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# CHARACTERIZATIONS OF THE KKM SPACES

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Abstract. A KKM space is an abstract convex space satisfying an abstract form of the original KKM theorem and its 'open' version. We introduce several characterizations of KKM spaces as abstract convex spaces satisfying those properties of matching, intersection, geometric or section, Fan-Browder type fixed point, maximal element, and others. Certain applications of those characterizations are indicated. Finally, some results in the fixed point theory on KKM uniform spaces or abstract convex uniform spaces are added.

## 1. Introduction

The KKM theory, first called by the author [21], is the study on applications of equivalent formulations of a celebrated theorem due to Knaster, Kuratowski, and Mazurkiewicz (simply, the KKM theorem) [19] and their generalizations. This theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences; see [23] and references therein.

At the beginning, the theory was mainly concerned with convex subsets of topological vector spaces as in the works of Fan [4-10]. Later, it has been extended to convex spaces by Lassonde [20], and to spaces having certain families of contractible subsets (simply, $C$-spaces or $H$-spaces) by Horvath [12-15]. This line of generalizations of earlier works is followed by the present author for generalized convex spaces or $G$-convex spaces; see [23-25,28,35]. Moreover, there have appeared several variations of such spaces by other authors; see [30].

Recently, in [26,27,29-33], the author introduced the concepts of abstract convex spaces and KKM spaces, which seem to be more adequate to establish the KKM theory for various purposes. A KKM space is an abstract convex space satisfying an abstract form of the original KKM theorem and its "open" version. In fact, our new concept of KKM spaces is a common generalization

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of many of known abstract convexities without any linear structure developed in connection mainly with the fixed point theory and the KKM theory.

In the present survey, we introduce several characterizations of KKM spaces as abstract convex spaces satisfying one of the properties of KKM, matching, intersection, geometric or section, Fan-Browder type fixed point, maximal element, and others. Certain earlier applications of such characterizations are indicated in each section. It is also noted that many of the results are mutually equivalent. Finally, some results in the fixed point theory on KKM uniform spaces or abstract convex uniform spaces are added.

This survey is a supplement to [29,32].

## 2. Abstract convex spaces and the KKM spaces

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set $D$. Multimaps are also called simply maps.

Recall the following in [26,29-33]:

**Definition.** An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space $E$, a nonempty set $D$, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the $\Gamma$-*convex hull* of $D'$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset $X$ of $E$ is called a $\Gamma$-*convex subset* of $(E, D; \Gamma)$ relative to $D'$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D'\rangle})$ is called a $\Gamma$-*convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset $X$ of $E$ is said to be $\Gamma$-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, $X$ is $\Gamma$-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

**Example 2.1.** We give examples of abstract convex spaces:

(1) The original KKM theorem [19] is for the triple $(\Delta_n \supset V; \mathrm{co})$, where $\Delta_n$ is the standard $n$-simplex, $V$ the set of its vertices $\{e_i\}_{i=0}^n$, and $\mathrm{co} : \langle V\rangle \multimap \Delta_n$ the convex hull operation.

(2) A triple $(X \supset D; \Gamma)$, where $X$ and $D$ are subsets of a t.v.s. $E$ such that $\mathrm{co}\, D \subset X$ and $\Gamma := \mathrm{co}$. Fan's celebrated KKM lemma [4] is for $(E \supset D; \mathrm{co})$, where $D$ is a nonempty subset of $E$.

(3) A *convex space* $(X \supset D; \Gamma)$ [23] is a triple where $X$ is a subset of a vector space, $\mathrm{co}\, D \subset X$, and each $\Gamma_A$ is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde

for $X = D$; see [20]. However he obtained several KKM type theorems w.r.t. $(X \supset D; \Gamma)$.

(4) A triple $(X \supset D; \Gamma)$ is called an *H-space* by Park [23,34] if $X$ is a topological space, $D$ a nonempty subset of $X$, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, $\omega$-connected) subsets of $X$ indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma) := (X, X; \Gamma)$ is called a $c$-space by Horvath [14] or an $H$-space by Bardaro and Ceppitelli in 1988.

