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NEW ABSTRACT CONVEX SPACES FOR THE KKM THEORY

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Abstract. We introduce some classes of abstract convex spaces, namely, abstract convex minimal spaces, minimal KKM spaces, and G-convex minimal spaces. Each of these subclasses is convenient for the KKM theory and contains G-convex spaces properly. The class of G-convex spaces contains the classes of L-spaces, FC-spaces, and others. A number of examples and related matters are also added.

1. Introduction

The KKM theory, originally called by the author [12], is nowadays the study of applications of various equivalent formulations of the Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in 1929 [9]. In the last decade, the theory has been extensively studied for generalized convex spaces (simply, G-convex spaces) in a sequence of papers of the author; for details, see [13-17] and references therein.

Since the concept of G-convex spaces first appeared in 1993 [26], a number of modifications or imitations of the concept have followed. Such examples are L-spaces due to Ben-El-Mechaiekh et al. [2], spaces having property (H) due to Huang [8], FC-spaces due to Ding [4,5], and others. It is known that all of such examples are particular forms of G-convex spaces; see [23].

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In our previous work [18], we introduced a new concept of abstract convex spaces and multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ having certain KKM property. These new spaces and multimap classes are known to be adequate to establish the KKM theory; see [18-22]. Especially, in [22], we generalized and simplified known results of the theory on convex spaces, H-spaces, G-convex spaces, and others. It is noticed there that the class of abstract convex spaces $(E, D; \Gamma)$ satisfying the KKM principle play the major role in the KKM theory. Therefore, it seems to be quite natural to call such spaces the KKM spaces. In our work [24], we showed that a large number of well-known results in the KKM theory on G-convex spaces also hold on the KKM spaces.

Moreover, apparently motivated by the author's earlier works, Alimohammady et al. [1] introduced the notion of minimal G-convex spaces and obtained the open and closed versions of the KKM principle in this new setting. Their method is just replacing the topological structure in the relevant results by the more general minimal structure.

In our previous work [25], we introduced a new concept of abstract convex minimal spaces which can be also useful to establish some results in the KKM theory. With this new concept, we obtained generalizations of the KKM principle and some of their applications. In fact, the KKM type maps were used to obtain coincidence theorems, the Fan-Browder type fixed point theorems, the Fan intersection theorem, and the Nash equilibrium theorem on abstract convex minimal spaces. These generalize corresponding previously known results.

Now it is evident that the class of abstract convex spaces contains many subclasses on which it is convenient to establish the KKM theory. In the present paper, we introduce such new subclasses of the class of abstract convex spaces, namely, abstract convex minimal spaces, minimal KKM spaces, and generalized convex minimal spaces. Each of these contains G-convex spaces properly. Recall that the class of G-convex spaces contains the classes of L-spaces, FC-spaces, and others. Some related matters are also discussed.

2. Abstract convex spaces

In this section, we recall definitions of abstract convex spaces given in [18-22]: Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

Definitions. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\}.$$

[co is reserved for the convex hull in topological vector spaces.] A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D' \rangle})$ itself is an abstract convex space called a *subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

If E is given a topology, then the abstract convex space $(E, D; \Gamma)$ is called *an abstract convex topological space*.

We already gave plenty of examples of abstract convex spaces in [18,22].

Definitions. [1] A family $\mathcal{M}$ of subsets of a set X is called a *minimal structure* on X if $\emptyset, X \in \mathcal{M}$. In this case, $(X, \mathcal{M})$ is called a *minimal space*. Any element of $\mathcal{M}$ is called an *m-open set* of X and a complement of an m-open set is called an *m-closed set* of X. For minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a function $f : X \to Y$ is said to be *continuous* (more precisely, *m-continuous* or $(\mathcal{M}, \mathcal{N})$-*continuous*) if $f^{-1}(V) \in \mathcal{M}$ for each $V \in \mathcal{N}$.

Definition. If E is given a minimal structure, then the abstract convex space $(E, D; \Gamma)$ is called *an abstract convex minimal space*.

Examples. 1. Any topological space is a minimal space and not conversely.

2. Any topological vector space is a minimal vector space. There is some linear minimal space which is not a topological vector space; see [1].

3. A triple $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consisting of a topological [resp., minimal] space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$ and a standard n-simplex Δ_n, is an abstract convex topological [resp., minimal] space by putting $\Gamma_A := \phi_A(\Delta_n)$.

3. The KKM spaces

Recall the following in [18,22]:

Definitions. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is said to have the *KKM property* and called *a* $\mathfrak{K}$-*map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E,Z) \subset \mathfrak{KC}(E,Z) \cap \mathfrak{KO}(E,Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical.

