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REMARKS ON THE PARTIAL KKM PRINCIPLE

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Abstract. The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In this paper, we show that some previously given examples of spaces satisfying the partial KKM principle are also KKM spaces. Consequently, it is not known yet whether there is a space which is not a KKM space but satisfies the partial KKM principle. Moreover, we note that the KKM theory on hyperconvex metric spaces can be extended to much more general theory. Finally, we give a metatheorem on the properties of spaces satisfying the partial KKM principle.

1. Introduction

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem due to Knaster-Kuratowski-Mazurkiewicz [23]. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In our recent works [37-39], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to KKM spaces and spaces satisfying the partial KKM principle.

In this paper, we consider mutual relations between the KKM spaces and spaces satisfying the partial KKM principle. We show that some previously given examples of spaces satisfying only the partial KKM principle are also KKM spaces. One of such examples is the topological convexity due to Horvath [13]. Consequently, it is not known yet whether there is an abstract convex space which is not a KKM space but satisfies the partial KKM principle. Moreover, we note that the KKM theory on hyperconvex metric spaces can be extended to much more general theory. Finally, we give a metatheorem on the properties of spaces satisfying the partial KKM principle. This metatheorem unifies results previously given in many works on the KKM theory.

This paper is a supplement to [37-39].

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2. Abstract convex spaces and the KKM spaces

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Recall the following in [32-39]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_N \mid N \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example 2.1. The following are known examples of abstract convex spaces:

(1) The original KKM theorem [23] is for the triple $(\Delta_n, V; \mathrm{co})$, where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and $\mathrm{co} : \langle V\rangle \multimap \Delta_n$ the convex hull operation.

(2) A triple $(X, D; \Gamma)$, where X and D are subsets of a t.v.s. E such that $\mathrm{co}\, D \subset X$ and $\Gamma := \mathrm{co}$. Fan's celebrated KKM lemma [8] is for $(E, D; \mathrm{co})$, where D is a nonempty subset of E.

(3) A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$; see [24]. However he obtained several KKM type theorems w.r.t. $(X, D; \Gamma)$.

(4) A triple $(X, D; \Gamma)$ is called an H-*space* if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D\rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D\rangle$. If $D = X$, $(X; \Gamma)$ is called a C-space by Horvath [11] or an H-space by Bardaro and Ceppitelli in 1988.

(5) Hyperconvex metric spaces due to Aronszajn and Panitchpakdi are particular cases of C-spaces; see [12].

(6) Hyperbolic spaces due to Reich and Shafrir [43] are also particular cases of C-spaces. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic.

(7) Any topological semilattice $(X, \le)$ with path-connected interval introduced by Horvath and Llinares [14]; see also [25].

(8) A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$,

there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. Recall that all examples (1)-(7) are G-convex spaces. For details, see references of [26,29-31].

(9) A ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [35,36]. The so-called FC-spaces can be made into L-spaces. Later ϕ_A-spaces are called GFC-spaces in [18].

(10) Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_H(A)$, where $conv_H(A)$ is the intersection of all closed convex subsets of H that contain A; see Kirk and Panyanak [21]. Then $(H, X; \Gamma)$ is an abstract convex space.

(11) According to Horvath [13], a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.

(12) A $\mathbb{B}$-space due to Briec and Horvath [2] is an abstract convex space.

Note that each of (2)-(12) has a large number of concrete examples.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KO}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$]; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

Similarly, we might temporarily define that a *partial KKM space* is an abstract convex space satisfying the partial KKM principle.

In our recent works [37-39], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to partial KKM spaces.

Example 2.2. We give known examples of KKM spaces:

(1) Every G-convex space is a KKM space [30].

(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space [34].

(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$; see [34]. But L^* is not a G-convex space [38].

(4) For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X \rangle$; see [13].

(5) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space [2, Corollary 2.2].

Example 2.3. We give examples of partial KKM spaces:

(1) All KKM spaces.

(2) For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, $(X; \Gamma)$ is a partial KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$; see [13].

This example (2) was presented in some of our previous works. However, we show that (2) is also a KKM space in Section 4.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

It is not known yet whether there is a partial KKM space which is not a KKM space. This is why we named partial KKM spaces temporarily.