(5) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\Delta_J$ is the face of $\Delta_n$ corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For details, see [23-25,28,35] and references therein.

(6) A *$\phi_A$-space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space $X$, a nonempty set $D$, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular $n$-simplexes) for $A \in \langle D \rangle$ with $|A| = n + 1$. Every $\phi_A$-space can be made into a $G$-convex space; see [30]. More recently, $\phi_A$-spaces are called GFC-spaces by Khanh et al. [17].

(7) A convexity space $(E, \mathcal{C})$ in the classical sense is an abstrct convex space whenever $E$ is given a topology; see [36], where the bibliography lists 283 papers.

(8) Suppose $X$ is a closed convex subset of a complete $\mathbb{R}$-tree $M$, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_M(A)$; see Kirk and Panyanak [18]. Then $(M \supset X; \Gamma)$ is an abstract convex space.

(9) According to Horvath [16], a convexity on a topological space $X$ is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of $X$, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.

(10) A $\mathbb{B}$-space due to Briec and Horvath [1] is an abstract convex space.

Note that each of (3)-(9) has a large number of concrete examples.

**Definition.** Let $(E, D; \Gamma)$ be an abstract convex space. A multimap $G : D \multimap E$ is called a *KKM map* if

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle.$$

**Example 2.2.** Granas [11] gave examples of KKM maps as follows:

(1) *Variational problems.* Let $C$ be a convex subset of a vector space $E$ and $\phi : C \to \mathbb{R}$ is a convex function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C \mid \phi(y) \leq \phi(x)\} \quad \text{for } x \in C$$

is a KKM map.

(2) *Best approximation.* Let $C$ be a convex subset of a vector space $E$, $p$ a seminorm on $E$, and $f : C \to E$ a function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C \mid p(f(y) - y) \leq p(f(y) - x)\} \quad \text{for } x \in C$$

is a KKM map.

(3) *Variational inequalities.* Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space, $C$ a convex subset of $H$, and $f : C \to H$ a function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C \mid \langle f(y), y - x \rangle \leq 0\} \quad \text{for } x \in C$$

is a KKM map.

**Example 2.3.** For a $\phi_A$-space $(X, D; \{\phi_A\})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each } \ A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is a KKM map on a $G$-convex space $(X, D; \Gamma)$.

**Definition.** The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

In our recent work [29,31,32], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are closely related to KKM spaces or abstract convex spaces satisfying the partial KKM principle.

**Example 2.4.** We give examples of KKM spaces:

(1) Every $G$-convex space is a KKM space [25].

(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space [31].

(3) The extended long line $L^*$ is a KKM space $(L^* \supset D; \Gamma)$ with the ordinal space $D := [0, \Omega]$; see [31]. But $L^*$ is not a $G$-convex space.

(4) For a closed convex subset $X$ of a complete $\mathbb{R}$-tree $M$, and $\Gamma_A := conv_M(A)$ for each $A \in \langle X \rangle$, the triple $(M \supset X; \Gamma)$ satisfies the partial KKM principle; see Kirk and Panyanak [18]. Later we found that $(M \supset X; \Gamma)$ is a KKM space.

(5) For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X \rangle$; see [16].
(6) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space; see [1].

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ $H$-space $\Longrightarrow$ $G$-convex space $\Longleftrightarrow$ $\phi_A$-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle
$\Longrightarrow$ Abstract convex space.

It is not known yet whether there is a space satisfying the partial KKM principle which is not a KKM space.