Further, when $(Z,\mathcal{M})$ is a minimal space, an $m\mathfrak{KC}$-map is defined for m-closed-valued maps G, and an $m\mathfrak{KO}$-map for m-open-valued maps G. In this case, we have

$$\mathfrak{K}(E,Z) \subset m\mathfrak{KC}(E,Z) \cap m\mathfrak{KO}(E,Z).$$

Examples. 1. Every abstract convex space in our sense has a map $F \in \mathfrak{K}(E,Z)$ for any nonempty set Z and for any class of KKM maps $G : D \multimap Z$ with respect to F. In fact, for each $x \in E$, choose $F(x) := Z$ or $F(x)$ contains some $z_0 \in Z$.

2. Further examples were given in Section 5 of [18].

Definitions. For an abstract convex topological space $(E, D; \Gamma)$, the *KKM principle* is the statement $1_E \in \mathfrak{KC}(E,E) \cap \mathfrak{KO}(E,E)$. A *KKM space* is an abstract convex topological space satisfying the KKM principle [24].

For an abstract convex minimal space $(E, D; \Gamma)$, the *KKM principle* is the statement $1_E \in m\mathfrak{KC}(E,E) \cap m\mathfrak{KO}(E,E)$. A *minimal KKM space* (or simply, *$mKKM$ space*) is an abstract convex minimal space satisfying the KKM principle.

Examples. We give examples of KKM spaces:

1. Every generalized convex space is a KKM space; see [14,15].

2. A connected ordered space $(X, \leq)$ can be made into an abstract convex topological space $(X \supset D; \Gamma)$ for any nonempty $D \subset X$ by defining $\Gamma_A := [\min A,\ \max A] = \{x \in X \mid \min A \leq x \leq \max A\}$ for each $A \in \langle D \rangle$. Further, it is a KKM space; see [21, Theorem 5(i)].

3. The extended long line L^* can be made into a KKM space $(L^* \supset D; \Gamma)$; see [21]. In fact, L^* is constructed from the ordinal space $D := [0, \Omega]$ consisting of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology. Recall that L^* is a generalized arc obtained from $[0, \Omega]$ by placing a copy of the interval $(0, 1)$ between each ordinal α and its successor $\alpha + 1$ and we give L^* the order topology. Now let $\Gamma : \langle D \rangle \multimap L^*$ be the one as in 2.

In our previous work [22], for G-convex spaces, there exist more than 15 equivalent formulations of the KKM principle such as Alexandroff-Pasynkoff theorem, Fan type matching theorem, Tarafdar type intersection theorem, geometric or section properties, Fan-Browder type fixed point theorems, maximal element theorems, analytic alternatives, Fan type minimax inequalities, variational inequalities, and others. This is also true for KKM spaces.

In our forthcoming paper [24], we show that some of well-known results in the KKM theory on G-convex spaces also hold on the KKM spaces. Examples of such results are theorems of Sperner and Alexandroff-Pasynkoff, the Horvath type fixed point theorem, the Fan-Browder type coincidence theorems, the Fan type minimax inequalities, variational inequalities, the von Neumann type minimax theorem, and the Nash type equilibrium theorem.

4. Generalized convex spaces

Recall the following appeared in [13-17]:

Definition. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is an n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

The original KKM principle [9] is for the triple $(\Delta_n, V; \mathrm{co})$, where V denotes the set of vertices and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation, and Fan's celebrated lemma [7] is for $(E, D; \mathrm{co})$, where D is a nonempty subset of a topological vector space E. These are the origins of our G-convex space $(X, D; \Gamma)$. Note that any KKM type theorem on $(X; \Gamma)$ can not generalize the KKM principle and the Fan lemma.

Definition. A *generalized convex minimal space* or a *G-convex minimal space* $(X, D; \Gamma)$ consists of a minimal space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. See [1].

Examples. 1. A G-convex space is a G-convex minimal space, and the converse does not hold; for example, see [1].

2. A G-convex space is a KKM space and the converse does not hold; for example, the extended long line L^* is a KKM space $(L^* \supset D; \Gamma)$, but not a G-convex space.

In fact, since $\Gamma\{0, \Omega\} = L^*$ is not path connected, for $A := \{0, \Omega\} \in \langle L^* \rangle$ and $\Delta_1 := [0, 1]$, there does not exist a continuous function $\phi_A : [0, 1] \to \Gamma_A$ such that $\phi_A\{0\} \subset \Gamma\{0\} = \{0\}$ and $\phi_A\{1\} \subset \Gamma\{\Omega\} = \{\Omega\}$. Therefore $(L^* \supset D; \Gamma)$ is not G-convex.

