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Comments on abstract convexity structures on topological spaces

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ABSTRACT

All results in "Some properties of abstract convexity structures on topological spaces" by S.-w. Xiang and H. Yang [S.-w. Xiang, H. Yang, Some properties of abstract convexity structures on topological spaces, Nonlinear Analysis 67 (2007) 803–808] and "A further characteristic of abstract convexity structures on topological spaces" by S.-w. Xiang and S. Xia [S.-w. Xiang, S. Xia, A further characteristic of abstract convexity structures on topological spaces, J. Math. Anal. Appl. 335 (2007) 716–723] are shown to be consequences of known ones or can be stated in more general forms.

1. Introduction

Since the appearance of generalized convex (simply, G-convex) spaces in 1993, the concept has been challenged by several authors who aimed to obtain more general concepts. In fact, a number of modifications or imitations of the concept have followed; for example, L-spaces [1], spaces having property (H) [2], FC-spaces [3,4], M-spaces and L-spaces in [5,6], and others. It is known that all of such examples belong to the class of ϕ_A-spaces [7,8] and are particular forms of G-convex spaces. Recently all of the above-mentioned spaces are unified to the class of abstract convex spaces in [9,10].

On the other hand, Xiang and Yang [11] and Xiang and Xia [12] established some relationships among the abstract convexity, the selection property, and the fixed point property. They showed that if a convexity structure $\mathcal{C}$ defined on a topological space has the selection property [resp., the weak selection property] then $\mathcal{C}$ satisfies the H-condition [resp., H_0-condition]. Moreover, they showed that, in an *l.c.* compact metric space, the selection property implies the fixed point property. Note that their terminology has their own meaning.

In the present short note, our aim is to show that all results in [11,12] are either consequences of known ones or can be stated in more general forms in the frame of G-convex spaces.

2. Abstract convex spaces

In this section, we follow mainly [13,14,9,10,15,7].

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the *Γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\}.$$

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A subset X of E is called a *Γ-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D' \rangle})$ itself is an abstract convex space called a *subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be *Γ-convex* if, for any $A \in \langle X \cap D \rangle$, we have $\Gamma_A \subset X$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. In [9,10], we gave plenty of examples of abstract convex spaces. Here we give only two of them as follows:

1. Usually, a *convexity space $(E, \mathcal{C})$ in the classical sense* consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see [16], where the bibliography lists 283 papers. From now on, we assume that E is a topological space. For any subset $X \subset E$, its *$\mathcal{C}$-convex hull* is defined and denoted by $\mathrm{Co}_{\mathcal{C}} X := \bigcap \{Y \in \mathcal{C} \mid X \subset Y\}$. We say that X is *$\mathcal{C}$-convex* if $X = \mathrm{Co}_{\mathcal{C}} X$. Now we can consider the map $\Gamma : \langle E \rangle \multimap E$ given by $\Gamma_A := \mathrm{Co}_{\mathcal{C}} A$. Then $(E, \mathcal{C})$ becomes our abstract convex space $(E; \Gamma)$.
2. A *generalized convex space* or a *G-convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

   Here, Δ_n is the standard n-simplex with vertices $\{e^i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e^{i_0}, e^{i_1}, \ldots, e^{i_k}\}$.

   There are a large number of examples of G-convex spaces; see [15] and the references therein.