## 3. Matching property

From now on, a triple $(X, D; \Gamma)$ denotes an abstract convex space unless explicitly stated otherwise. The (partial) KKM principle is equivalent to the Fan type matching property:

**Theorem 3.1.** *An abstract convex space $(X, D; \Gamma)$ satisfies the partial KKM principle iff for any map $S : D \multimap X$ satisfying*
(3.1.1) *$S(z)$ is open for each $z \in D$; and*
(3.1.2) *$X = \bigcup_{z \in M} S(z)$ for some $M \in \langle D \rangle$,*
*there exists an $N \in \langle D \rangle$ such that*

$$\Gamma_N \cap \bigcap_{z \in N} S(z) \neq \emptyset.$$

*An abstract convex space $(X, D; \Gamma)$ is a KKM space iff the above condition also holds for closed-valued maps $S$.*

*Proof.* (Necessity) Let $G : D \multimap X$ be a map given by $G(z) := X \setminus S(z)$ for $z \in D$. Then $G$ has closed [resp., open] values. Suppose, on the contrary to the conclusion, that for any $N \in \langle D \rangle$, we have $\Gamma_N \cap \bigcap_{z \in N} S(z) = \emptyset$; that is, $\Gamma_N \subset X \setminus \bigcap_{z \in N} S(z) = \bigcup_{z \in N}(X \setminus S(z)) = G(N)$. Then $G$ is a KKM map. Since $(X, D; \Gamma)$ satisfies the (partial) KKM principle, there exists a $\widehat{y} \in \bigcap_{z \in N} G(z) = \bigcap_{z \in N}(X \setminus S(z))$. Hence $\widehat{y} \notin S(z)$ for all $z \in N$. This violates condition (3.1.2).

(Sufficiency) Let $G : D \multimap X$ be a KKM map with closed [resp., open] values. Suppose $\bigcap_{z \in M} G(z) = \emptyset$ for some $M \in \langle D \rangle$. Then $\bigcup_{z \in M} G^c(z) = \bigcup_{z \in M}(X \setminus G(z)) = X$. Therefore, by the sufficiency assumption, there exists an

$N \in \langle D \rangle$ such that $\Gamma_N \cap \bigcap_{z\in N} G^c(z) \neq \emptyset$. Since $G$ is a KKM map, $\Gamma_N \subset G(N)$. Therefore, we have a contradiction $G(N) \cap (G(N))^c \neq \emptyset$. □

From Theorem 3.1, we have the following:

**Corollary 3.2.** *Let $(X, D; \Gamma)$ satisfy the partial KKM principle and $S : D \multimap X$ be a map such that*

(1) *$S(z)$ is open for each $z \in D$;*

(2) *$S^-(y)$ is nonempty for each $y \in X$ (that is, $S$ is surjective); and*

(3) *$X \setminus S(M)$ is compact for some $M \in \langle D \rangle$.*

*Then there exists an $N \in \langle D \rangle$ such that*

$$\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset.$$

*Proof.* Note that (2) and (3) imply (3.1.2). □

**Remark.** The origin of Corollary 3.2 goes back to Fan [9,10] for a convex set $X = D$.

From now on, proofs not appeared in Sections 4-8 can be seen in [32].

## 4. Another intersection property

The (partial) KKM principle is equivalent to another intersection property:

**Theorem 4.1.** *An abstract convex space $(X, D; \Gamma)$ satisfies the partial KKM principle iff for any maps $S : D \multimap X,\ T : X \multimap X$ satisfying*

(4.1.1) *$S$ has closed values;*

(4.1.2) *for each $x \in X$, $\mathrm{co}_\Gamma(D \setminus S^-(x)) \subset X \setminus T^-(x)$; and*

(4.1.3) *$x \in T(x)$ for each $x \in X$,*

*$\{S(z)\}_{z\in D}$ has the finite intersection property.*

*An abstract convex space $(X, D; \Gamma)$ is a KKM space iff the above condition also holds for any open-valued map $S$.*

From Theorem 4.1 or Corollary 3.2, we immediately have another whole intersection property:

**Corollary 4.2.** *Let $(X, D; \Gamma)$ satisfy the partial KKM principle and $S : D \multimap X,\ T : X \multimap X$ be maps such that*

(1) *$S$ has closed values;*

(2) *for each $x \in X$, $\mathrm{co}_\Gamma(D \setminus S^-(x)) \subset X \setminus T^-(x)$;*

(3) *$x \in T(x)$ for each $x \in X$; and*

(4) *$\bigcap_{z\in M} S(z)$ is compact for some $M \in \langle D \rangle$.*

*Then*

$$\bigcap_{z\in D} S(z) \neq \emptyset.$$

**Remark.** The first particular form of Corollary 4.2 is due to Tarafdar [37] for a convex set $X = D$. Another particular forms of Corollary 4.2 also appear in [12,22] and others.

