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From the KKM principle to the Nash equilibria

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ABSTRACT

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In this paper, we clearly show that a sequence of statements from the partial KKM principle to the Nash equilibria can be obtained for any KKM spaces. This unifies previously known several proper examples of such sequences for particular types of KKM spaces.

Keywords: Abstract convex space; G-convex space; KKM space; (partial) KKM principle; minimax theorem; von Neumann-Fan intersection theorem; Nash equilibrium theorem.

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1 Introduction

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In our recent works [32-34], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are closely related to KKM spaces or spaces satisfying the partial KKM principle. Moreover, a number of such results are equivalent to each other.

On the other hand, some authors studied particular types of KKM spaces and deduced the Nash type equilibrium theorem from the partial KKM principle; for example, [1,4,12,15,21], explicitly, and many more in the references, implicitly. Therefore, in order to avoid unnecessary repetitions for each particular type of KKM spaces, it would be necessary to state clearly them for spaces satisfying the partial KKM principle.

In this paper, we clearly show that a sequence of statements from the partial KKM principle to the Nash equilibria can be obtained for any abstract convex space satisfying the partial KKM principle. This unifies previously known several proper examples of such sequences for particular types of KKM spaces.

This paper is a supplement to [32-34], where some particular results or more details can be found.

2 Abstract Convex Spaces and KKM Spaces

Multimaps are also called simply maps. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [28-35]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.
For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.
When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example 2.1. The following are abstract convex spaces:

(1) The triple $(\Delta_n, V; \mathrm{co})$ in the original KKM theorem [13], where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation.

(2) A triple $(X, D; \Gamma)$, where X and D are subsets of a t.v.s. E such that $\mathrm{co}\, D \subset X$ and $\Gamma := \mathrm{co}$. Fan's celebrated KKM lemma [3] is for $(E, D; \mathrm{co})$, where D is a nonempty subset of E.

(3) A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde [14] for $X = D$. However he obtained several KKM type theorems w.r.t. $(X, D; \Gamma)$.

(4) A triple $(X, D; \Gamma)$ is called an *H-space* by Park [18] if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma) := (X, X; \Gamma)$ is called a C-space by Horvath [5].

(5) Hyperconvex metric spaces due to Aronszajn and Panitchpakdi are particular cases of C-spaces; see [6].

(6) Hyperbolic spaces due to Reich and Shafrir [36] are also particular cases of C-spaces. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic.

(7) Any topological semilattice $(X, \leq)$ with path-connected interval introduced by Horvath and Llinares [8]. See also [18].

(8) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.
Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

For $X = D$, G-convex spaces reduce to L-spaces due to Ben-El-Mechaiekh et al. Recall that all examples (1)-(7) are G-convex spaces. For details, see references of [18,20,21,23-25,27].

(9) A ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D\rangle$ with $|A| = n+1$. Every ϕ_A-space can be made into a G-convex space, and the so-called FC-spaces are particular forms of L-spaces; see [27,31]. Recently, ϕ_A-spaces are called GFC-spaces by Khanh et al. [10].

(10) Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X\rangle$, $\Gamma_A := conv_H(A)$, where $conv_H(A)$ is the intersection of all closed convex subsets of H that contain A; see Kirk and Panyanak [11]. Then $(H \supset X; \Gamma)$ is an abstract convex space.

(11) According to Horvath [7], a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$. Then $(X; \Gamma)$ with $\Gamma_A := [[A]]$ for each $A \in \langle X\rangle$ is an abstract convex space.

(12) A $\mathbb{B}$-space due to Briec and Horvath [1] is an abstract convex space.

Note that each of (2)-(12) has a large number of concrete examples and that all examples (1)-(9) are G-convex spaces.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map*.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.
An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

In our recent works [32-34], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Example 2.2. We give examples of KKM spaces:

(1) Every G-convex space is a KKM space [23].

(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space [30].

(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$; see [30]. But L^* is not a G-convex space.

(4) For a closed convex subset X of a complete $\mathbb{R}$-tree H, and $\Gamma_A := conv_H(A)$ for each $A \in \langle X\rangle$, the triple $(H \supset X; \Gamma)$ satisfies the partial KKM principle; see Kirk and Panyanak [11]. Later we found that $(H \supset X; \Gamma)$ is a KKM space [35].

