members of the family intersects, any KKM map $G : D \to \mathcal{A}(H)$ need not have any coercivity condition (that is, G has the trivial coercivity condition); see [18]:

(3) For a closed convex subset X of a complete $\mathbb{R}$-tree H, and $\Gamma_A := conv_H(A)$ for each $A \in \langle X\rangle$, the triple $(H \supset X; \Gamma)$ satisfies the partial KKM principle; see Kirk and Panyanak [25]. A coercivity condition for closed-valued KKM map $G : X \multimap H$ is that X is geodesically bounded [25, Theorem 4.2].

Recall that the first coercivity condition was adopted by Fan 1969 [19] in his well-known KKM lemma. Motivated by this and the above observations, we define as follows:

Definition. A *Fan map* $G : D \multimap E$ on an abstract convex space $(E, D; \Gamma)$ is a closed-valued KKM map having a coercivity condition. (Hence, we have $\bigcap_{y\in D} G(y) \neq \emptyset$.)

Example. Even in a KKM space, there is a KKM map which is not a Fan map. Let $(\mathbb{R};\ \mathrm{co})$ be a real line with the convex hull operation $\mathrm{co} : \langle\mathbb{R}\rangle \multimap \mathbb{R}$. Then it is a KKM space. Define $G(x) := [x, \infty)$ for $x \in \mathbb{R}$. Then $G : \mathbb{R} \multimap \mathbb{R}$ is a KKM map, but not a Fan map since $\bigcap_{x\in\mathbb{R}} G(x) = \emptyset$.

Theorem 3 can be reformulated as for the case $(0)'$–$(\mathrm{XI})'$ and hence we have another dozen equivalent statements. We give only the following example corresponding to $(\mathrm{XI})'$:

Theorem 4 (*Minimax Inequality*)**.** *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, $f : D \times X \to \overline{\mathbb{R}}$, $g : X \times X \to \overline{\mathbb{R}}$ extended real-valued functions, and $\gamma \in \overline{\mathbb{R}}$ such that*
(1) *for each $z \in D$, $\{y \in X \mid f(z, y) \le \gamma\}$ is closed;*
(2) *for each $y \in E$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \gamma\} \subset \{x \in X \mid g(x, y) > \gamma\}$; and*
(3) *the map $G : D \multimap E$ defined by $G(z) := \{y \in X \mid f(z, y) \le \gamma\}$ has a coercivity condition.*

Then
 (i) *there exists a $y_0 \in E$ such that $f(z, y_0) \le \gamma$ for all $z \in D$; and*
 (ii) *if $\gamma := \sup_{x\in X} g(x, x)$, then we have*

$$\inf_{y\in X} \sup_{z\in D} f(x, y) \le \sup_{x\in X} g(x, x).$$

Proof. Each $G(z)$ is closed by (1). From (2), it is routine to show that $G : D \multimap X$ is a KKM map. Then $\{G(z)\}_{z \in D}$ has the finite intersection property. Now by (3), G is a Fan map and hence we have $\bigcap_{z \in D} G(x) \neq \emptyset$. The conclusion follows as in Theorem 3. $\square$

5. Applications of the partial KKM principle on compact spaces

An abstract convex space $(E, D; \Gamma)$ is said to be *compact* if E is a compact topological space. From now on, for simplicity, we are mainly concerned with compact abstract convex spaces $(X; \Gamma)$ satisfying the partial KKM principle. For example, any compact G-convex space, any compact H-space, or any compact Lassonde type convex space is such a space. Of course, all results in this section can be generalized to non-compact case by assuming proper coercivity conditions for relevant KKM maps.

Recall that an extended real-valued function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is open for each $r \in \overline{\mathbb{R}}$.

For an abstract convex space $(E \supset D; \Gamma)$, a function $f : E \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in E \mid f(x) > r\}$ [resp., $\{x \in E \mid f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbb{R}}$.

In what follows, we simply concerned with real-valued functions, which can be replaced by functions having values in certain ordered sets or ordered vector spaces.

Consider the following statements for compact abstract convex spaces $(X; \Gamma)$ satisfying the partial KKM principle:

(XII) **Minimax inequality.** Theorem 4 *for the particular case when $X := E = D$ is compact.*

(XIII) **Minimax inequality.** *Let $f, g : X \times X \to \mathbb{R}$ be functions and $\gamma \in \mathbb{R}$ such that*

(13.1) *for any $x, y \in X$, $f(x, y) \leq g(x, y)$ and $g(x, x) \leq \gamma$;*
(13.2) *for each $x \in X$, $\{y \in X \mid f(x, y) > \gamma\}$ is open in X; and*
(13.3) *for each $y \in X$, $\{x \in X \mid g(x, y) > \gamma\}$ is Γ-convex on X.*