Recently, we were concerned with particular subclasses or variants of G-convex spaces as follows [7,8]:

Definition. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a *ϕ_A-space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Example. We give examples of ϕ_A-spaces as follows:

1. [1] An L-space, which is a G-convex space $(E; \Gamma)$.
2. [17] An MC-space, which is known to be a G-convex space.
3. [2] A topological space Y with property (H), that is, if, for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$.
4. [3,4] An FC-space $(Y, \{\varphi_N\})$, that is, Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ where some elements in N may be the same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$.
5. [5,6] Particular forms of G-convex spaces are introduced.
6. Any G-convex space is a ϕ_A-space. Conversely, a ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ can be made into a G-convex space $(X, D; \Gamma)$ [7,8]. Therefore, G-convex spaces and ϕ_A-spaces are essentially the same.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

3. Definitions in [11,12]

In this section, we introduce the definitions given in [11,12]:

Definition. For a pair $(Y, \mathcal{C})$, where Y is a topological space and $\mathcal{C}$ is a family of subsets of Y, $\mathcal{C}$ is called a *convexity structure* if

(1) the empty set $\emptyset$ is in $\mathcal{C}$;
(2) $\mathcal{C}$ is stable for intersections, that is, if $\mathcal{D} \subset \mathcal{C}$ is nonempty, then $\bigcap_{A \in \mathcal{D}} A$ is in $\mathcal{C}$.

The *convex hull* conv is defined as

$$\mathrm{conv}(A) = \bigcap \{D \in \mathcal{C} \mid A \subset D\}, \quad A \subset Y.$$

A subset C of Y is said to be *convex* if $C \in \mathcal{C}$. It is clear that C is convex if and only if $\mathrm{conv}(C) = C$.

A metric space (Y, d) with a convexity structure $\mathcal{C}$ is called an *l.c.* space if $\{y \in Y \mid d(y, E) < \varepsilon\} \in \mathcal{C}$ for any $\varepsilon > 0$ and any $E \in \mathcal{C}$.

A pair $(X, \mathcal{C})$ is said to have the *selection property* with respect to a topological space S if every multi-valued map $F : S \multimap X$ admits a single-valued continuous selection whenever F is lower semicontinuous and nonempty closed convex-valued. [Such F is called *a Michael map*.]

A pair $(X, \mathcal{C})$ is said to have the *weak selection property* with respect to a topological space S if every multi-valued map $F : S \multimap X$ admits a single-valued continuous selection whenever F has nonempty convex values and preimages relatively open in X. [Such F is called *a Browder map*.]

We say that a pair $(Y, \mathcal{C})$ satisfies the H-*condition* [resp., H_0-*condition*] if the convexity structure $\mathcal{C}$ has the following property, resp.:

(H) For each finite subset $\{y_0, \dots, y_n\} \subset Y$, there exists a continuous mapping $f : \Delta_N \to \overline{\mathrm{conv}}\{y_0, \dots, y_n\}$, where Δ_N is the standard n-simplex, such that $f(\Delta_J) \subset \overline{\mathrm{conv}}\{y_j \mid j \in J\}$ for each nonempty subset $J \subset N = \{0, 1, \dots, n\}$, where $\overline{\mathrm{conv}}$ denotes the closed convex hull.

(H_0) For each finite subset $\{y_0, \dots, y_n\} \subset Y$, there exists a continuous mapping $f : \Delta_N \to \mathrm{conv}\{y_0, \dots, y_n\}$ such that $f(\Delta_J) \subset \mathrm{conv}\{y_j \mid j \in J\}$ for each nonempty subset $J \subset N = \{0, 1, \dots, n\}$.

For these definitions, we note the following remarks:

(i) It would be better to define a convexity structure as for a convexity space in the classical sense. Otherwise the case $\mathcal{C} = \{\emptyset\}$ will be meaningless since there would be the only convex subset $\emptyset$. So we will use the convexity structure in the sense of convexity spaces in the classical sense (that is, we assume $Y \in \mathcal{C}$ instead of $\emptyset \in \mathcal{C}$).

(ii) A convexity space $(Y, \mathcal{C})$ is a particular form of our abstract convex space $(E, D; \Gamma)$ with $Y = E = D$ and $\Gamma_A := \mathrm{Co}_{\mathcal{C}} A = \mathrm{conv}(A) = \bigcap\{B \in \mathcal{C} \mid A \subset B\}$ for $A \in \langle Y \rangle$. Then $(Y, \mathcal{C})$ becomes our abstract convex space $(Y; \Gamma)$.