(5) For Horvath's convex space $(X; \Gamma)$ with the weak Van de Vel property is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X\rangle$; see [7,35].

(6) A $\mathbb{B}$-space due to Briec and Horvath [1] is a KKM space.

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{gathered}\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space} \\ \Longrightarrow H\text{-space} \Longrightarrow G\text{-convex space} \Longleftrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\ \Longrightarrow \text{Space satisfying the partial KKM principle} \\ \Longrightarrow \text{Abstract convex space.}\end{gathered}$$

It is not known yet whether there is a space satisfying the partial KKM principle that is not a KKM space.

3 From the KKM Principle to the Nash Equilibria

For an abstract convex space $(E, D; \Gamma)$, let us consider the following statements:

(A) The KKM principle. *For any closed-valued [resp., open-valued] KKM map $G : D \multimap E$, the family $\{G(z)\}_{z\in D}$ has the finite intersection property.*

(B) The Fan-Browder fixed point property. *Let $S : E \multimap D$, $T : E \multimap E$ be maps satisfying*
(B.1) *$S^-(z)$ is open [resp., closed] for each $z \in D$;*
(B.2) *for each $x \in E$, $\text{co}_\Gamma S(x) \subset T(x)$; and*
(B.3) *$E = \bigcup_{z\in M} S^-(z)$ for some $M \in \langle D\rangle$.*
Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

Theorem 1. A characterization of the KKM spaces. *For an abstract convex space $(E, D; \Gamma)$, the statements* (A) *and* (B) *are equivalent.*

Proof. (A) $\Longrightarrow$ (B). Suppose $(E, D; \Gamma)$ satisfies the KKM principle (A). Define a map $G : D \multimap E$ by $G(z) := E \setminus S^-(z)$ for each $z \in D$. Then each $G(z)$ is closed [resp., open] by (B.1), and

$$\bigcap_{z\in M} G(z) = E \setminus \bigcup_{z\in M} S^-(z) = E \setminus E = \emptyset$$

by (B.3). Therefore, the family $\{G(z)\}_{z\in D}$ does not have the finite intersection property, and hence, G is not a KKM map by (A). Thus, there exists an $N \in \langle D\rangle$ such that $\Gamma_N \not\subset G(N) = \bigcup\{E \setminus S^-(z) \mid z \in N\}$. Hence, there exists an $x_0 \in \Gamma_N$ such that $x_0 \in S^-(z)$ for all $z \in N$; that is, $N \subset S(x_0)$. Therefore, $x_0 \in \Gamma_N \subset \text{co}_\Gamma S(x_0) \subset T(x_0)$ by (B.2).

(B) $\Longrightarrow$ (A). Let $G : D \multimap E$ be a KKM map with closed [resp., open] values. Suppose the family $\{G(z)\}_{z\in D}$ does not have the finite intersection property; that is, there exists an $M \in \langle D\rangle$ such that $\bigcap_{z\in M} G(z) = \emptyset$. Define a map $S : E \multimap D$ by $S^-(z) := E \setminus G(z)$ for $z \in D$ and a map $T : E \multimap E$ by $T(x) := \text{co}_\Gamma S(x)$ for each $x \in E$. Note that $E = \bigcup_{z\in M}(E \setminus G(z)) = \bigcup_{z\in M} S^-(z)$. Then all of the requirements (B.1)–(B.3) are satisfied. Hence there exists an $x_0 \in E$ such that $x_0 \in T(x_0)$. Then $x_0 \in T(x_0) = \text{co}_\Gamma S(x_0)$ and hence, there exists an $N \in \langle S(x_0)\rangle$ such that $x_0 \in \Gamma_N \subset \text{co}_\Gamma S(x_0)$. Therefore, for each $z \in N$, we have $x_0 \in S^-(z)$ or $x_0 \notin G(z)$; that is, $\Gamma_N \not\subset G(N)$. Hence G is not a KKM map, a contradiction. Therefore $(E, D; \Gamma)$ satisfies the KKM principle (A).

□

Remark. There are several more statements equivalent to the KKM principle for abstract convex spaces; see [34].

