FURTHER GENERALIZATIONS OF THE GALE-NIKAIDO-DEBREU THEOREM

SEHIE PARK

Abstract. A fixed point theorem on compact compositions of acyclic maps on admissible (in the sense of Klee) convex subsets of a t.v.s. is applied to generalize Gwinner's extensions of the Walras excess demand theorem and of the Gale-Nikaido-Debreu theorem.

1. Introduction

In 1981, Gwinner [Gw] displayed relations and connections between some of the most fundamental results of modern nonlinear analysis in the form of a circular tour. His tour starts in a traditional way, but also ends with the classical Knaster-Kuratowski-Mazurkiewicz (simply KKM) theorem [KK]; thus eight results in [Gw] are in some wide sense equivalent to the KKM theorem. Nowadays there are nearly one hundred such equivalent results; see [P6] and references therein. Especially, in [Gw], an infinite dimensional analogue of the Walras' excess demand theorem was given, which is equivalent to the Fan-Glicksberg fixed point theorem.

On the other hand, in our previous works [P4,5], we applied a fixed point theorem for compact compositions of acyclic maps on admissible (in the sense of Klee) convex subsets of a t.v.s. to obtain a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorem.

2000 Mathematics Subject Classification. 47H10, 49J53, 54C60, 54H25, 90A14, 91A13.

Key words and phrases. Kakutani map, acyclic map, admissible set (in the sense of Klee), Walras theorem, Gale-Nikaido-Debreu theorem.

In the present paper, our fixed point theorem is applied to generalize Gwinner's extensions of the Walras excess demand theorem and of the Gale-Nikaido-Debreu theorem. A new generalization of the Walras theorem is shown to be equivalent to one of our fixed point theorems. We follow Gwinner's method.

2. Preliminaries

All spaces are assumed to be Hausdorff and a t.v.s. means a topological vector space.

A *multimap* or *map* $F : X \multimap Y$ is a function from a set X into the set 2^Y of nonempty subsets of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$. For any $B \subset Y$, the (lower) inverse of B under F is defined by $F^-(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}$.

For topological spaces X and Y, a map $F : X \multimap Y$ is said to be *closed* if its graph

$$\mathrm{Gr}(F) := \{(x, y) \mid y \in F(x),\ x \in X\}$$

is closed in $X \times Y$, and *compact* if $F(X)$ is contained in a compact subset of Y.

$F : X \multimap Y$ is said to be *upper semicontinuous* (*u.s.c.*) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X; *lower semicontinuous* (*l.s.c.*) if, for each open set $B \subset Y$, $F^-(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c.

If F is u.s.c. with closed values and if Y is regular, then F is closed. The converse is true whenever Y is compact.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a t.v.s., convex $\Longrightarrow$ star-shaped $\Longrightarrow$ contractible $\Longrightarrow$ ω-connected $\Longrightarrow$ acyclic $\Longrightarrow$ connected, and not conversely.

For topological spaces X and Y, a map $F : X \multimap Y$ is called a *Kakutani map* whenever Y is a convex subset of a t.v.s. and F is u.s.c. with compact convex values; and an *acyclic map* whenever F is u.s.c. with compact acyclic values.

Let $\mathbb{V}(X, Y)$ be the class of all acyclic maps $F : X \multimap Y$, and $\mathbb{V}_c(X, Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces.

The following is a particular form of our previous work [P1,2,6, PS]:

Theorem 1. *Let X be a nonempty convex subset of a locally convex t.v.s. E and $T \in \mathbb{V}_c(X, X)$. If T is compact, then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well-known that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are ℓ^p, $L^p(0, 1)$, H^p for $0 < p < 1$, and many others; see [P2,6] and references therein.

The following particular form of a fixed point theorem due to the author [P3,5] is the basis of our arguments in this paper. We give its simple proof for the completeness.

Theorem 2. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact map $T \in \mathbb{V}_c(X, X)$ has a fixed point.*

Proof. Let $\mathcal{V}$ be a fundamental system of neighborhoods of the origin 0 of E. Since T is closed and compact, it is sufficient to show that for any $V \in \mathcal{V}$, there exists an $x_V \in X$ such that $(x_V + V) \cap T(x_V) \neq \emptyset$.