Then

(i) *there exists a $y_0 \in X$ such that*

$$f(x, y_0) \leq \gamma \quad \text{for all } x \in X;$$

(ii) *if $\gamma := \sup_{x \in X} g(x, x)$, then*

$$\inf_{y \in X} \sup_{x \in E} f(x, y) \leq \sup_{x \in X} g(x, x).$$

(XIV) **Minimax inequality.** *Let $f, g : X \times X \to \mathbb{R}$ be functions such that*

(14.1) *$f(x, y) \leq g(x, y)$ for each $(x, y) \in X \times X$,*
(14.2) *for each $x \in X$, $f(x, \cdot)$ is l.s.c. on X; and*
(14.3) *for each $y \in X$, $g(\cdot, y)$ is quasiconcave on E.*

Then we have

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} g(x, x).$$

(XV) **Variational inequality.** *Let $p, q : X \times X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ functions satisfying*

(15.1) *$p(x, y) \leq q(x, y)$ for each $(x, y) \in X \times X$, and $q(x, x) \leq 0$ for all $x \in X$;*
(15.2) *for each $x \in X$, $p(x, \cdot) + h(\cdot)$ is l.s.c. on X; and*
(15.3) *for each $y \in X$, $q(\cdot, y) - h(\cdot)$ is quasiconcave on X.*

Then there exists a $y_0 \in X$ such that

$$p(x, y_0) + h(y_0) \leq h(x) \quad \text{forall } x \in X.$$

(XVI) **Variational inequality.** *Let $f, g : X \times X \to \mathbb{R}$ be a function satisfying*

(16.1) *for any $x, y \in X$, $f(x, y) \leq g(x, y)$;*
(16.2) *for each $x \in X$, $\{y \in X \mid f(x, y) < f(y, y)\}$ is open; and*
(16.3) *for each $y \in X$, $\{x \in X \mid g(x, y) < g(y, y)\}$ is Γ-convex.*

Then

(i) *there exists a $y_0 \in X$ such that*

$$f(x, y_0) \geq f(y_0, y_0) \quad \text{for all } x \in X; \text{ and}$$

(ii) *we have*

$$\sup_{y \in X} \inf_{x \in X} f(x, y) \geq \inf_{x \in X} f(x, x).$$

(XVII) **Variational inequality.** *Let $f, g : X \times X \to \mathbb{R}$ be functions satisfying*

(17.1) *$f \leq g$ on the diagonal $\Delta := \{(x, x) \mid x \in X\}$ and $g \leq f$ on $(X \times X) \setminus \Delta$;*
(17.2) *for each $x \in X$, $y \mapsto g(y, y) - g(x, y)$ is l.s.c. on X; and*
(17.3) *for each $y \in X$, $x \mapsto f(x, y)$ is quasiconcave on X.*

Then there exists a $y_0 \in X$ *such that*

$$f(y_0, y_0) \geq f(x, y_0) \quad \text{for all } x \in X.$$

Theorem 5. *For a compact abstract convex space* $(X; \Gamma)$ *satisfying the partial KKM principle, the statements* (XII)–(XVII) *hold.*

Proof. (XII) holds since it is particular to Theorem 4.
(XIII) $\Longleftarrow$ (XII). Note that (13.1) and (13.3) imply condition (2) of Theorem 4. Therefore, by (XII), we have the conclusion (i). If $\sup_{x\in E} g(x, x) = +\infty$, then the inequality in the conclusion (ii) holds automatically. If $\gamma = \sup_{x\in X} g(x, x) < +\infty$, then by (XII), we have the conclusion (ii).
(XIV) $\Longleftarrow$ (XIII). Observe that $\sup_{x\in E} f(x, y)$ is by (14.2) a l.s.c. function of y on the compact space X, and therefore its minimum exists. Therefore, by (XIII), we have the conclusion.
(XV) $\Longleftarrow$ (XIV). Let

$$f(x, y) := p(x, y) + h(y) - h(x), \qquad g(x, y) := q(x, y) + h(y) - h(x)$$

for $(x, y) \in X \times Y$. Then f and g satisfy the requirements of (XIV). Furthermore, $g(x, x) = q(x, x) \leq 0$ for all $x \in E$. Therefore, by (XIV), the conclusion follows.
(XVI) $\Longleftarrow$ (XIII). In (XIII), put $\gamma = 0$ and replace $f(x, y)$ and $g(x, y)$ by $f(y, y) - f(x, y)$ and $g(y, y) - g(x, y)$, resp. Then we have the conclusion.
(XVII) $\Longleftarrow$ (XVI). Define $p, q : X \times X \to \mathbb{R}$ by