3. A G-convex minimal space $(X, D; \Gamma)$ is a minimal KKM space in view of the following:

Proposition 1. [1] *Let $(E, D; \Gamma)$ be a generalized convex minimal space and $F : D \multimap E$ a KKM map with m-closed values [resp., m-open values]. Then $\{F(z)\}_{z\in D}$ has the finite intersection property.*

Essentially, the proof of Proposition 1 [1, Theorems 3.2 and 3.5] is the one in [14,15] with slight modifications.

It is obvious that most facts on G-convex spaces (e.g. in [14]) can be extended to corresponding ones on G-convex minimal spaces.

In the category of topological vector spaces or C-spaces, the concepts of locally convex spaces, LC-spaces, Φ-spaces, subsets of the Zima-Hadžić type, admissible subsets, and Klee approximable sets are quite well-known. They were introduced in order to generalize known fixed point theorems.

In our previous work [17], we extended those concepts to G-convex uniform spaces and established the mutual relations among them as follows:

Proposition 2. *In the class of G-convex uniform spaces, the following hold:*
(1) *Any LG-space is of the Zima-Hadžić type.*
(2) *Every LG-space is locally G-convex whenever every singleton is Γ-convex.*
(3) *Any nonempty subset of a locally G-convex space is a Φ-set.*
(4) *Any Zima-Hadžić type subset of a G-convex uniform space such that every singleton is Γ-convex is a Φ-set.*
(5) *Every Φ-space is admissible. More generally, every nonempty compact Φ-subset is Klee approximable.*

Note that Proposition 2 can be extended to the KKM uniform spaces.

5. Spaces having a family $\{\phi_A\}_{A\in\langle D\rangle}$

In this section, we deal with particular subclasses or variants of G-convex spaces as follows:

Definition. A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A*-space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

Similarly, a *minimal* ϕ_A*-space* can be defined whenever X is a minimal space.

Examples. 1. [2] An L-structure on a topological space E is given by a nonempty set-valued map $\Gamma : \langle E\rangle \to E$ verifying

($*$) for each $A \in \langle E\rangle$, say $A = \{x_0, x_1, \dots, x_n\}$, there exists a continuous function $f^A : \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0, 1, \dots, n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_i \mid i \in J\})$.

The pair (E, Γ) is then called an *L-space*, and $X \subset E$ is said to be *L-convex* if $\forall A \in \langle X \rangle$, $\Gamma(A) \subset X$. Note that an L-space (E, Γ) is a particular form of a G-convex space $(E, D; \Gamma)$ with $E = D$.

2. [8] A topological space Y is said to have property (H) if, for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$.

Let X be a nonempty set and Y be a topological space with property (H). $T : X \to 2^Y$ is said to be a generalized R-KKM mapping if for each $\{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ such that

$$\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

for all $\{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$.

3. [4,5] $(Y, \{\varphi_N\})$ is said to be an FC-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset D of $(Y, \{\varphi_N\})$ is said to be a FC-subspace of Y if for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$ where $\Delta_k = \text{co}\{e_{i_j} \mid j = 0, \ldots, k\}$.

4. For minimal spaces, we can also define L-spaces, spaces having property (H), and FC-spaces.

5. We can consider a ψ_A-space $(X, D; \{\psi_A\}_{A \in \langle D \rangle})$, similar to a ϕ_A-space, where $\psi_A : [0,1]^n \to X$ is continuous for each $A \in \langle D \rangle$ with $|A| = n+1$. Such types of spaces are given by Michael [11], Llinares [10], and Cain and Gonzáles [3]. For each $n \geq 0$, considering continuous functions $g_n : \Delta_n \to [0,1]^n$ given by

$$g_n : u = \sum_{i=0}^{n} \lambda_i(u) e_i \mapsto (\lambda_0(u), \ldots, \lambda_{n-1}(u))$$

for $u \in \Delta_n$ and by putting $\phi_A := \psi_A g_n$, a ψ_A-space becomes a ϕ_A-space.

6. Any G-convex minimal space is a minimal ϕ_A-space. The converse also holds:

Proposition 3. *A minimal ϕ_A-space $(X, D; \{\phi_A\})$ can be made into a G-convex minimal space $(X, D; \Gamma)$.*

Proof. This can be done at least in three ways.

(1) For each $A \in \langle D \rangle$, by putting $\Gamma_A := X$, we obtain a trivial G-convex minimal space $(X, D; \Gamma)$.

(2) Let $\{\Gamma^\alpha\}_\alpha$ be the family of maps $\Gamma^\alpha : \langle D \rangle \multimap X$ giving a G-convex minimal space $(X, D; \Gamma^\alpha)$. Note that, by (1), this family is not empty. Then, for each α

and each $A \in \langle D \rangle$ with $|A| = n+1$, we have

$$\phi_A(\Delta_n) \subset \Gamma^\alpha_A \ \text{ and } \ \phi_A(\Delta_J) \subset \Gamma^\alpha_J \quad \text{for} \ \ J \subset A.$$

Let $\Gamma := \bigcap_\alpha \Gamma^\alpha$, that is, $\Gamma_A = \bigcap_\alpha \Gamma^\alpha_A$. Then

$$\phi_A(\Delta_n) \subset \Gamma_A \ \text{ and } \ \phi_A(\Delta_J) \subset \Gamma_J \quad \text{for} \ \ J \subset A.$$

Therefore, $(X, D; \Gamma)$ is a G-convex minimal space.