## 5. Geometric or section properties

In this section, we show that the (partial) KKM principle is equivalent to two geometric forms. The following is usually called the section property:

**Theorem 5.1.** *An abstract convex space* $(X, D; \Gamma)$ *satisfies the partial KKM principle iff for any sets* $A \subset D \times X,\ B \subset X \times X$ *satisfying*

(5.1.1) $\{y \in X \mid (z, y) \in A\}$ *is closed for each* $z \in D$*;*

(5.1.2) *for each* $y \in X,\ \mathrm{co}_\Gamma\{z \in D \mid (z, y) \notin A\} \subset \{x \in X \mid (x, y) \notin B\}$*; and*

(5.1.3) $(x, x) \in B$ *for each* $x \in X$*,*

*for each* $N \in \langle D\rangle$*, there exists an* $x_0 \in X$ *such that* $N \times \{x_0\} \subset B$*.*

*An abstract convex space* $(X, D; \Gamma)$ *is a KKM space iff the above condition also holds for any set* $A \subset D \times X$ *satisfying*

$(5.1.1)'$ $\{y \in X \mid (z, y) \in A\}$ *is open for each* $z \in D$

*instead of* (5.1.1)*.*

**Corollary 5.2.** *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle and* $A \subset D \times X,\ B \subset X \times X$ *be two sets satisfying* (5.1.1)-(5.1.3)*. Further if*

(5.1.4) $\{y \in X \mid (z_0, y) \in A\}$ *is compact for some* $z_0 \in D$*,*

*then there exists an* $x_0 \in X$ *such that* $D \times \{x_0\} \subset A$*.*

**Remark.** If $X = D$ is a convex subset of a topological vector space and if $A = B$, Corollary 5.2 reduces to Fan's 1961 Lemma [4, Lemma 4]. He obtained his result from his own generalization of the KKM theorem and applied it to a direct proof of the Tychonoff fixed point theorem. Other interesting applications of his useful lemma to fixed points, minimax theorems, equilibrium points, extension of monotone sets, potential theory, and others have been made by Fan [6] and many others; see [23].

The following geometric property is equivalent to the (partial) KKM principle:

**Theorem 5.3.** *An abstract convex space* $(X, D; \Gamma)$ *satisfies the partial KKM principle iff for any sets* $A \subset D \times X,\ B \subset X \times X$ *satisfying*

(5.3.1) $\{y \in X \mid (z, y) \in A\}$ *is open for each* $z \in D$*;*

(5.3.2) *for each* $y \in X$, $\mathrm{co}_\Gamma \{z \in D \mid (z, y) \in A\} \subset \{x \in X \mid (x, y) \in B\}$*; and*

(5.3.3) *there exists an* $M \in \langle D \rangle$ *such that for any* $x \in X$, $(z, x) \in A$ *for some* $z \in M$,

*there exists an* $x_0 \in X$ *such that* $(x_0, x_0) \in B$.

*An abstract convex space* $(X, D; \Gamma)$ *is a KKM space iff the above condition also holds when the set in* (5.3.1) *is closed.*

**Corollary 5.4.** *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle and* $A \subset D \times X$, $B \subset X \times X$ *be two sets satisfying* (5.3.1) *and* (5.3.2)*. Further if*

(1) *for each* $y \in X$*, there exists a* $z \in D$ *such that* $(z, y) \in A$*; and*

(2) $\{y \in X \mid (z_0, y) \notin A\}$ *is compact for some* $z_0 \in D$,

*then there exists an* $x_0 \in X$ *such that* $(x_0, x_0) \in B$.