$$p(x, y) := f(y, y) - f(x, y), \qquad q(x, y) := g(y, y) - g(x, y).$$

Then (1) $p(x, y) \leq q(x, y)$ and $q(x, x) = 0$ for all $x, y \in X$ by (17.1). Moreover (2) for each $x \in X$, $\{y \in X \mid p(x, y) < p(y, y)\} = \{y \in X \mid f(y, y) < f(x, y)\}$ is open by (17.2). Further, for each $y \in X$, $\{x \in X \mid q(x, y) < q(y, y)\} = \{x \in X \mid g(y, y) < g(x, y)\}$ is Γ-convex by (17.3). Therefore, by (XVI) with (p, q) instead of (f, g), we have a $y_0 \in X$ such that $p(x, y_0) \geq p(y_0, y_0)$ for all $x \in X$. Note that

$$\begin{aligned} p(x, y_0) \geq p(y_0, y_0) &\Longleftrightarrow f(y_0, y_0) - f(x, y_0) \geq f(y_0, y_0) - f(y_0, y_0) = 0 \\ &\Longleftrightarrow f(y_0, y_0) \geq f(x, y_0) \quad \text{for all} x \in X. \end{aligned}$$

This completes the proof of Theorem 5. □

Remark. From (XVI) or (XVII) we can obtain the Fan type best approximation theorems which directly implies the Brouwer fixed point theorem; see [9,12,13]. Recall that the Brouwer theorem is equivalent to the original KKM theorem and the KKM theorem for G-convex spaces; see [3,9]. Therefore, in the category of G-convex spaces, all of (0)–(XVII) are equivalent to the Brouwer fixed point theorem.

6. Applications of the partial KKM principle on product spaces

In this section, we are mainly concerned with Cartesian products (of abstract convex spaces) satisfying the partial KKM principle. For simplicity, we assume all spaces are compact. Of course, all results in this section can be generalized to non-compact case by assuming proper coercivity conditions for relevant KKM maps.

Let $(X; \Gamma_1)$ and $(Y; \Gamma_2)$ be abstract convex spaces. For their product, we can define $\Gamma_{X\times Y}(A) := \Gamma_1(\pi_1(A)) \times \Gamma_2(\pi_2(A))$ for $A \in \langle X \times Y\rangle$, where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are projections. Recall that if $(X; \Gamma_1)$ and $(Y; \Gamma_2)$ are G-convex spaces, so is their product and hence is a KKM space.

Let us consider the following statements:

(XVIII) **The basic minimax theorem.** *Let* $(E; \Gamma) := (X \times Y; \Gamma_{X\times Y})$ *be the product abstract convex space,* $f, s, t, g : X \times Y \to \mathbb{R}$ *be four functions,*

$$\mu := \inf_{y\in Y} \sup_{x\in X} f(x, y) \quad \text{and} \quad \nu := \sup_{x\in X} \inf_{y\in Y} g(x, y).$$

Suppose that

(18.1) $f(x, y) \leq s(x, y) \leq t(x, y) \leq g(x, y)$ *for each* $(x, y) \in X \times Y$;
(18.2) *for each* $r < \mu$ *and* $y \in Y$, $\{x \in X \mid s(x, y) > r\}$ *is* Γ_1*-convex; for each* $r > \nu$ *and* $x \in X$, $\{y \in Y \mid t(x, y) < r\}$ *is* Γ_2-convex;
(18.3) *for each* $r > \nu$, *there exists a finite set* $\{x_i\}_{i=1}^m \subset X$ *such that*

$$Y = \bigcup_{i=1}^{m} \text{Int}\, \{y \in Y \mid f(x_i, y) > r\};$$

(18.4) *for each* $r < \mu$, *there exists a finite set* $\{y_j\}_{j=1}^n \subset Y$ *such that*

$$X = \bigcup_{j=1}^{n} \text{Int}\, \{x \in X \mid g(x, y_j) < r\}.$$

If $(E;\Gamma)$ *satisfies the partial KKM principle, then we have*

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

(XIX) **Generalized von Neumann–Sion minimax theorem.** *Let* $(X;\Gamma_X)$ *and* $(Y;\Gamma_Y)$ *be compact abstract convex spaces,* $(E;\Gamma) := (X\times Y;\Gamma_{X\times Y})$ *the product abstract convex space, and* $f,s,t,g : X\times Y\to\mathbb{R}$ *be functions satisfying*

(19.1) $f(x,y)\le s(x,y)\le t(x,y)\le g(x,y)$ *for each* $(x,y)\in X\times Y$;
(19.2) *for each* $x\in X$, $f(x,\cdot)$ *is l.s.c. and* $t(x,\cdot)$ *is quasiconvex on* Y; *and*
(19.3) *for each* $y\in Y$, $s(\cdot,y)$ *is quasiconcave and* $g(\cdot,y)$ *is u.s.c. on* X.