3. Further examples of KKM spaces

The following are given in [33, Theorem 4.2 and Corollary 4.3]:

Theorem 3.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, and $F : E \multimap Z$. Suppose that for any $A \in \langle D \rangle$, the set $\overline{F(\Gamma_A)}$ in its induced topology is a normal space. Then $F \in \mathfrak{KC}(E, Z)$ if and only if $F \in \mathfrak{KO}(E, Z)$.*

Corollary 3.2. *Let $(E, D; \Gamma)$ be an abstract convex space and Z a normal space. Then $\mathfrak{KO}(E, Z) \subset \mathfrak{KC}(E, Z)$.*

From Theorem 3.1, we have the following:

Theorem 3.3. *Let $(E, D; \Gamma)$ be a partial KKM space such that for any $A \in \langle D \rangle$, the set $\overline{\Gamma_A}$ in its induced topology is a normal space. Then $(E, D; \Gamma)$ is a KKM space.*

Proof. Put $E = Z$ and $F = 1_E$ in Theorem 3.1. Then $1_E \in \mathfrak{KC}(E, E)$ implies $1_E \in \mathfrak{KO}(E, E)$. Then the partial KKM space becomes a KKM space.

Corollary 3.4. *A partial KKM space $(E, D; \Gamma)$ is a KKM space if one of the following conditions holds:*

(1) *E is metrizable.*

(2) *E is Hausdorff and, for any $A \in \langle D \rangle$, the set Γ_A is compact.*

Proof. Each of (1) and (2) implies that, for any $A \in \langle D \rangle$, the set $\overline{\Gamma_A}$ in its induced topology is a normal space. Then Theorem 3.3 works.

Theorem 3.3 and Corollary 3.4 are useful to find KKM spaces systematically:

Example 3.1.

(1) Let $X := \Delta_n$ in the triple $(\Delta_n, V; \mathrm{co})$. The original KKM theorem [23] says that $1_X \in \mathfrak{KC}(X, X)$. Since each co$A$ is Hausdorff and compact, by Theorem 3.3, we have $1_X \in \mathfrak{KO}(X, X)$. This was known by Kim and Shih-Tan in 1987, independently, by different methods from ours.

(2) If D is a nonempty subset of a topological vector space E (not necessarily Hausdorff), Fan's KKM lemma [8] says that $1_E \in \mathfrak{KC}(E, E)$ for $(E, D; \mathrm{co})$. From Theorem 3.3, we have $1_E \in \mathfrak{KO}(E, E)$

(3) For a Lassonde type *convex space* $(X, D; \Gamma)$, it is known that $1_X \in \mathfrak{KC}(X, X)$ by Lassonde [24]. Since each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology, by Theorem 3.3, we have $1_X \in \mathfrak{KO}(X, X)$.

(4) For a triple $(H, X; \Gamma)$, where H is a hyperconvex metric space, $X \subset H$, and $\Gamma_A := \bigcap\{B \mid B \text{ is a closed ball containing } A\}$ for $A \in \langle X \rangle$, Khamsi [17] showed $1_H \in \mathfrak{KC}(H, H)$. Since H is a metric space, by Corollary 3.4(1), the triple is a KKM space. Horvath [12] showed that a hyperconvex metric space is an *l.c.* space, a particular type of H-space.

(5) Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and suppose $G : X \multimap H$ has nonempty closed values. Suppose also that for each finite $A \in \langle X \rangle$,

$$conv_H(A) \subset G(A).$$

Then $\{G(x)\}_{x \in X}$ has the finite intersection property; that is, $1_H \in \mathfrak{KC}(H, H)$ (Kirk and Panyanak [21]).

Note that, since H is a metric space, this also holds for open-valued map G by Corollary 3.4(1); that is, $1_H \in \mathfrak{KO}(H, H)$. Hence, $(H, X; conv_H)$ is a KKM space.

4. Topological convexities of Horvath

In this section, all spaces are assumed to be Hausdorff.

According to a recent work of Horvath [13], a convexity on a set X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions. Uniform convex spaces, that is uniform topological spaces for which the convex hull operator is a uniformly continuous map, are also introduced in [13]. Here convex hulls of finite sets is called a *polytope*.

With these new concepts, Horvath obtained remarkable generalizations of very important results on selections and fixed points as follows: Uniform convex spaces with homotopically trivial polytopes are absolute extensors for the class of metric spaces; if they are completely metrizable then a continuous selection theorem *à la*

Michael holds. Upper semicontinuous maps have approximate selections and fixed points, under the usual assumptions.