(iii) The selection property [resp., the weak selection property] would be better to call the Michael [resp., Browder] selection property since the property is originated from the classical works of Michael [resp., Browder]. We can easily generalize these selection properties to our abstract convex spaces; see Theorem 4.1.

(iv) A convexity space $(Y, \mathcal{C})$ satisfying the H-condition [resp., H_0-condition] in [11] or [12] can be replaced by a G-convex space $(Y; \Gamma)$ with $\Gamma = \overline{\mathrm{conv}}$ [resp., $\Gamma = \mathrm{conv}$], a particular form of our G-convex space $(X, D; \Gamma)$ with $Y = X = D$.

Note that the concepts of the *l.c.* metric spaces, the Michael [resp., Browder] selection property, and the H-condition [resp., H_0-condition] can be easily extended to abstract convex spaces. We will freely use them.

4. Comments on results in [11]

In this section, we state the more general versions of the results in [11]:

Theorem 4.1. *If an abstract convex space $(E, D; \Gamma)$ has the Michael selection property with respect to a simplex, then it is a G-convex space and hence we have a ϕ_A-space*

$$(E, D; \{\phi_A\}_{A \in \langle D \rangle}).$$

Proof. Without loss of generality we may assume that Γ has closed values. Let $A = \{y_0, y_1, \dots, y_n\} \in \langle D \rangle$. For each $x = \sum_{i=0}^n t_i e^i \in \Delta_n$ with $0 \le t_i \le 1$ and $\sum_{i=0}^n t_i = 1$, let

$$\chi(x) := \left\{ i \mid x = \sum_{i=0}^{n} t_i e^i,\ t_i > 0 \right\} \quad \text{and} \quad T(x) := \Gamma\{y_j \mid j \in \chi(x)\}.$$

Then the multimap $T : \Delta_n \multimap E$ is lower semicontinuous as in the proof of [11, Theorem 1]. Note that T has closed Γ-convex values. Since $(E, D; \Gamma)$ has the Michael selection property with respect to a simplex, there exists a continuous selection $\phi_A : \Delta_n \to \Gamma(A)$ of T such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. □

Corollary 4.2 (*[11, Theorem 1]*). *If a convexity space $(Y, \mathcal{C})$ has the selection property with respect to a simplex, then it satisfies the H-condition.*

Therefore, a convexity space $(Y, \mathcal{C})$ having the selection property with respect to any Hausdorff compact space is actually a particular form of G-convex spaces; see [11, Corollary 1].

From the original KKM theorem and its well-known "open"-valued version, we have the following; see [13,14,9] and the references therein:

Theorem 4.3 (*The KKM Principle*). *Let $(E, D; \Gamma)$ be a G-convex space. Then for any multimap $G : D \multimap E$ satisfying*

(1) *G has closed [resp. open] values, and*
(2) *G is a KKM map,*

$\{G(y)\}_{y \in D}$ has the finite intersection property.

Proof. Let $N = \{z_0, z_1, \dots, z_n\} \subset D$. Then we have a continuous function $\phi_N : \Delta_n \to \Gamma_N$. Since G is a KKM map, for each vertex e_i of Δ_n, we have $\phi_N(e_i) \in G(z_i)$ for $0 \le i \le n$. Then $e_i \mapsto \phi_N^{-1}G(z_i)$ is a closed [resp., open] valued map such that $\Delta_k = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1}G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the KKM principle [resp., its "open"-valued version], $\Delta_n \supset \bigcap_{i=0}^{n} \phi_N^{-1}G(z_i) \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \left(\bigcap_{z\in N} G(z)\right) \neq \emptyset$. □

Note that [11, Theorem 2] is a very particular form of Theorem 4.3.
From Theorem 4.3, we have the following as in [13,14]:

Theorem 4.4 (*The Fan–Browder Fixed Point Theorem*). *Let $(E, D; \Gamma)$ be a G-convex space, and $G : E \multimap D$, $H : E \multimap E$ multimaps satisfying*

(1) *for each $x \in E$, $M \in \langle G(x)\rangle$ implies $\Gamma_M \subset H(x)$; and*
(2) *$E = G^-(N)$ for some $N \in \langle D\rangle$.*
(3) *G^- has open [resp., closed] values.*

Then H has a fixed point $\bar{x} \in E$, that is, $\bar{x} \in H(\bar{x})$.