**Remark.** If $X = D$ is a convex subset of a topological vector space and if $A = B$, Corollary 5.4 reduces to Fan [8, Theorem 2]. In this case, (5.3.2) merely tells that $\{x \in X \mid (x, y) \in A\}$ is convex.

## 6. The Fan-Browder type fixed point theorems

The (partial) KKM principle is equivalent to the Fan-Browder type fixed point theorem:

**Theorem 6.1.** *An abstract convex space* $(X, D; \Gamma)$ *satisfies the partial KKM principle iff for any maps* $S : D \multimap X$, $T : X \multimap X$ *satisfying*

(6.1.1) $S(z)$ *is open for each* $z \in D$*;*

(6.1.2) *for each* $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$*; and*

(6.1.3) $X = \bigcup_{z \in M} S(z)$ *for some* $M \in \langle D \rangle$,

$T$ *has a fixed point* $x_0 \in X$*; that is,* $x_0 \in T(x_0)$.

*An abstract convex space* $(X, D; \Gamma)$ *is a KKM space iff the above condition also holds for any map* $S : D \multimap X$ *satisfying*

(6.1.1)$'$ $S(z)$ *is closed for each* $z \in D$

*instead of* (6.1.1).

*Proof.* (Necessity) Suppose $(X, D; \Gamma)$ satisfies the (partial) KKM principle. Define a map $G : D \multimap X$ by $G(z) := X \backslash S(z)$ for each $z \in D$. Then each $G(z)$ is closed [resp., open] by (6.1.1) [resp., (6.1.1)$'$], and

$$\bigcap_{z \in M} G(z) = X \backslash \bigcup_{z \in M} S(z) = X \backslash X = \emptyset$$

by (6.1.3). Therefore, the family $\{G(z)\}_{z \in D}$ does not have the finite intersection property, and hence, $G$ is not a KKM map by the (partial) KKM principle. Thus, there exists an $N \in \langle D \rangle$ such that $\Gamma_N \not\subset G(N) = \bigcup \{X \backslash S(z) \mid z \in N\}$. Hence, there exists an $x_0 \in \Gamma_N$ such that $x_0 \in S(z)$ for all $z \in N$; that is,

$N \subset S^-(x_0)$. Therefore, $x_0 \in \Gamma_N \subset \mathrm{co}_\Gamma S^-(x_0) \subset T^-(x_0)$ by (6.1.2). This implies $x_0 \in T(x_0)$.

(Sufficiency) Let $G : D \multimap X$ be a KKM map with closed [resp., open] values. Suppose the family $\{G(z)\}_{z\in D}$ does not have the finite intersection property; that is, there exists an $M \in \langle D\rangle$ such that $\bigcap_{z\in M} G(z) = \emptyset$. Define a map $S : D \multimap X$ by $S(z) = X\backslash G(z)$ for $z \in D$ and a map $T : X \multimap X$ by $\mathrm{co}_\Gamma S^-(y) = T^-(y)$ for each $y \in X$. Then all of the requirements (6.1.1)–(6.1.3) [resp., (6.1.1)′–(6.1.3)] are satisfied. Hence there exists an $x_0 \in X$ such that $x_0 \in T(x_0)$. Then $x_0 \in T^-(x_0) = \mathrm{co}_\Gamma S^-(x_0)$ and hence, there exists an $N \in \langle S^-(x_0)\rangle$ such that $x_0 \in \Gamma_N \subset \mathrm{co}_\Gamma S^-(x_0)$. Therefore, for each $z \in N$, we have $x_0 \in S(z)$ or $x_0 \notin G(z)$; that is, $\Gamma_N \not\subset G(N)$. Hence $G$ is not a KKM map, a contradiction. Therefore $(X, D; \Gamma)$ satisfies the (partial) KKM principle. □