In the present section, we give new examples of KKM spaces by exploiting results from [13].

For definition of the Van de Vel property of a uniform convex space $(X, \mathcal{U}, \mathcal{C})$, see [13, Definition 2.3].

The following two definitions are given [13, Definitions 5.1 and 5.2]:

Definition. [13] Let X be a topological space endowed with a convexity $\mathcal{C}$; we say that $(X, \mathcal{C})$ has the weak Van de Vel property if polytopes are compact and are, with the induced convexity, uniform convex spaces with the Van de Vel property.

Definition. [13] An indexed family of subsets $\{A_0, \ldots, A_m\}$ of a set X is a KKM family with respect to a convexity $\mathcal{C}$ on X if there exists a finite set of points $\{p_0, \ldots, p_m\} \subset X$ such that for all set of indices $J \subset \{0, \ldots, m\}$ we have $[[\{p_j \mid j \in J\}]] \subset \bigcup_{j \in J} A_j$.

The convexity $\mathcal{C}$ is KKM w.r.t. a family $\mathcal{A}$ of subsets of X if for all indexed family $\{A_0, \ldots, A_m\} \subset \mathcal{A}$ which is KKM w.r.t. $\mathcal{C}$ we have $\bigcap_{i=1}^m A_i \neq \emptyset$.

Using such new concepts, Horvath deduced the following [13, Theorem 5.1]:

Theorem 4.1. [13] *If $(X, \mathcal{C})$ has the weak Van de Vel property then $\mathcal{C}$ is KKM w.r.t. the family of closed sets.*

Theorem 4.2. *If $(X, \mathcal{C})$ has the weak Van de Vel property then $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$.*

Proof. By Theorem 4.1, (X, Γ) is a partial KKM space. Note that each polytope $\Gamma_A := [[A]]$ is compact by the definition of the weak Van de Vel property. Therefore, by Corollary 3.4(2), the corresponding $(X; \Gamma)$ is a KKM space.

This shows that Example 2.3(2) is a KKM space.

Corollary 4.3. *If $(X, \mathcal{C})$ has the weak Van de Vel property and if polytopes are metrizable then $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$.*

This is a restatement of [13, Proposition 5.1].

Since we do not have any proper example of partial KKM spaces yet, in [37-39], such spaces are called abstract convex spaces satisfying the partial KKM principle.

5. Remarks on hyperconvex metric spaces

Let (M, d) be a metric space. Motivated by Khamsi [17] and others, for a bounded subset $A \subset M$, we set

$\mathrm{ad}(A) := \bigcap \{B \mid B \text{ is a closed ball such that } A \subset B\}$.

$\mathcal{A}(M) := \{A \subset M \mid A = \mathrm{ad}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.

For $x \in M$ and $\varepsilon > 0$, let

$$B(x, \varepsilon) := \{y \in M \mid d(x, y) \leq \varepsilon\}.$$

We introduce a new definition as in [40]:

Definition. An abstract convex space $(M, D; \Gamma)$ is called simply a *metric space* if (M, d) is a metric space, D is a nonempty set, and $\Gamma : \langle D \rangle \to \mathcal{A}(M)$ is a map having admissible values.

A Γ-convex subset of $(M \supset D; \Gamma)$ is said to be *subadmissible* by some authors.

Example 5.1. We give examples of metric spaces $(M, D; \Gamma)$ and KKM maps on them:

(1) $(M \supset X; \Gamma)$ where $\Gamma_A := \mathrm{ad}(A)$; see [17]. A map $G : X \multimap M$ is called a KKM map if $\Gamma_A \subset G(A)$ for each $A \in \langle X \rangle$.

(2) For each $A = \{a_0, a_1, \ldots, a_n\} \in \langle D \rangle$, choose a subset $B = \{x_0, x_1, \ldots, x_n\} \in \langle M \rangle$ and define $\Gamma_A := \mathrm{ad}(B)$. Then $(M, D; \Gamma)$ becomes a metric space. For this metric space, the so-called generalized $gKKM$ mapping in [6] simply becomes a KKM map.