Since $\overline{T(X)}$ is a compact subset of the admissible subset X, there exist a continuous map $h : \overline{T(X)} \to X$ and a finite dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{T(X)}$ and $h(\overline{T(X)}) \subset L \cap X$. Let $M := h(\overline{T(X)})$. Then M is a compact subset of L and hence $P := \operatorname{co} M$ is a compact convex subset of $L \cap X$. Note that $h : \overline{T(X)} \to P$ and $F|_P : P \multimap \overline{T(X)}$. Since $h \circ (T|_P) \in \mathbb{V}_c(P, P)$, by Theorem 1, it has a fixed point $x_V \in P$; that is, $x_V \in hT(x_V)$ and hence $x_V = h(y)$ for some $y \in T(x_V) \subset \overline{T(X)}$. Since $y - h(y) \in V$, we have $y \in h(y) + V = x_V + V$. Therefore, $(x_V + V) \cap F(x_V) \neq \emptyset$. $\square$

Recall that a real-valued function $f : X \to \mathbf{R}$ on a topological space is *lower* [resp. *upper*] *semicontinuous* (*l.s.c.*) [resp. *u.s.c.*] if $\{x \in X \mid f(x) > r\}$ [resp.

$\{x \in X : f(x) < r\}$ is open for each $r \in \mathbf{R}$. If X is a convex set in a vector space, then f is *quasiconcave* [resp. *quasiconvex*] if $\{x \in X \mid f(x) > r\}$ [resp. $\{x \in X \mid f(x) < r\}$] is convex for each $r \in \mathbf{R}$.

3. Main results

We begin with the following generalization of Gwinner's extension of the Walras theorem in [Gw]:

Theorem 3. *Let K and L be compact convex subsets of t.v.s. E and F, resp., such that K is admissible. Let $c \in \mathbf{R}$, $f : K \times L \to \mathbf{R}$ a continuous function, and $T : K \multimap L$ a multimap. Suppose*

(1) *for each $y \in L$, $f(\cdot, y)$ is quasiconvex;*
(2) *T is an acyclic map; and*
(3) *for each $x \in K$ and $y \in T(x)$, we have $f(x, y) \geq c$.*

Then there exists a Walras equilibrium, that is, there exist $\overline{x} \in K$, $\overline{y} \in L$ such that

$$\overline{y} \in T(\overline{x}) \quad \textit{and} \quad f(x, \overline{y}) \geq c \quad \textit{for all} \quad x \in K.$$

Proof of Theorem 3 using Theorem 2. Define a multimap $S : L \multimap K$ by

$$S(y) := \{x \in K \mid f(x, y) \leq f(x', y) \text{ for all } x' \in K\}.$$

Then, for each $y \in L$, $S(y)$ is nonempty since the continuous function $f(\cdot, y)$ attains its minimum on the compact set K. Moreover, $S(y)$ is closed and convex. Since f is continuous, S has closed graph in the compact set $K \times L$ and hence u.s.c. Therefore S is an acyclic map. Now, by Theorem 2, the composition $ST : K \multimap K$ has a fixed point $\overline{x} \in ST(\overline{x})$, and hence, there exists $\overline{y} \in T(\overline{x}) \subset L$ such that $\overline{x} \in S(\overline{y})$. Therefore, in view of (3), we have

$$c \leq f(\overline{x}, \overline{y}) \leq f(x, \overline{y}) \quad \text{for all} \quad x \in K.$$

This completes our proof. □

Remark. Condition (3) is usually called the *Walras law.*

Since every convex subset of a locally convex t.v.s. is admissible, from Theorem 3, we immediately have the following:

Corollary 3.1. *Theorem 3 is also valid even if E is a locally convex t.v.s.*

Remarks. 1. Corollary 3.1 is equivalent to Theorem 1. This can be shown by simply following Gwinner [Gw, Proof of Theorem 7 using Theorem 8].

2. For a Kakutani map T instead of an acyclic map, Corollary 3.1 reduces to [Gw, Theorem 8], where local convexity of F is redundant. This is called, in [Z], the main theorem of mathematical economics of Gale [G], Nikaido [Ni], and Debreu [D2].

To specialize Theorem 3 towards the Gale-Nikaido-Debreu theorem, as in [Gw], we boil down the function f to a bilinear form $\langle \cdot, \cdot \rangle$ for a dual system $(E, F, \langle \cdot, \cdot \rangle)$ of t.v.s. E and F.