For an abstract convex space $(E, D; \Gamma)$, let us consider the following partial conditions of (A) and (B):

(A)′ The partial KKM principle. *For any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.*

(B)′ The corresponding partial condition of (B) [whenever S has open values].

From Theorem 1, we have the following:

Theorem 2. A characterization of spaces satisfying the partial KKM principle. *For an abstract convex space $(E, D; \Gamma)$, (A)′ and (B)′ are equivalent.*

Recall that an extended real-valued function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is open for each $r \in \overline{\mathbb{R}}$.

The following is known:

Lemma. *Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be any family of abstract convex spaces. Let $X := \prod_{i \in I} X_i$ be equipped with the product topology and $D = \prod_{i \in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D \rangle$, define $\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$. Then $(X, D; \Gamma)$ is an abstract convex space.*
Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be a family of G-convex spaces. Then $(X, D; \Gamma)$ is a G-convex space.

An abstract convex space $(E, D; \Gamma)$ is said to be *compact* if E is a compact topological space. From now on, for simplicity, we are mainly concerned with compact abstract convex spaces $(E; \Gamma)$ satisfying the partial KKM principle. For example, any compact G-convex space, any compact H-space, or any compact convex space is such a space.
For an abstract convex space $(E; \Gamma)$, a real-valued function $f : E \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp. *quasiconvex*] if $\{x \in E \mid f(x) > r\}$ [resp., $\{x \in E \mid f(x) < r\}$] is Γ-convex (relative to itself) for each $r \in \overline{\mathbb{R}}$.

From (B)′, we have the following:

Theorem 3. Generalized von Neumann-Sion minimax theorem. *Let $(X; \Gamma_1)$ and $(Y; \Gamma_2)$ be compact abstract convex spaces, $(E; \Gamma) := (X \times Y; \Gamma_{X \times Y})$ the product abstract convex space defined as in Lemma, and $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions satisfying*
(3.1) *$f(x, y) \le g(x, y)$ for each $(x, y) \in X \times Y$;*
(3.2) *for each $x \in X$, $f(x, \cdot)$ is l.s.c. and $g(x, \cdot)$ is quasiconvex on Y; and*
(3.3) *for each $y \in Y$, $f(\cdot, y)$ is quasiconcave and $g(\cdot, y)$ is u.s.c. on X.*
If $(E; \Gamma)$ satisfies the partial KKM principle, then we have

$$\min_{y \in Y} \sup_{x \in X} f(x, y) \le \max_{x \in X} \inf_{y \in Y} g(x, y).$$

Here, $\Gamma_{X \times Y}(A) := \Gamma_1(\pi_1(A)) \times \Gamma_2(\pi_2(A))$ for $A \in \langle X \times Y \rangle$.

Proof. Note that $y \mapsto \sup_{x\in X} f(x,y)$ is l.s.c. on Y and $x \mapsto \inf_{y\in Y} g(x,y)$ is u.s.c. on X. Therefore, the both sides of the inequality exist. Suppose that there exists a real c such that

$$\max_x \inf_y g(x,y) < c < \min_y \sup_x f(x,y).$$

For the compact abstract convex space $(E;\Gamma) := (X \times Y; \Gamma_{X\times Y})$, define a map $T : X \times Y \multimap X \times Y$ by

$$T(x,y) = \{\bar{x} \in X \mid f(\bar{x},y) > c\} \times \{\bar{y} \in Y \mid g(x,\bar{y}) < c\}$$

for $(x,y) \in X\times Y$. Then $T(x,y)$ is nonempty and Γ-convex for each $(x,y) \in X\times Y$ and $T^-(x,y)$ is open. Since E is compact, it is covered by a finite number of $T^-(x,y)'$s. Therefore, by (B)$'$, we have an $(x_0,y_0) \in X\times Y$ such that $(x_0,y_0) \in T(x_0,y_0)$. Therefore, $c < f(x_0,y_0) \le g(x_0,y_0) < c$, a contradiction. □

If $f = g$, Theorem 3 reduces to the following:

Corollary 3.1. *Let $(X;\Gamma)$ and $(Y;\Gamma')$ be compact abstract convex spaces and $f : X \times Y \to \mathbb{R}\cup\{+\infty\}$ a real function such that*
(1) *for each $x \in X$, $f(x,\cdot)$ is l.s.c. and quasiconvex on Y; and*
(2) *for each $y \in Y$, $f(\cdot,y)$ is u.s.c. and quasiconcave on X.*
If $(X \times Y;\Gamma)$ satisfies the partial KKM principle, then
(i) *f has a saddle point $(x_0,y_0) \in X \times Y$; and*
(ii) *we have*

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y).$$

Remark. Corollary 3.1 generalizes historically well-known minimax theorems due to von Neumann, Kakutani, Nikaidô, Sion, Komiya, Bielawski, Horvath, and Park; see [21,24].