Definition. A metric space (H, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

Results of Aronszajn and Panitchpakti [1, Theorem 1′] and Isbell [16, Theorem 1.1] are combined in the following:

Lemma 5.1. *A hyperconvex metric space is complete and (freely) contractible.*

The following is easy to prove:

Lemma 5.2. *An admissible subset of a hyperconvex metric space is hyperconvex.*

Definition. An abstract convex space $(H, D; \Gamma)$ is called simply a *hyperconvex metric space* if (H, d) is a hyperconvex metric space, D is a nonempty set, and $\Gamma : \langle D \rangle \to \mathcal{A}(H)$ is a map having admissible values such that

$$A, B \in \langle D \rangle, \ A \subset B \text{ implies } \Gamma_A \subset \Gamma_B.$$

There should be no confusion between a hyperconvex metric space $H = (H, d)$ and $(H, D; \Gamma)$.

Theorem 5.3. *Any hyperconvex metric space $(H, D; \Gamma)$ is a KKM space.*

Proof 1. For each $A \in \langle D \rangle$, Γ_A is hyperconvex by Lemma 5.2 and hence contractible by Lemma 5.1. Therefore, $(H, D; \Gamma)$ is an H-space and hence a G-convex space. It is well-known that any G-convex space is a KKM space.

Proof 2. Recall that $1_H \in \mathfrak{KC}(H, H)$ is well-known by Horvath [12] and subsequent works on the H-space theory. Moreover, $1_H \in \mathfrak{KC}(H, H)$ implies $1_H \in \mathfrak{KO}(H, H)$ by Theorem 3.1 or Corollary 3.2.

Example 5.2.

(1) As a consequence of Theorem 5.3, we obtain Khamsi's KKM theorem for a particular Γ and for particular KKM maps with finitely closed values; see [17]. In fact, by replacing the original topology of H by its finitely generated extension, we can eliminate "finitely".

(2) From Theorem 5.3, we obtain another particular forms in [4, Theorems 2 and 3].

Here we recall some known results on the original hyperconvex metric space (H, d) related to the KKM principle as follows:

Example 5.3.

(1) In [27], from the partial KKM principle for hyperconvex metric space due to Khamsi [17], the author deduced a Fan type matching theorem for open covers, a coincidence theorem, a Fan-Browder type fixed point theorem, a Brouwer-Schauder-Rothe type fixed point theorem, and other results for hyperconvex metric space.

(2) In [28], the author continued the study in [27] and obtain noncompact versions of the partial KKM principle, the matching theorem for open covers, the Fan-Browder type coincidence theorems and other results. Especially, we obtain the Schauder type fixed point theorems for compact maps on hyperconvex spaces and a generalization of a theorem due to Kirk [20] on the location of fixed point sets.

(3) In [22], its authors defined *finitely metrically closed sets* and *generalized metric KKM mapping* and gave a generalized metric KKM principle for hyperconvex metric spaces. This is a particular form of the partial KKM principle. From this principle, they deduced a Fan type minimax inequality, a Fan type best approximation theorem, a Fan-Browder type fixed point theorem, a Fan type geometric or section property, a maximal element theorem, a von Neumann type minimax theorem, a von Neumann type intersection theorem, and a Nash type equilibrium theorem for hyperconvex metric spaces. Note that these results were already known for more general H-spaces or G-convex spaces.

(4) In [46], it is shown that the above generalized metric KKM principle for finitely metrically closed-valued KKM map is equivalent to its "open"-valued version. This is an example of our Theorem 3.1.

(5) In [15], its authors repeated the above equivalency and reformed them to a routine Fan type matching theorem. This is applied to some fixed point theorems, intersection theorems, and best approximation theorems for hyperconvex metric spaces.

(6) In [44], its authors gave almost same contents of [15].

(7) In [42], its authors generalized and improved known fixed point theorems in hyperconvex metric spaces by applying the selection theorem of Ben-El-Mechaiekh and Oudadess and other results on C-spaces or G-convex spaces. In fact, they obtained generalizations of fixed point theorems for nonexpansive multimaps, the Kakutani type maps, and the Fan-Browder type maps. Some additional new observations were also stated for the Caristi-Kirk-Browder type theorems. In this paper, it was clearly noted that most works including [22] on the KKM theory for hyperconvex metric spaces are simple consequences of much more general results on C-spaces due to Horvath [12,13].

(8) In [47], its author gave a coincidence theorem, a minimax theorem, a section theorem, an intersection theorem and two existence theorems of solutions for generalized quasi-variational inequalities in hyperconvex metric spaces.

(9) In [45], its authors presented two continuous selection theorems in hyperconvex metric spaces and applied them to study fixed point and coincidence point problems as well as variational inequality problems in hyperconvex metric spaces.