If $(E;\Gamma)$ *satisfies the partial KKM principle, then we have*

$$\min_{y\in Y}\sup_{x\in X} f(x,y) \le \max_{x\in X}\inf_{y\in Y} g(x,y).$$

(XX) **The von Neumann–Sion minimax theorem.** *Let* $(E;\Gamma) := (X\times Y;\Gamma_{X\times Y})$ *be a product abstract convex space, where* $(X;\Gamma_X)$ *and* $(Y;\Gamma_Y)$ *are compact abstract convex spaces, and* $f,g : X\times Y\to\mathbb{R}$ *be functions satisfying*

(20.1) $f(x,y)\le g(x,y)$ *for each* $(x,y)\in X\times Y$;
(20.2) *for each* $x\in X$, $f(x,\cdot)$ *is l.s.c. and* $g(x,\cdot)$ *is quasiconvex on* Y; *and*
(20.3) *for each* $y\in Y$, $f(\cdot,y)$ *is quasiconcave and* $g(\cdot,y)$ *is u.s.c. on* X.

If $(E;\Gamma)$ *satisfies the partial KKM principle, we have*

$$\min_{y\in Y}\sup_{x\in X} f(x,y) \le \max_{x\in X}\inf_{y\in Y} g(x,y).$$

Let $\{(X_i,D_i;\Gamma_i)\}_{i\in I}$ be any family of abstract convex spaces. Let $E := \prod_{i\in I} X_i$ be equipped with the product topology and $D := \prod_{i\in I} D_i$. For each $i\in I$, let $\pi_i : D\to D_i$ be the projection. For each $A\in\langle D\rangle$, define $\Gamma(A) := \prod_{i\in I}\Gamma_i(\pi_i(A))$.

Then the following is known:

Lemma 1. (1) *$(E,D;\Gamma)$ is an abstract convex space.*
(2) *If* $\{(X_i,D_i;\Gamma_i)\}_{i\in I}$ *is a family of G-convex spaces, then* $(E,D;\Gamma)$ *is a G-convex space.*
(3) *If* $\{(X_i;\Gamma_i)\}_{i\in I}$ *is a family of G-convex spaces, then the product of* Γ*-convex subsets is also* Γ*-convex in the product G-convex space.*

From now on, we consider only product spaces of a finite family of compact abstract convex spaces for simplicity:

(XXI) **Collective fixed point theorem.** *Let* $\{(X_i;\Gamma_i)\}_{i\in I}$ *be a finite family of compact abstract convex spaces such that* $(E;\Gamma) = (\prod_{i\in I} X_i;\Gamma)$ *satisfies the partial KKM principle, and for each* $i\in I$, $T_i : E\multimap X_i$ *a* Φ*-map. Then there exists a point* $x\in X$ *such that* $x\in T(x) := \prod_{i\in I} T_i(x)$; that is, $x_i = \pi_i(x)\in T_i(x)$ *for each* $i\in I$.

Given a Cartesian product $E = \prod_{i=1}^n X_i$ of sets, let $X^i = \prod_{j\ne i} X_j$ and $\pi_i : E\to X_i, \pi^i : E\to X^i$ be the projections; we write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x,y\in X$, we let

$$[y_i,x^i] := (x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_n).$$

(XXII) **The von Neumann–Fan intersection theorem.** *Let* $\{(X_i;\Gamma_i)\}_{i\in I}$ *be a finite family of compact abstract convex spaces such that* $(E;\Gamma) = (\prod_{i\in I} X_i;\Gamma)$ *satisfies the partial KKM principle and, for each* $i\in I$, *let* A_i *and* B_i *are subsets of* E *satisfying*

(22.1) *for each* $x^i\in X^i$, $\emptyset\ne \mathrm{co}_{\Gamma_i} B_i(x^i)\subset A_i(x^i)$; *and*
(22.2) *for each* $y_i\in X_i$, $B_i^-(y_i)$ *is open in* X^i.

Then we have $\bigcap_{i\in I} A_i\ne\emptyset$.

(XXIII) **The Fan type analytic alternative.** *Let* $\{(X_i;\Gamma_i)\}_{i\in I}$ *be a finite family of compact abstract convex spaces such that* $(E;\Gamma) = (\prod_{i\in I} X_i;\Gamma)$ *satisfies the partial KKM principle and, for each* $i\in I$, *let* $f_i,g_i : E = X^i\times X_i\to\mathbb{R}$ *be real functions satisfying*

(23.1) $f_i(x)\le g_i(x)$ *for each* $x\in X$;
(23.2) *for each* $x^i\in X^i$, $x_i\mapsto g_i[x^i,x_i]$ *is quasiconcave on* X_i; *and*
(23.3) *for each* $x_i\in X_i$, $x^i\mapsto f_i[x^i,x_i]$ *is l.s.c. on* X^i.