For a convex cone P of E, the *dual cone* is defined by

$$P^+ := \{y \in F \mid \langle p, y \rangle \geq 0,\ p \in P\}.$$

Theorem 4. *Let $(E, F, \langle \cdot, \cdot \rangle)$ be a dual system of t.v.s. E and F such that the bilinear form $\langle \cdot, \cdot \rangle$ is continuous on compact subsets of $E \times F$. Let K and L be compact convex subsets of t.v.s. E and F, resp., such that K is admissible; and P the convex cone $\bigcup\{rK \mid r \geq 0\}$. Let $T : K \multimap L$ be an acyclic map such that $\langle x, y \rangle \geq 0$ for all $x \in K$ and $y \in T(x)$. Then there exists $\overline{x} \in K$ such that $T(\overline{x}) \cap P^+ \neq \emptyset$.*

Proof of Theorem 4 using Theorem 3. Put $f(x, y) := \langle x, y \rangle$ and $c = 0$ in Theorem 3. Note that, for each $y \in Y$, $f(\cdot, y)$ is quasi-convex. Since all of the requirements of Theorem 3 are satisfied, there exist $\overline{x} \in K$, $\overline{y} \in L$ such that

$$\overline{y} \in T(\overline{x}) \ \text{ and } \ \langle x, \overline{y} \rangle \geq 0 \ \text{ for all } \ x \in K.$$

Hence $\langle x, \overline{y} \rangle \geq 0$ for all $x \in P$. Therefore $\overline{y} \in P^+$. $\square$

Corollary 4.1. *Theorem 4 is also valid even if E is a locally convex t.v.s. instead of the admissibility of K.*

Remarks. 1. For a Kakutani map T instead of an acyclic map, Corollary 4.1 reduces to [Gw, Corollary to Theorem 8], where local convexity of F is redundant.

2. With the choice $P := \{x \in \mathbf{R}^n \mid x_i \geq 0;\ i = 1, 2, \ldots, n\}$ and $K = L := \{x \in P \mid x_1 + x_2 + \cdots + x_n = 1\}$ (the standard simplex), the Gale-Nikaido-Debreu theorem ([G, Principal Lemma], [Ni, Theorem 16.6], [D2, 5.6(1)]) can be immediately obtained from Corollary 4.1.

4. Historical remarks

John von Neumann's 1928 minimax theorem [N1] and 1937 intersection lemma [N2] have numerous generalizations and applications. Kakutani's 1941 fixed point theorem [K] was to give simple proofs of the above-mentioned results. John Nash [N] obtained his 1951 equilibrium theorem based on the Brouwer or Kakutani fixed point theorem. In 1952, Fan [F] and Glicksberg [Gl] extended the Kakutani theorem to locally convex Hausdorff topological vector spaces. This result was applied by its authors to generalize the von Neumann intersection lemma and the Nash equilibrium theorem. Further generalizations were followed by Ma [M] and others. For the literature, see [P6] and references therein.

An upper semicontinuous (u.s.c.) multimaps with nonempty compact convex values is called a *Kakutani map*. The Fan-Glicksberg theorem was extended by Himmelberg [H] in 1972 for compact Kakutani maps instead of assuming compactness of domains. In 1990, Lassonde [L] extended the Himmelberg theorem to multimaps factorizable by Kakutani maps through convex sets in Hausdorff topological vector spaces. Moreover, Lassonde applied his theorem to game theory and obtained a von Neumann type intersection theorem for finite number of sets and a Nash type equilibrium theorem comparable to Debreu's social equilibrium existence theorem [D1].

On the other hand, in 1946, the Kakutani fixed point theorem was extended for acyclic maps by Eilenberg and Montgomery [EM]. This result was applied by Park [P3] to give acyclic versions of the social equilibrium existence theorem due to Debreu [D1], saddle point theorems, minimax theorems, and the Nash equilibrium theorem. Moreover, Park [P2,4] obtained a fixed point theorem for compact compositions of acyclic maps defined on admissible (in the sense of Klee) convex subsets of topological vector spaces. This new theorem was applied in [P4] to deduce acyclic versions of the von Neumann intersection lemma, the minimax theorem, the Nash equilibrium theorem, and others. Further, in [P5], Park obtained a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorems.