Given a cartesian product $X = \prod_{i=1}^n X_i$ of sets, let $X^i = \prod_{j\neq i} X_j$ and $\pi_i : X \to X_i$, $\pi^i : X \to X^i$ be the projections; we write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x, y \in X$, we let

$$[x^i, y_i] := (x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n).$$

Theorem 4. Generalized von Neumann-Fan intersection theorem. *Let $\{(X_i;\Gamma_i)\}_{i=1}^n$ be a family of compact abstract convex spaces such that $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ satisfies the partial KKM principle and, for each i, let A_i and B_i are subsets of X satisfying the following:*
(4.1) *for each $y \in X$, $B_i(y) := \{x \in X \mid [x^i, y_i] \in B_i\}$ is open; and*
(4.2) *for each $x \in X$, $\emptyset \neq \mathrm{co}_\Gamma B_i(x) \subset A_i(x) := \{y \in X \mid [x^i,y_i] \in A_i\}$.*
Then we have $\bigcap_{i=1}^n A_i \neq \emptyset$.

Proof. Define maps $T, S : X \multimap X$ by $T(x) := \bigcap_{i=1}^n A_i(x)$ and $S(x) := \bigcap_{i=1}^n B_i(x)$ for $x \in X$. From (4.2), we have

$$\mathrm{co}_\Gamma S(x) = \mathrm{co}_\Gamma(\bigcap_{i=1}^n B_i(x)) \subset \bigcap_{i=1}^n \mathrm{co}_\Gamma B_i(x) \subset \bigcap_{i=1}^n A_i(x) = T(x)$$

for each $x \in X$. For each $x \in E$ and each i, there exists a $y^{(i)} \in B_i(x)$ by (4.2), or $[x^i, y_i^{(i)}] \in B_i$. Hence, we have $(y_1^{(1)}, \ldots, y_n^{(n)}) \in \bigcap_{i=1}^n B_i(x)$. This shows $S(x) \neq \emptyset$. Moreover, $S^-(y) = \bigcap_{i=1}^n B_i(y)$ is open for each $y \in E$ by (4.1). Since X is compact, it is covered by a finite number of $S^-(y)$'s. Hence, all the requirements of (B)$'$ are satisfied. Therefore, there exists an $x^0 \in T(x^0) = \bigcap_{i=1}^n A_i(x^0)$, that is, $x^0 \in A_i$ for all i. □

Remark. Theorem 4 generalizes historically well-known intersection theorems due to von Neumann, Fan, Bielawski, Kirk et al., and Park; see [21,24].

Theorem 5. Generalized Nash equilibrium theorem. *Let $\{(X_i; \Gamma_i)\}_{i=1}^n$ be a family of compact abstract convex spaces such that $(X; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ satisfies the partial KKM principle and, for each i, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions such that*
(5.0) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*
(5.1) *for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is quasiconcave on X_i;*
(5.2) *for each $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ is u.s.c. on X_i; and*
(5.3) *for each $x_i \in X_i$, $x^i \mapsto g_i[x^i, x_i]$ is l.s.c. on X^i.*
Then there exists a point $\hat{x} \in X$ such that

$$f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i[\hat{x}^i, y_i] \quad \textit{for all } i.$$