Let $\{t_i\}_{i\in I}$ *be a family of real numbers. Then either*

(a) *there exist an* $i\in I$ *and an* $x^i\in X^i$ *such that*

$$f_i[x^i,y_i]\le t_i \quad \text{for all } y_i\in X_i; \text{ or}$$

(b) *there exists an* $x\in E$ *such that*

$$g_i(x) > t_i \quad \text{for all } i\in I.$$

(XXIV) **Generalized Nash–Fan type equilibrium theorem.** *Let* $\{(X_i;\Gamma_i)\}_{i\in I}$ *be a finite family of compact abstract convex spaces such that* $(E;\Gamma) = (\prod_{i\in I} X_i;\Gamma)$ *satisfies the partial KKM principle and, for each* $i\in I$, *let* $f_i,g_i : E = X^i\times X_i\to\mathbb{R}$ *be real functions such that*

(24.0) $f_i(x)\le g_i(x)$ *for each* $x\in E$;

(24.1) *for each* $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ *is quasiconcave on* X_i;
(24.2) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is u.s.c. on* X_i; *and*
(23.3) *for each* $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ *is l.s.c. on* X^i.

Then there exists a point $\hat{x} \in X$ *such that*

$$g_i(\hat{x}) \geq \max_{y_i \in X_i} f_i[\hat{x}^i, y_i] \quad \text{for all } i \in I.$$

(XXV) **Generalized Nash–Fan type equilibrium theorem.** *Let* $\{(X_i; \Gamma_i)\}_{i\in I}$ *be a finite family of compact abstract convex spaces such that* $(E; \Gamma) = (\prod_{i\in I} X_i; \Gamma)$ *satisfies the partial KKM principle and, for each* $i \in I$, *let* $f_i : E \to \mathbb{R}$ *be a function such that*

(25.1) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is quasiconcave on* X_i;
(25.2) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is u.s.c. on* X_i; *and*
(25.3) *for each* $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ *is l.s.c. on* X^i.

Then there exists a point $\hat{x} \in X$ *such that*

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i[\hat{x}^i, y_i] \quad \text{for all } i \in I.$$

(XXVI) **The von Neumann–Sion minimax theorem.** *Let* $(X; \Gamma)$ *and* $(Y; \Gamma')$ *be compact abstract convex spaces and* $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ *a real function such that*

(26.1) *for each* $x \in X$, $f(x, \cdot)$ *is l.s.c. and quasiconvex on* Y; *and*
(26.2) *for each* $y \in Y$, $f(\cdot, y)$ *is u.s.c. and quasiconcave on* X.

If $(E; \Gamma)$ *satisfies the partial KKM principle, then*

(i) f *has a saddle point* $(x_0, y_0) \in X \times Y$; *and*
(ii) *we have*

$$\max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \max_{x\in X} f(x, y).$$

Theorem 6. *For a compact product abstract convex space* $(E; \Gamma)$ *satisfying the partial KKM principle, the statements* (XVIII)–(XXVI) *hold.*

Proof. (XVIII) $\Longleftarrow$ (V)$'$. Suppose that there exists a real c such that

$$\nu = \sup_x \inf_y g(x, y) < c < \inf_y \sup_x f(x, y) = \mu.$$

For the abstract convex space

$$(E, D; \Gamma) := (X \times Y, \{(x_i, y_j)\}_{i,j}; \ \Gamma_{X\times Y}),$$

define two maps $S : E \multimap D$, $T : E \multimap E$ by

$$S^-(x_i, y_j) := \mathrm{Int}\,\{x \in X \mid g(x, y_j) < c\} \times \mathrm{Int}\,\{y \in Y \mid f(x_i, y) > c\},$$
$$T(x, y) := \{\bar{x} \in X \mid s(\bar{x}, y) > c\} \times \{\bar{y} \in Y \mid t(x, \bar{y}) < c\},$$

for $(x_i, y_j) \in D$ and $(x, y) \in E$, resp. Then each $T(x, y)$ is nonempty and Γ-convex and E is covered by open sets $S^-(x_i, y_j)$'s. Moreover,

$$S(x, y) \subset \{(x_i, y_j) \mid g(x, y_j) < c, \ f(x_i, y) > c\}$$
$$\subset \{(\bar{x}, \bar{y}) \mid s(\bar{x}, y) > c, \ t(x, \bar{y}) < c\} \subset T(x, y).$$

This implies $\mathrm{co}_\Gamma S(x, y) \subset T(x, y)$ for all $(x, y) \in E$. Therefore, by (V)$'$, we have an $(x_0, y_0) \in X \times Y$ such that $(x_0, y_0) \in T(x_0, y_0)$. Therefore, $c < s(x_0, y_0) \leq t(x_0, y_0) < c$, a contradiction.
(XIX) $\Longleftarrow$ (XVIII). Note that $y \mapsto \sup_{x\in X} f(x, y)$ is l.s.c. on Y and $x \mapsto \inf_{y\in Y} g(x, y)$ is u.s.c. on X. Therefore, the both sides of the inequality exist. Then all the requirements of (XVIII) are satisfied.
(XX) $\Longleftarrow$ (XIX). Put $f = s$ and $t = g$ in (XIX).
(XXI) $\Longleftarrow$ (V)$'$. Let $S_i : E \multimap X_i$ be the companion map corresponding to the Φ-map T_i. Define $S, T : E \multimap E$ by

$$S(x) := \prod_{i\in I} S_i(x), \qquad T(x) := \prod_{i\in I} T_i(x) \quad \text{for each } x \in X.$$