Proof. For each $z \in D$, define $T(z) := E \setminus G^-(z)$. Then $\bigcap_{z\in N} T(z) = E \setminus \bigcup_{z\in N} G^-(z) = \emptyset$ by (2), that is, the values of the map $T : D \multimap E$ do not have the finite intersection property. Then T cannot be a KKM map by Theorem 4.3; that is, $\Gamma_M \not\subset T(M)$ for some $M \in \langle D\rangle$. Hence, there exists $\bar{x} \in \Gamma_M$ such that $\bar{x} \notin T(M)$. Then, $\bar{x} \in G^-(\bar{z})$ for all $z \in M$; that is, $M \subset G(x)$. This implies $\bar{x} \in \Gamma_M \subset H(\bar{x})$ by (1). Therefore, $\bar{x} \in H(\bar{x})$. □

Corollary 4.5 (*[11, Theorem 3]*). *Let $(Y, \mathcal{C})$ be a convexity space satisfying H-condition and X a convex compact subset of $(Y, \mathcal{C})$. Then any Browder map $F : X \multimap X$ has a fixed point.*

Proof. In fact, any convex subset X of $(Y, \mathcal{C})$ has a convexity structure $\mathcal{C}_X := \{X \cap C \mid C \in \mathcal{C}\}$. Note that $(X, \mathcal{C}_X)$ itself can be made into a G-convex space $(X; \Gamma|_{\langle X\rangle})$. Now Theorem 4.4 works. □

The following continuous selection theorem for multimaps with noncompact domain holds in view of [13, Lemma 1]:

Lemma 4.6. *Let X be a normal space, $(Y, D; \Gamma)$ a G-convex space, and $S : X \multimap D$ a map such that $X = \bigcup\{\mathrm{Int}\, S^-(y) \mid y \in A\}$ for some $A \in \langle D\rangle$. Then there exists a continuous function $s : X \to \Gamma_A$ such that $s(x) \in \Gamma(A \cap S(x))$ for all $x \in X$. In fact, if $|A| = n + 1$, then $s = \phi_A \circ p$, where $\phi_A : \Delta_n \to \Gamma_A$ and $p : X \to \Delta_n$ are continuous functions.*

Lemma 4.6′. *Let X be a Hausdorff compact space, $(Y, D; \Gamma)$ a G-convex space, and $S : X \multimap D$ a map such that $X = \bigcup\{\mathrm{Int}\, S^-(y) \mid y \in D\}$. Then there exists a continuous function $s : X \to Y$ such that $s(x) \in \Gamma(S(x))$ for all $x \in X$.*

The following is an improved version of [11, Lemma 1]:

Corollary 4.7. *Let X be a Hausdorff compact space, $(Y, \mathcal{C})$ a convexity space satisfying the H-condition, and $F : X \multimap Y$ a map with nonempty convex values and open fibers. Then F has a continuous selection.*

Combining Theorems 4.1 and 4.4, we have

Theorem 4.8. *If an abstract convex space $(E, D; \Gamma)$ has the Michael selection property with respect to any simplex, and $G : E \multimap D$, $H : E \multimap E$ maps satisfying (1)–(3) of Theorem 4.4. Then H has a fixed point $\bar{x} \in E$, that is, $\bar{x} \in H(\bar{x})$.*

Note that [11, Theorem 4] is a very particular form of Theorem 4.8 for a compact convex subset X of a convexity space $(Y, \mathcal{C})$.
The following is a particular form of [15, Theorem 5.9]:

Theorem 4.9. *Let $(X \supset D; \Gamma)$ be a metric G-convex space such that*

(1) *D is dense in X; and*
(2) *every open ball is Γ-convex.*

Then every continuous compact map $f : X \to X$ has a fixed point.