Proof. For any $\varepsilon > 0$, we define

$$A_{\varepsilon,i} = \{x \in X \mid f_i(x) > \max_{y_i \in X_i} g_i[x^i, y_i] - \varepsilon\},$$

$$B_{\varepsilon,i} = \{x \in X \mid g_i(x) > \max_{y_i \in X_i} g_i[x^i, y_i] - \varepsilon\}$$

for each i. Then
(1) for each $x^i \in X^i$, $B_{\varepsilon,i}(x^i) \subset A_{\varepsilon,i}(x^i)$;
(2) for each $x^i \in X^i$, $A_{\varepsilon,i}(x^i)$ is Γ_i-convex;
(3) for each $x^i \in X^i$, $B_{\varepsilon,i}(x_i) \neq \emptyset$ since $x_i \mapsto g_i[x^i, x_i]$ is u.s.c. on the compact space X_i; and
(4) for each $x_i \in X_i$, $B_{\varepsilon,i}(x_i)$ is open since $x^i \mapsto g_i[x^i, x_i]$ is l.s.c. on X^i.
Therefore, by applying Theorem 4, we have

$$\bigcap_{i=1}^n A_{\varepsilon,i} \neq \emptyset \quad \text{for every} \quad \varepsilon > 0.$$

Since X is compact, there exists an $\hat{x} \in X$ such that $f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i[\hat{x}^i, y_i]$ for all i. □

From Theorem 5, we obtain the following form of the Nash equilibrium theorem for abstract convex spaces:

Corollary 5.1. *Let $\{(X_i; \Gamma_i)\}_{i=1}^n$ be a family of compact abstract convex spaces such that $(X; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ satisfies the partial KKM principle and, for each i, let $f_i : X \to \mathbb{R}$ be a function such that*

(1) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is quasiconcave on* X_i*;*
(2) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is u.s.c. on* X_i*; and*
(3) *for each* $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ *is l.s.c. on* X^i.
Then there exists a point $\hat{x} \in X$ *such that*

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i[\hat{x}^i, y_i] \quad \textit{for all } i.$$

Remark. 1. For G-convex spaces, Theorem 5 and Corollary 5.1 hold for not-necessarily finite family; see [23].
2. Corollary 5.1 generalizes well-known equilibrium theorems due to Nash, Fan, Bielawski, Kirk et al., and Park; see [16,17,21,24].
3. The point $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in X$ in the conclusion of Corollary 5.1 is called a *Nash point equilibrium*. This concept is a natural extension of the local maxima (for the case $n = 1$, $f = f_1$) and of the saddle points (for the case $n = 2$, $f_1 = -f$, $f_2 = f$.

4 Historical Notes and Related Results

(I) For the origins, particular forms, variants, corollaries, and applications of each statement in this paper, see [16,17,19-26,32,33].

(II) For any convex subset of a topological vector space, Theorems 3-5 were given by Fan in a sequence of his papers; see [20] and references therein.

(III) In 1998 [19], an acyclic version of the social equilibrium existence theorem of Debreu is obtained. This is applied to deduce acyclic versions of theorems on saddle points, minimax theorems, and the following Nash equilibrium theorem:

Corollary 3. [19] *Let* $\{X_i\}_{i=1}^n$ *be a family of acyclic polyhedra,* $X = \prod_{i=1}^n X_i$*, and for each* i*,* $f_i : X \to \overline{\mathbb{R}}$ *a continuous function such that*
(0) *for each* $x^i \in X^i$ *and each* $\alpha \in \overline{\mathbb{R}}$*, the set*

$$\{x_i \in X_i \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.
Then there exists a point $\widehat{a} \in X$ *such that*

$$f_i(\hat{a}) = \max_{y_i \in X_i} f_i[\widehat{a}^i, y_i] \quad \textit{for all} \quad i.$$

(IV) In 1999 [21], we obtained Theorems 3-5 for G-convex spaces. These results extended and unified a number of known results for particular types of G-convex spaces; see also [23,24]. Therefore, they hold also for Lassonde type convex spaces, Horvath's H-space, hyperconvex metric spaces, and others.

(V) In 2000 [22] and 2002 [26], we applied our fixed point theorem for compact compositions of acyclic maps on admissible (in the sense of Klee) convex subsets of a t.v.s. to obtain a cyclic

coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorem.
The following examples are generalized forms of quasi-equilibrium theorem or social equilibrium existence theorems which directly imply generalizations of the Nash type equilibrium existence theorem.