We show that T is a Φ-map with the companion map S. In fact, we have

$$x \in S^-(y) \Longleftrightarrow y \in S(x) = \prod_{i\in I} S_i(x) \Longleftrightarrow y_i \in S_i(x) \quad \text{for each } i \in I$$
$$\Longleftrightarrow x \in S_i^-(y_i) \quad \text{for each } \ i \in I.$$

Since each $S_i^-(y_i)$ is open and I is finite, we have

(i) for each $y \in X, S^-(y) = \bigcap_{i\in I} S_i^-(y_i)$ is open.

Note that

$$M \in \langle S(x)\rangle \Longrightarrow \pi_i(M) \in \langle S_i(x)\rangle \Longrightarrow \Gamma_i(\pi_i(M)) \subset T_i(x),$$

and hence

$$\Gamma_M = \prod_{i\in I}\Gamma_i(\pi_i(M)) \subset \prod_{i\in I} T_i(x) = T(x).$$

Therefore, we have

(ii) for each $x \in X,\ M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$.

Moreover, let $x \in E$. Since $S_i : E \multimap X_i$ is the companion map corresponding to the Φ-map T_i, for each $i \in I$, there exists $j = j(i)$ such that

$$x \in S_i^-(y_{i,j}) \Longrightarrow y_{i,j} \in S_i(x) \Longrightarrow y \in \prod_{i\in I} S_i(x) = S(x) \Longrightarrow x \in S^-(y),$$

where $y := (y_{1,j(1)}, \dots, y_{n,j(n)})$. Since X is compact, we have

(iii) $E = \bigcup_{z\in M} S^-(z)$ for some $M \in \langle D\rangle$.

Since $(E; \Gamma)$ satisfies the KKM principle, by (V)$'$, the Φ-map T has a fixed point. This completes our proof.

(XXII) $\Longleftarrow$ (XXI). We apply (XXI) with multimaps $S_i, T_i : X \multimap X_i$ given by $S_i(x) := B_i(x^i)$ and $T_i(x) := A_i(x^i)$ for each $x \in X$. Then for each $i \in I$ we have the following:

(i) For each $x \in E$, we have $\emptyset \neq \mathrm{co}_{\Gamma_i} S_i(x) \subset T_i(x)$.

(ii) For each $y_i \in X_i$, we have

$$x \in S_i^-(y_i) \iff y_i \in S_i(x) = B_i(x^i) \iff [x^i, y_i] \in B_i \subset X^i \times X_i = E.$$

Hence,

$$S_i^-(y_i) = \{x = [x^i, x_i] \in E \mid x^i \in B_i(y_i),\ x_i \in X_i\} = B_i(y_i) \times X_i.$$

Note that $S_i^-(y_i)$ is open in $E = X^i \times X_i$ and that T_i is a Φ-map. Therefore, by (XXI), there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x}) = A_i(\widehat{x}^i)$ for all $i \in I$. Hence $\widehat{x} = [\widehat{x}^i, \widehat{x}_i] \in \bigcap_{i\in I} A_i \neq \emptyset$.

(XXIII) $\Longleftarrow$ (XXII). Suppose that (a) does not hold; that is, for any $i \in I$ and any $x^i \in X^i$, there exists an $x_i \in X_i$ such that $f_i[x^i, x_i] > t_i$. Let

$$A_i := \{x \in E \mid g_i(x) > t_i\} \quad \text{and} \quad B_i = \{x \in E \mid f_i(x) > t_i\}$$

for each $i \in I$. Then

(23.4) for each $x^i \in X^i$, $\emptyset \neq B_i(x^i) \subset A_i(x^i)$;
(23.5) for each $x^i \in X^i$, $A_i(x^i)$ is Γ_i-convex; and
(23.6) for each $y_i \in X_i$, $B_i(y_i)$ is open in X^i.

Therefore, by (XXII), there exists an $x \in \bigcap_{i\in I} A_i$. This is equivalent to (b).

(XXIV) $\Longleftarrow$ (XXIII). Since each X_i is compact, by (24.2), for any $\varepsilon > 0$, $t_i := \max_{y_i\in X_i} f_i[x^i, y_i] - \varepsilon$ exists for all $x^i \in X^i$ and $i \in I$. Hence (XXIII)(a) does not hold. Then by (XXIII), there exists an $\hat{x} \in E$ such that $g_i(\hat{x}) > t_i = \max_{y_i\in X_i} f_i[\hat{x}^i, y_i] - \varepsilon$ for all $i \in I$. Since $\varepsilon > 0$ is arbitrary, the conclusion follows.