**Corollary 6.2.** *Let $(X, D; \Gamma)$ satisfy the partial KKM principle and $S : D \multimap X$, $T : X \multimap X$ be two maps satisfying* (6.1.1)-(6.1.2). *If*

(1) *for each $y \in X$, $S^-(y) \neq \emptyset$; and*

(2) *$X \setminus S(z_0)$ is compact for some $z_0 \in D$,*

*then $T$ has a fixed point $x_0 \in X$.*

**Corollary 6.3.** *Let $(X, D; \Gamma)$ satisfy the partial KKM principle, $X$ be compact, and $S : X \multimap D$, $T : X \multimap X$ two maps satisfying*

(1) *for each $x \in X, \mathrm{co}_\Gamma S(x) \subset T(x)$; and*

(2) *$X = \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in D\}$.*

*Then $T$ has a fixed point $x_0 \in X$.*

**Corollary 6.4.** *Let $(X, D; \Gamma)$ satisfy the partial KKM principle, $X$ be compact, and $S : X \multimap D$ a map satisfying*

(1) *for each $x \in X$, $S(x)$ is nonempty; and*

(2) *for each $z \in D$, $S^-(z)$ is open.*

*Then there exists an $\widehat{x} \in X$ such that $\widehat{x} \in \mathrm{co}_\Gamma S(\widehat{x})$.*

The following simplified form of Corollary 6.2 or 6.3 is also a Fan-Browder type fixed point theorem:

**Corollary 6.5.** *Let $(X; \Gamma)$ satisfy the partial KKM principle, $X$ be compact, and $T : X \multimap X$ a map satisfying*

(1) *for each $x \in X$, $T(x)$ is $\Gamma$-convex; and*

(2) *$X = \bigcup\{\mathrm{Int}\, T^-(y) \mid y \in X\}$.*

*Then $T$ has a fixed point.*

**Remarks.** (1) For a convex subset $X$ of a topological vector space $E$, if $T^-(y)$ itself is open, then Corollary 6.5 reduces to the original Browder theorem [3].

(2) Note that Browder's result is a reformulation of Fan's geometric lemma [4] in the form of a fixed point theorem and its proof was based on the Brouwer

fixed point theorem and the partition of unity argument. Since then it is known as the Fan-Browder fixed point theorem.

(3) Browder [3] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems.

## 7. The existence theorems of maximal elements

Any binary relation $R$ in a set $X$ can be regarded as a map $T : X \multimap X$ and conversely by the following obvious way:

$$y \in T(x) \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a map $T$ if $T(x_0) = \emptyset$.

The following shows that the (partial) KKM principle is equivalent to the non-existence of certain finite covers:

**Theorem 7.1.** *An abstract convex space* $(X, D; \Gamma)$ *satisfies the partial KKM principle iff for any maps* $S : X \multimap D,\ T : X \multimap X$ *satisfying*

(7.1.1) $S^-(z)$ *is open for each* $z \in D$*;*

(7.1.2) *for each* $x \in X,\ \mathrm{co}_\Gamma S(x) \subset T(x)$*; and*

(7.1.3) *for each* $x \in X,\ x \notin T(x)$*,*

$X$ *can not be covered by a finite number of* $S^-(z)$*'s,* $z \in D$*.*

*An abstract convex space* $(X, D; \Gamma)$ *is a KKM space iff the above condition also holds for any map* $S : X \multimap D$ *satisfying*

(7.1.1)$'$ $S^-(z)$ *is closed for each* $z \in D$

*instead of* (7.1.1)*.*

**Remark.** Theorems 3.1, 4.1, 5.1, 5.3, 6.1, and 7.1 are all characterizations of the KKM spaces. This means that there are no abstract convex spaces other than KKM spaces satisfying any of the properties of matching, intersection, geometric or section, Fan-Browder type fixed point, or maximal element. Similarly, Theorems 3.1, 4.1, 5.1, 5.3, 6.1, 7.1 and their Corollaries characterize abstract convex spaces satisfying the partial KKM principle.