Theorem 6. [22] *Let $\{X_i\}_{i=1}^n$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, $S_i : X \multimap K_i$ a closed map, and $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ u.s.c. functions for each i.*
Suppose that for each i,
(i) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*
(ii) *the function M_i defined on X by*

$$M_i(x) = \max_{y \in S_i(x)} g_i[x^i, y] \quad \text{for} \quad x \in X$$

is l.s.c.; and
(iii) *for each $x \in X$, the set*

$$\{y \in S_i(x) \mid f_i[x^i, y] \geq M_i(x)\}$$

is acyclic.
If X is admissible in $E = \Pi_{j=1}^n E_j$, then there exists an $\widehat{x} \in K$ such that for each i,

$$\widehat{x}_i \in S_i(\widehat{x}) \quad \text{and} \quad f_i[\widehat{x}^i, \widehat{x}_i] \geq g_i[\widehat{x}^i, y] \quad \text{for all} \quad y \in S_i(\widehat{x}).$$

Theorem 5. [26] *Let X_0 be a topological space and $\{X_i\}_{i=1}^n$ be a family of convex sets, each in a t.v.s. E_i. For each $i = 0, 1, \ldots, n$, let $S_i : X^i \multimap X_i$ be a closed map with compact values, and $f_i, g_i : X = \prod_{i=0}^n X_i \to \mathbf{R}$ u.s.c. real-valued functions.*
Suppose that for each i,
(i) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*
(ii) *the function $M_i : X^i \to \mathbf{R}$ defined by*

$$M_i(x^i) = \max_{y_i \in S_i(x^i)} g_i[x^i, y_i] \quad \text{for} \quad x^i \in X^i$$

is l.s.c.; and
(iii) *for each $x^i \in X^i$, the set*

$$\{x_i \in S_i(x^i) \mid f_i[x^i, x_i] \geq M_i(x^i)\}$$

is acyclic.
If X^0 is admissible in $E^0 = \prod_{j=1}^n E_j$ and if all the maps S_i are compact except possibly S_n and S_n is u.s.c., then there exists an equilibrium point $\widehat{x} \in X$; that is,

$$\widehat{x}_i \in S_i(\widehat{x}^i) \quad \text{and} \quad f_i(\widehat{x}) \geq \max_{y_i \in S_i(x^i)} g_i[\widehat{x}^i, y] \quad \text{for all} \quad i \in \mathbb{Z}_{n+1}.$$

(VI) Since 1996 [9], many authors have published some results of the present paper for hyperconvex metric spaces. For example, Kirk, Sims, and Yuan in 2000 [12] established the KKM theorem, its equivalent formulations, fixed point theorems, and the Nash theorem for hyperconvex metric spaces. However, already in 1993, Horvath [6] found that hyperconvex metric spaces are a particular type of C-spaces.

(VII) In 2001 [25], based on a collective fixed point theorem due to Park, we obtained generalized forms of the von Neumann–Sion type minimax theorem, the Fan–Ma intersection theorem, the Fan–Ma type analytic alternative, and the Nash–Ma equilibrium theorem for G-convex spaces.

(VIII) In 2001, for any topological semilattice $(X, \leq)$ with path-connected interval introduced by Horvath and Llinares [8], the KKM theorem, the Fan-Browder theorem, and the Nash theorem are shown by Luo [15].

(IX) Cain and González [2] cosidered relationship among some subclasses of the class of G-convex spaces and introduced a subclass of L-spaces. In 2007, González et al. [4] showed that G-convex spaces and L-spaces satisfy the partial KKM principle. They added that L-spaces satisfy the properties of the Fan type minimax inequality, Fan-Browder type fixed point, and the Nash type equilibrium. All such results are already known for more general G-convex spaces.

(X) In 2008, for $\mathbb{B}$-spaces, Briec and Horvath [1] showed that some theorems mentioned in this paper hold; that is, Fan-Browder fixed point theorem, Himmelberg type (in fact, Browder type and Kakutani type) fixed point theorems, Fan type minimax inequality, existence of Nash equilibria, and others. Note that $\mathbb{B}$-spaces are KKM spaces [1, Corollary 2.2], and its authors are based on the Peleg type multiple KKM theorem.

(XI) Finally, recall that there are several hundred published works on the KKM theory and we can cover only a part of it. For more historical background for the related fixed point theory and for more involved or related results in this paper, see the references of [32-35] and the literature therein.

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