Recall that Leon Warlas is regarded as the creator of mathemaical economics through his basic work [W]. However, only during the 1950s did his ideas appear in a precise mathematical form following the works of Arrow and Debreu [AD], Gale [G], and Nikaido [Ni]; see [Z]. As a continuation of applications of our fixed point theorem, in the present paper, we generalize Gwinner's extensions of the Walras excess demand theorem and of the Gale-Nikaido-Debreu theorem in [Gw]. A new generalization of the Walras theorem (Corollary 3.1) is shown to be equivalent to one of our fixed point theorems (Theorem 1).

References

[AD] K. Arrow and G. Debreu, *Existence of an equilibrium for a competitive economy*, Econometrica **22** (1954), 265–290 [= Chap.4, Mathematical Economics: Twenty Papers of Gerald Debreu, Cambridge Univ. Press, 1983].

[D1] G. Debreu, *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci. USA **38** (1952), 886–893 [= Chap.2, Mathematical Economics: Twenty Papers of Gerald Debreu, Cambridge Univ. Press, 1983].

[D2] G. Debreu, *Theory of Value*, Wiley, New York, 1959.

[EM] S. Eilenberg and D. Montgomery, *Fixed point theorems for multivalued transformations*, Amer. J. Math. **68** (1946), 214–222.

[F] K. Fan, *Fixed point and minimax theorems in locally convex linear spaces*, Proc. Nat. Acad. Sci. USA **38** (1952), 121–126.

[G] D. Gale, *The law of supply and demand*, Math. Scand. **3** (1955), 155–169.

[Gl] I.L. Glicksberg, *A further generalizations of the Kakutani fixed point theorem, with applications to Nash equilibrium points*, Proc. Amer. Math. Soc. **3** (1952), 170–174.

[Gw] J. Gwinner, *On fixed points and variational inequalities — A circular tour*, Nonlinear Anal., TMA **5** (1981), 565–583.

[H] C.J. Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207.

[K] S. Kakutani, *A generalization of Brouwer's fixed-point theorem*, Duke Math. J. **8** (1941), 457–459.

[KK] B. Knaster, K. Kuratowski, S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-Dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.

[L] M. Lassonde, *Fixed points of Kakutani factorizable multifunctions*, J. Math. Anal. Appl. **97** (1983), 46–60.

[M] T.-W. Ma, *On sets with convex sections*, J. Math. Anal. Appl. **27** (1969), 413–416.

[N] J. Nash, *Non-cooperative games*, Ann. Math. **54** (1951), 286–293.

[N1] J. von Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928), 295–320.

[N2] ———, *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergeb. eines Math. Kolloq. **8** (1937), 73–83. [= Rev. Economic Studies XIII, No.33 (1945–46), 1–9].

[Ni] H. Nikaido, *Convex Structures and Economic Theory*, Academic Press, New York, 1968.

[P1] S. Park, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Korean Math. Soc. **31** (1994), 493–519.

[P2] ———, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), 803–829. *Corrections*, ibid. **36** (1999), 829–832.

[P3] ———, *Remarks on a social equilibrium existence theorem of G. Debreu*, Appl. Math. Lett. **11**(5) (1998), 51–54.

[P4] ———, *Acyclic versions of the von Neumann and Nash equilibrium theorems*, J. Comp. Appl. Math. **113** (2000), 83–91.

[P5] ———, *Remarks on acyclic versions of generalized von Neumann and Nash equilibrium theorems*, Appl. Math. Letters **15** (2002), 641–647.

[P6] ———, *One hundred years of the Brouwer fixed point theorem*, Appendix in: Theory of the KKM spaces, to appear.

[PS] S. Park, S.P. Singh, and B. Watson, *Some fixed point theorems for composites of acyclic maps*, Proc. Amer. Math. Soc. **121** (1994), 1151–1158.

[W] L. Walras, *Eléments d'economie politique pure*, Corbaz, Lausanne, 1874.

[Z] E. Zeidler, *Nonlinear Functional Analysis and its Applications, IV: Applications to Mathematical Physics*, Springer-Verlag, New York/Berlin, 1988.

The National Academy of Sciences, Republic of Korea, Seoul 137–044; and
Department of Mathematical Sciences, Seoul National University, Seoul 151–747, KOREA
E-mail address: shpark@math.snu.ac.kr