Corollary 4.10 (*[11, Theorem 5]*). *Let a convexity space $(Y, \mathcal{C})$ be an l.c. metric space such that every singleton $\{x\}$ is convex and X a compact convex subset of Y. If $(Y, \mathcal{C})$ has the selection property [with respect to a simplex], then X has the fixed point property.*

Proof. In fact, $(X, \mathcal{C}_X)$ itself can be made into a metric G-convex space by Theorem 4.1 with $X = D$. Moreover, every open ball $B_\varepsilon(y) = \{x \in X \mid d(x, y) = d(y, \{x\}) < \varepsilon\} \in \mathcal{C}$ is convex since $(X, \mathcal{C}_X)$ is an $l.c.$ metric space and each singleton $\{x\}$ is convex. Then the conclusion follows from Theorem 4.9. □

In [11, Theorem 5], it is assumed that $(Y, \mathcal{C})$ has the selection property with respect to itself.
The following corrects [11, Lemma 2]:

Lemma 4.11. *Let X be a Hausdorff compact space, $(Y; \Gamma)$ an l.c. metric G-convex space, and $F : X \multimap Y$ a lower semicontinuous map with nonempty Γ-convex values. Then for any $\varepsilon > 0$ there exists a continuous function $g : X \to Y$ such that*

$$d(g(x), F(x)) < \varepsilon \quad \text{for all}, x \in X.$$

Proof. Follow the proof of [11, Lemma 2] and apply Lemma 4.6′. □

By following the proof of [11, Theorem 6], we obtain corrected forms of [11, Theorems 6 and 7] as follows, resp.:

Theorem 4.12. *Let X be a Hausdorff compact space and $(Y; \Gamma)$ a complete l.c. metric G-convex space. Then any Michael map $F : X \multimap Y$ has a continuous selection.*

Theorem 4.13. *Let $(Y; \Gamma)$ be an abstract convex l.c. complete metric space. Then Y has the Michael selection property with respect to any Hausdorff compact space if and only if $(Y; \overline{\Gamma})$ is a G-convex space.*

Finally in [11], examples of G-convex spaces $(Y; \Gamma)$ are given as Property 1. For further examples, see [15] and the references therein.

5. Comments on results in [12]

Recall that any pair $(Y, \mathcal{C})$ satisfying the H_0-condition becomes a particular form $(Y; \text{conv})$ of a G-convex space. In [12], its authors are mainly concerned with pairs $(Y, \mathcal{C})$ satisfying the H_0-condition. Therefore all results in [12] are consequences of our G-convex space theory.
In this section, we state comments on the results in [12] or give the more general versions of all results in [12].

Theorem 5.1. *A convexity space $(Y, \mathcal{C})$ becomes a G-convex space $(Y; \text{conv})$ if and only if it has the Browder selection property with respect to any standard simplex.*

Note that Theorem 5.1 strengthens [12, Theorem 3.1], which is the *if* part of Theorem 5.1. The *only if* part is a consequence of Lemma 4.6′.
Similarly, the following strengthens [12, Corollary 3.1]:

Theorem 5.2. *A convexity space $(Y, \mathcal{C})$ becomes a G-convex space $(Y; \text{conv})$ if and only if it has the Browder selection property with respect to any Hausdorff compact topological space.*

Remarks. The following are comments on results in [12]:

(1) [12, Theorem 3.1] is strengthened to Theorem 5.1.
(2) [12, Theorem 3.2] is a consequence of the KKM principle 4.3 for L-spaces.
(3) [12, Theorem 3.3] is a Fan–Browder fixed point Theorem 4.4 for L-spaces.
(4) [12, Theorem 3.4], where X should be Hausdorff, is a consequence of Lemma 4.6′.
(5) [12, Theorem 3.5], where the Hausdorffness is necessary, is similar to Theorem 5.2.