From Theorem 7.1, we can deduce some results on maximal elements as follows:

**Corollary 7.2.** *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle and* $S : X \multimap D,\ T : X \multimap X$ *be two maps satisfying* (7.1.1)-(7.1.3)*. If*

(7.1.4) $X \setminus S^-(M)$ *is compact for some* $M \in \langle D \rangle$*,*

*then there exists an* $\widehat{x} \in X$ *such that* $S(\widehat{x}) = \emptyset$*.*

Corollary 6.4 is equivalent to the following simple consequence of Corollary 7.2:

**Corollary 7.3.** *Let $(X, D; \Gamma)$ satisfy the partial KKM principle, $X$ be compact, and $S : X \multimap D$ a map satisfying*
(1) $x \notin \mathrm{co}_\Gamma S(x)$ *for each* $x \in X$*; and*
(2) $S^-(z)$ *is open for each* $z \in D$.
*Then there exists an* $\widehat{x} \in X$ *such that* $S(\widehat{x}) = \emptyset$.

Corollary 7.3 is used by Borglin and Keiding [2] and Yannelis and Prabhakar [38] to the existence of maximal elements in mathematical economics.

Until recently, all of the conditions in Theorems 3.1, 4.1, 5.1, 5.3, 6.1, 7.1 and their Corollaries are known for $G$-convex spaces only.

## 8. The KKM type theorems

From the partial KKM principle, we have a whole intersection property:

**Theorem 8.1.** *Let $(X, D; \Gamma)$ satisfy the partial KKM principle and $G : D \multimap X$ be a closed-valued KKM map. If*
(8.1.1) $\bigcap_{z \in M} G(z)$ *is compact for some* $M \in \langle D \rangle$,
*then we have*

$$\bigcap_{z \in D} G(z) \neq \emptyset.$$

Note that Theorem 8.1 properly generalizes the KKM lemma due to Fan [4]. Corollary 3.2 can be stated in its contrapositive form and in terms of the complement $G(z)$ of $S(z)$ in $X$. Then we obtain Theorem 8.1. Conversely, we can deduce Corollary 3.2 from Theorem 8.1.

From Theorem 8.1, we can deduce the following:

**Theorem 8.2.** *Let $(X, D; \Gamma)$ satisfy the partial KKM principle, $K$ be a nonempty compact subset of $X$, and $G : D \multimap X$ a map satisfying*
(8.2.1) $\bigcap_{z \in D} G(z) = \bigcap_{z \in D} \overline{G(z)}$ [*that is, $G$ is transfer closed-valued*];
(8.2.2) $\overline{G}$ *is a KKM map; and*
(8.2.3) *either*
(i) $K := \bigcap \{\overline{G(z)} \mid z \in M\}$ *for some* $M \in \langle D \rangle$*; or*
(ii) *for each* $N \in \langle D \rangle$*, there exists a compact* $\Gamma$*-convex subset* $L_N$ *of* $X$ *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$K := L_N \cap \bigcap \{\overline{G(z)} \mid z \in D'\}.$$

*Then* $K \cap \bigcap \{G(z) \mid z \in D\} \neq \emptyset$.

From this theorem we can deduce its equivalent formulations in the following forms for abstract convex spaces satisfying the partial KKM principle:

Analytic alternatives (a basis of various equilibrium problems).
Fan type minimax inequalities.
Variational inequalities, and others.

Consequently, in [32], we obtained characterizations of abstract convex spaces satisfying the partial KKM principle consisting of 11 theorems and their corollaries; for characterizations for compact abstract convex spaces of the form $(X;\Gamma)$ satisfying the partial KKM principle, we deduced 15 theorems and their corollaries. Moreover, we noticed there that, for a compact $G$-convex space $(X;\Gamma)$, each of these 15 theorems and their corollaries is equivalent to the original KKM theorem.