(3) Let $N \in \langle D \rangle$ with $|N| = n+1$. For each $M \in \langle D \rangle$ with $N \subset M$, $M = \{a_0, \dots, a_m\}$ and $N = \{a_{i_0}, \dots, a_{i_n}\}$, there exists a subset $\phi_M(\Delta_n^M)$ of X such that $\Delta_n^M := \text{co}\{e_{i_j} \mid j = 0, \dots, n\} \subset \Delta_m$. Now let

$$\Gamma_N = \Gamma(N) := \bigcup_{M \supset N} \phi_M(\Delta_n^M).$$

Then $\Gamma : \langle D \rangle \multimap X$ is well-defined and $(X, D; \Gamma)$ becomes a G-convex minimal space: For each $A \in \langle D \rangle$ with $|A| = n+1$, there exists a continuous map $\phi_A : \Delta_n \multimap \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Therefore, G-convex minimal spaces and minimal ϕ_A-spaces are essentially same.

Examples. Let $\Delta_3 = \text{co}\, V$ where $V = \{e_0, e_1, e_2, e_3\}$.

1. We have a G-convex space $(\Delta_3, V; \text{co})$ where $\text{co} : \langle V \rangle \multimap \Delta_3$ is the convex hull operator.

2. Let $(\Delta_3, V; \Gamma)$ be another G-convex space given by $\Gamma\{e_0, e_1\} := \text{co}\{e_0, e_1, e_2\}$ and $\Gamma(A) := \text{co}\, A$ for all other $A \in \langle V \rangle$.

Let $(\Delta_3, V; \{\phi_A\})$ be a ϕ_A-space where $\phi_A(\Delta_n) = \Gamma(A)$. Then

$$\phi_{\{e_0,e_1\}}(\Delta_1) = \phi_{\{e_0,e_1\}}(\text{co}\,\{e_0, e_1\}) = \Gamma\{e_0, e_1\} = \text{co}\,\{e_0, e_1, e_2\},$$

where we may assume $\phi_{\{e_0,e_1\}}$ is a surjective space-filling curve such that $\phi_{\{e_0,e_1\}}(e_0) = \{e_0\}$ and $\phi_{\{e_0,e_1\}}(e_1) = \{e_1\}$. Then it is easily checked that Γ itself is the one in the proof (3) of Proposition 3 corresponding to $\{\phi_A\}$.

Example. The extended long line $(L^* \supset D; \Gamma)$ is not G-convex, and hence not a ϕ_A-space.

Recall that, in the recent study on abstract convex spaces in [18-22], many basic theorems on G-convex spaces are further generalized.

For a G-convex minimal space $(X, D; \Gamma)$, a multimap $G : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset G(A)$ for each $A \in \langle D \rangle$.

Proposition 4. *For a minimal ϕ_A-space $(X, D; \{\phi_A\})$, any map $T : D \multimap X$ satisfying*

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each} \ \ A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is a KKM map on a G-convex minimal space $(X, D; \Gamma)$.

Proof. Define $\Gamma : \langle D \rangle \multimap X$ by $\Gamma_A := T(A)$ for each $A \in \langle D \rangle$. Then $(X, D; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n + 1$, we have a continuous function $\phi_A : \Delta_n \to T(A) =: \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle D \rangle$ and hence $T : D \multimap X$ is a KKM map on a G-convex minimal space $(X, D; \Gamma)$.

Similarly, we have the following:

Proposition 5. *A generalized R-KKM map* $T : X \to 2^Y$ *in* [6, 7] *is simply a KKM map for some G-convex space* $(Y, X; \Gamma)$.

Contrary to Proposition 5, Ding in [6] claimed as follows: "The above class of generalized *R-KKM* mappings include those classes of *KKM* mappings, *H-KKM* mappings, *G-KKM* mappings, generalized *G-KKM* mappings, generalized *S-KKM* mappings, *GLKKM* mappings and *GMKKM* mappings defined in topological vector spaces, *H*-spaces, *G*-convex spaces, *G-H*-spaces, *L*-convex spaces and hyperconvex metric spaces, respectively, as true subclasses."

Therefore, all of the KKM type theorems on such variants are simple consequences of our G-convex space theory. Consequently, all results in [6] are artificial disguised forms of known ones having no proper examples.

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