(XXV) $\Longleftarrow$ (XXIV). Put $f_i = g_i$.

(XXVI) $\Longleftarrow$ (XXV). Let $f_1(x, y) := -f(x, y)$ and $f_2(x, y) := f(x, y)$. Then all of the requirements of (XXV) are satisfied. Therefore, by (XXV), there exists a point $(x_0, y_0) \in X \times Y$ such that

$$f_1(x_0, y_0) = \max_{y\in Y} f_1(x_0, y) \quad \text{and} \quad f_2(x_0, y_0) = \max_{x\in X} f_2(x, y_0).$$

Therefore, we have

$$-f(x_0, y_0) = f_1(x_0, y_0) \geq f_1(x_0, y) = -f(x_0, y) \quad \text{for all } y \in Y,$$

and

$$f(x_0, y_0) = f_2(x_0, y_0) \geq f_2(x, y_0) = f(x, y_0) \quad \text{for all } x \in X.$$

Hence

$$f(x, y_0) \leq f(x_0, y_0) \leq f(x_0, y) \quad \text{for all } (x, y) \in X \times Y.$$

Therefore

$$\max_{x\in X} f(x, y_0) \leq f(x_0, y_0) \leq \min_{y\in X} f(x_0, y).$$

This implies

$$\min_{y\in X}\max_{x\in X} f(x,y) \le f(x_0,y_0) \le \max_{x\in X}\min_{y\in X} f(x,y).$$

On the other hand, we have trivially

$$\min_{y\in X} f(x,y) \le \max_{x\in X} f(x,y)$$

and hence

$$\max_{x\in X}\min_{y\in X} f(x,y) \le \min_{y\in X}\max_{x\in X} f(x,y).$$

Therefore, we have the conclusion.

This completes our proof of Theorem 6. □

Remark. In the proof of Theorem 6, we notice the following:

1. Note that each of (XVIII)–(XX) and (XXI)–(XXVI) follows from the Fan–Browder fixed point theorem (V)′ which is equivalent to the partial KKM principle. Note also that
$$(\mathrm{V})' \Longrightarrow (\mathrm{XXI}) \Longrightarrow (\mathrm{XXII}) \Longrightarrow (\mathrm{XXIII}) \Longrightarrow (\mathrm{XXIV}) \Longrightarrow (\mathrm{XXV}) \Longrightarrow (\mathrm{XXVI}).$$
2. Recall that (XXII) implies the classical von Neumann intersection lemma [28] which is known to be equivalent to the Brouwer fixed point theorem; see [3]. Therefore, in the category of G-convex spaces, (0)–(XVII), (XXI), and (XXII) are all equivalent to the Brouwer theorem, which has numerous equivalent formulations; see [5,12].
3. In 2006, Torres-Martínez [29] showed that a particular type of the Nash equilibrium theorem [30,31] and hence (XXV) implies the Brouwer theorem. Therefore, in the category of G-convex spaces, (0)–(XVII), (XXI)–(XXV) are all equivalent to the Brouwer theorem. But there are no evidences for that any of (XVIII)–(XX) and (XXVI) implies the Brouwer theorem.
4. Note that $(0)' \Longrightarrow (\mathrm{V})' \Longrightarrow (\mathrm{XXII}) \Longrightarrow (\mathrm{XXV})$ can be called "from the KKM principle to the Nash equilibria" [simply, "K to N"]; see [17].

7. Further applications and historical notes

(1) For the origins, variants, corollaries, and applications of each statement in this paper, see [3,6,9,12–14].

(2) For any convex subset of a topological vector space, most statements in this paper were given by Fan in a sequence his papers from 1961 [19]; see [3,5] and references therein.

In 1983, Fan listed various fields in mathematics which have applications of KKM maps, as follows (see [3]):

> Potential theory; Pontryagin spaces or Bochner spaces in inner product spaces; Operator ideals; Weak compactness of subsets of locally convex topological vector spaces; Function algebras; Harmonic analysis; Variational inequalities; Free boundary value problems; Convex analysis; Mathematical economics; Game theory; Mathematical statistics.

We may add the following fields to this list: Nonlinear functional analysis; Approximation theory; Optimization theory; Fixed point theory, and some others.

(3) In 1981, Gwinner [32] displayed relations and connections between some of the most fundamental results of modern nonlinear analysis in the form of a circular tour. His tour starts in a traditional way, but also ends with the classical KKM theorem [1]; thus eight results in [32] are in some wide sense equivalent to the KKM theorem. Nowadays there are nearly one hundred such equivalent results; see [3] and references therein.