Further applications of our theory on abstract convex spaces satisfying the partial KKM principle in the present paper are given in [31,33] as follows:

Best approximations.
The von Neumann type minimax theorem.
The von Neumann type intersection theorem.
The Nash type equilibrium theorem.
The Himmelberg fixed point theorem for KKM spaces.
Weakly KKM maps.

## 9. Fixed point theorems on abstract convex spaces

Once the author noted that the KKM theorem implies a large number of fixed point theorems on convex subsets of topological vector spaces or convex spaces. Similar arguments also work on the KKM spaces. Moreover, the unified fixed point theory in generalized convex uniform spaces in [28] can be extended to abstract convex uniform spaces or KKM uniform spaces in [33].

In this section, we introduce a typical fixed point theorem as a sample:

**Definition**. Let $(E,D;\Gamma;\mathcal{U})$ be an abstract convex uniform space where $(E,\mathcal{U})$ is a uniform space. A subset $K$ of $E$ is said to be *Klee approximable* if, for each entourage $U\in\mathcal{U}$, there exists a continuous function $h:K\to E$ satisfying conditions

(1) $(x,h(x))\in U$ for all $x\in K$;
(2) $h(K)\subset\Gamma_N$ for some $N\in\langle D\rangle$; and
(3) there exist continuous functions $p:K\to\Delta_n$ and $\phi_N:\Delta_n\to\Gamma_N$ with $|N|=n+1$ such that $h=\phi_N\circ p$.

Especially, for a subset $X$ of $E$, $K$ is said to be Klee approximable *into* $X$ whenever the range $h(K)\subset\Gamma_N\subset X$ for some $N\in\langle D\rangle$ in condition (2).

The following summarizes the mutual relations among the various subclasses of abstract convex uniform spaces; see [28,33]:

**Theorem 9.1.** *In the class of abstract convex uniform spaces* $(X, D; \Gamma; \mathcal{U})$*, the following hold:*

(1) *Any* $L\Gamma$*-space is of the Zima-Hadžić type.*

(2) *Every nonempty subset of an* $L\Gamma$*-space is locally* $\Gamma$*-convex whenever every singleton is* $\Gamma$*-convex.*

(3) *Any nonempty subset of a locally* $\Gamma$*-convex space is a* $\Phi$*-set.*

(4) *Any Zima-Hadžić type subset of an abstract convex uniform space such that every singleton is* $\Gamma$*-convex is a* $\Phi$*-set.*

(5) *Every* $G$*-convex* $\Phi$*-space is admissible. More generally, every nonempty compact* $\Phi$*-subset of a* $G$*-convex space is Klee approximable.*

**Definition**. Let $(E, D; \Gamma)$ be an abstract convex space, $X$ a nonempty subset of $E$, and $Y$ a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from $X$ into $Y$ as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that $\Gamma_N$ can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

This definition works for $G$-convex spaces or $\phi_A$-spaces.

The following is a new typical fixed point theorem containing a large number of particular cases in [28,33]:

**Theorem 9.2.** *Let* $(E, D; \Gamma; \mathcal{U})$ *be an abstract convex uniform space,* $X \subset Y$ *subsets of* $E$*, and* $F : Y \multimap Y$ *a map such that* $F|_X \in \mathfrak{B}(X, Y)$ *and* $F(X)$ *is Klee approximable into* $X$*. Then* $F$ *has the almost fixed point property.*

*Further if* $(E, \mathcal{U})$ *is Hausdorff,* $F$ *is closed, and* $\overline{F(X)}$ *is compact in* $Y$*, then* $F$ *has a fixed point* $x_0 \in Y$ *(that is,* $x_0 \in F(x_0)$*).*

Finally, recall that there are several hundred published works on the KKM theory and we can cover only an essential part of it. For the more historical background on the related fixed point theory, the reader can consult with [23]. For more involved or generalized versions of the results in this paper, see [20,22] for convex spaces, [12-15,34] for $H$-spaces, [23-25,28,35] for $G$-convex spaces, and [26,27,29-33] for abstract convex spaces, and references therein.

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