We borrow the following of Komiya [18]:

Lemma 5.3 (*[18, Theorem 1]*). *Let an abstract convex space $(X; \Gamma)$ be an l.c. metric space all of whose singletons are Γ-convex. Then the following are equivalent:*

(1) *$(X; \Gamma)$ has the fixed point property;*
(2) *$(X; \Gamma)$ has the Browder fixed point property;*
(3) *$(X; \Gamma)$ has the Kakutani fixed point property.*

Recall that (2) [resp., (3)] means that every Browder [resp., Kakutani] map $F : X \multimap X$ has a fixed point, where a Kakutani map is an upper semicontinuous multimap with nonempty closed Γ-convex values.

Theorem 5.4. *Let $(Y; \Gamma)$ be an l.c. metric G-convex space all of whose singletons are Γ-convex. Then any compact Γ-convex subset X has the properties (1)–(3) in Lemma 5.3.*

Proof. Any Γ-convex subset X of $(Y; \Gamma)$ itself is a G-convex space $(X; \Gamma|_{\langle X \rangle})$. Therefore, by Theorem 4.4, it has the Browder fixed point property. Since X is also an l.c. metric space all of whose singletons are Γ-convex, by Lemma 5.3, the conclusion follows. □

Corollary 5.5. *Let $(Y, \mathcal{C})$ be an l.c. metric convexity space all of whose singletons are convex, and X a convex compact subset of $(Y; \mathcal{C})$. If $(Y, \mathcal{C})$ has the Browder selection property with respect to any Hausdorff compact space, then X has the fixed point property.*

Proof. By Theorem 3.2, $(Y, \mathcal{C})$ can be made into an L-space $(Y; \Gamma)$. Now Theorem 5.4 works. □

Remarks. 1. Corollary 5.5 corrects and improves [12, Theorem 3.6].
2. The correct form of [11, Lemma 2] and [12, Lemma 3.1] is Lemma 4.11.
3. The correct form of [12, Lemma 3.2] is Theorem 4.12.

In view of Lemma 4.11 and Theorem 4.12, we have the following strengthened form of [12, Theorem 3.7]:

Theorem 5.6. *Let $(Y; \Gamma)$ be a complete l.c. metric G-convex space. If Y has the Browder selection property with respect to any Hausdorff compact space, then Y has the Michael selection property with respect to any Hausdorff compact space.*

Proof. The Browder selection property (Lemma 4.6′) implies Lemma 4.11, which is used to obtain Theorem 4.12. □

Remark. The converse also holds for the Browder maps having closed values.

Recall that a refined form of the Michael selection property for C-spaces is the following well-known result of Ben-El-Mechaiekh and Oudadess [19], where $(Y; \Gamma)$ is a C-space; that is, each Γ_A is ω-connected for $A \in \langle Y \rangle$ [20, Theorem 3].

Lemma 5.7 ([19, Theorem 3]). *Let X be a paracompact space, $Z \subset X$ with $\dim_X Z \leq 0$, $B \subset X$ countable, $(Y; \Gamma)$ a complete l.c. metric C-space such that $\Gamma_{\{y\}} = \{y\}$ for all $y \in Y$, and $T : X \multimap Y$ a l.s.c. map having nonempty values such that $T(x)$ is closed for $x \notin B$ and $\overline{T(x)}$ is Γ-convex for $x \notin Z$. Then T has a continuous selection $f : X \to Y$; that is, $f(x) \in T(x)$ for all $x \in X$.*

Finally, [12, Property 3.1] gives examples of L-spaces, but our G-convex spaces are not, contrary to [12, Property 3.1(iii)].

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