(4) Granas [5] studied generalizations and applications of early results in the KKM theory on convex subsets of topological vector spaces and unified classical results systematically. In fact, Granas expanded Fan's works systematically and established new topological methods in convex analysis and nonlinear analysis. Under the guidance of Granas at the Montreal School, the KKM theory had been extended to convex spaces by Lassonde in 1983 [6], to C-spaces (or H-spaces) by Horvath in 1984–93 [9,10], and many others. For other literature, see [3].

(5) Since 1993, the KKM theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the present author and others. In our previous work [9] and many others on G-convex spaces, there exist more than 15 equivalent formulations of the KKM principle such as Alexandroff–Pasynkoff theorem, Fan type matching theorem, Tarafdar type intersection theorem, geometric or section properties, Fan–Browder type fixed point theorems, maximal element theorems, analytic alternatives, Fan type minimax inequalities, variational inequalities, and others. These are also true for KKM spaces.

(6) In such investigations, we found that a large number of known or new theorems are also known to be equivalent to the Brouwer theorem; see [3]. For example, Horvath and Lassonde [33] obtained intersection theorems of the KKM type, Klee type, and Helly type, which are all equivalent to the Brouwer theorem.

In 2001, Park and Jeong [34] collected some equivalents of the Brouwer fixed point theorem which are closely related to the Euclidean spaces or n-simplexes or n-balls. Among them are the Sperner lemma, the KKM theorem, some intersection theorems, various fixed point theorems, an intermediate value theorem, various non-retract theorems, the non-contractibility of spheres, and others.

(7) The original proofs of the Nash theorem [30,31] were based on the Brouwer or Kakutani fixed point theorem. The first "K to N" was given by Fan [35] in 1966 for a product of compact convex subsets of topological vector spaces. Since then there have appeared many papers concerning the transition "K to N" for particular types of KKM spaces; see [17]. For example, for a topological semilattice $(X, \leq)$ with path-connected interval, Horvath and Llinares [21] showed "K to N". More recently, for $\mathbb{B}$-spaces, Briec and Horvath [27] showed "K to N" and other results including Himmelberg type (in fact, Browder type and Kakutani type) fixed point theorems, Fan type minimax inequality, and others.

(8) Instead of the KKM method, in our previous works [36,37], we applied a fixed point theorem for compact compositions of acyclic maps on admissible (in the sense of Klee) convex subsets of topological vector spaces to obtain a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorem. On the other hand, we obtained generalizations of the Gale–Nikaido–Debreu theorem by the KKM method in [38] and by the fixed point method in [39].

(9) In 2008 [12–14], we applied some equivalents of the KKM principle to obtain fixed point, minimax, and equilibrium theorems including the Nash equilibrium theorem. Especially, in [13], we showed that some of well-known results in the KKM theory on G-convex spaces also hold on the KKM spaces. Examples of such results are theorems of Sperner and Alexandroff–Pasynkoff, Horvath type fixed point theorem, Fan–Browder type coincidence theorems, Fan type minimax inequalities, variational inequalities, von Neumann type minimax theorem, Nash type equilibrium theorem, and Himmelberg type fixed point theorem.

(10) In 2008, Kulpa and Szymanski [23] deduced some results in this paper for a particular type of abstract convex space from a Fan type matching property. They also suggested a way of extending their results to a wider class of topological spaces called L^*-space satisfying the Fan matching property. This paper would suffice their suggestion.

(11) One of the most important applications of the KKM theory is to the fixed point theory on topological vector spaces or on abstract convex spaces. A unified account of fixed point theory on G-convex spaces was given in [40]. Further investigations on fixed point theory in abstract convex spaces or KKM spaces are given in [16]. Moreover, overall survey on this topic will be given in [41].

Earlier fixed point theorems in these works were applied to the following problems in the author's works in 1991–2007 (see [3] and MathSciNet):

> Best approximations; Variational inequalities; Quasi-variational inequalities; Leray–Schauder type alternatives; Existence of maximal elements; Minimax inequalities; Walras excess demand theorems; Generalized equilibrium problems; Generalized complementarity problems; Condensing maps; Openness of multimaps; Birkhoff–Kellogg type theorems; Saddle points in nonconvex sets; Acyclic or other versions of the Nash equilibrium theorems; Quasi-equilibrium theorems; Extensions of monotone sets; Eigenvector problems, and others.

(12) Some results in this paper are given in [12–14] in the same or slightly different form. Our aim in this paper is to strengthen the mutual relations among them. Moreover, in our previous work [18], we suggested a metatheorem indicating the contents of this paper without giving any proof. This paper contains a complete proof of the metatheorem.

(13) Finally, recall that there are several hundred published works on the KKM theory and we can cover only an essential part of them. Most of our previous works were devoted to unify some of them in general forms. For the more historical background for the related fixed point theory and for the more involved or generalized versions of the results in this paper, see the references of [12–14,16–18] and the literature therein.

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