(10) In [7], the partial KKM principle due to Khamsi [17] was applied to a Fan type minimax inequality, a best approximation theorem, and a von Neumann-Sion type minimax theorem for hyperconvex metric spaces. Modified versions of the Fan-Browder fixed point theorem and a coincidence theorem are added.

(11) In [5], from a well-known fact, its authors deduced a KKM type intersection property for a family of admissible subsets in a hyperconvex metric space and its routine consequences, that is, a KKM theorem, coincidence theorems, variational inequality theorems, and minimax inequality theorems. Note that these results are equivalent to each other in a wide sense for any abstract convex space.

There would be more examples of the same nature, but we will stop here. For other comments on related works, see [19,40].

6. A metatheorem on KKM spaces

In the KKM theory, it is routine to reformulate the (partial) KKM principle to the following equivalent forms; see [37,39].

Fan type matching property
Another intersection property
Geometric or section properties
The Fan-Browder type fixed point theorem
Existence theorem of maximal elements, and others

Any of these characterizes KKM spaces and partial KKM spaces; see [39].
Similarly we can deduce the following equivalent results for partial KKM spaces:

KKM type whole intersection property
Tarafdar type whole intersection property
Analytic alternatives (a basis of various equilibrium problems)
Fan type minimax inequalities
Variational inequalities, and others.

Consequently, for a compact partial KKM space $(X; \Gamma)$, we deduced 15 theorems from any of the characterizations of such spaces. Moreover, we noticed there that, for a compact G-convex space $(X; \Gamma)$, each of these 15 theorems and their corollaries is equivalent to the original KKM theorem.

Further applications of our theory on partial KKM spaces are given in [37-39] as follows:

Best approximations (under certain restrictions)
The von Neumann type minimax theorem
The von Neumann type intersection theorem
The Nash type equilibrium theorem
The Himmelberg type fixed point theorem for KKM spaces

Consequently, we have the following as is suggested in [37-39]:

Metatheorem 6.1. *For any partial KKM space, all theorems mentioned in this section hold.*

Details on this metatheorem are given in our works [39,41].

Example 5.1. (1) For any convex subset of a topological vector space, Metatheorem 6.1 was shown implicitly by Fan in a sequence his papers; see [29] and references therein.

(2) For G-convex spaces, Metatheorem 6.1 was shown by Park; see [29,30]. Therefore, it holds also Lassonde type convex spaces, Horvath's H-space, hyperconvex metric spaces, and others.

(3) Especially, for any topological semilattice $(X, \leq)$ with path-connected interval introduced by Horvath and Llinares [14], a part of Metatheorem 5.1 is shown by Luo [25].

(4) As we have seen in Section 5, most results in the KKM theory on hyperconvex metric spaces are consequences of Metatheorem 6.1.

(5) Cain and González [3] considered relationship among some subclasses of the class of G-convex spaces and introduced a subclass of L-spaces. González et al. [9] repeated to show that G-convex spaces and L-spaces satisfy the partial KKM principle. They added that L-spaces satisfy the properties of the Fan type minimax inequality, Fan-Browder type fixed point, and the Nash type equilibrium. All of such results are already known for more general G-convex spaces.

(6) For $\mathbb{B}$-spaces, Briec and Horvath [2] showed that some theorems mentioned in this section hold; that is, Fan-Browder fixed point theorem, Himmelberg type (in fact, Browder type and Kakutani type) fixed point theorems, Fan type minimax inequality, existence of Nash equilibria, and others.

(7) Recently, Khanh et al. [18] introduced GFC-spaces which are exactly same to our ϕ_A-spaces and proposed generalized KKM theorems and other related results. Most results are modifications of well-known ones.

(8) Some authors studied particular types of KKM spaces and deduced the Nash type equilibrium theorem from the partial KKM principle; for example, [2,9,25,31], explicitly, and many more implicitly. This deduction is a part of Metatheorem 6.1 and can be phrased as "From the KKM principle to the Nash equilibria". In fact, in our forthcoming work [41], we clearly show that a sequence of statements from the partial KKM principle to the Nash equilibria can be obtained for any partial KKM space.

(9) The so-called FC-spaces are G-convex spaces and hence KKM spaces; see [36]. Therefore all results in Section 6 are already well-known for FC-spaces in more general form. In 2009, the authors of [10] showed some of them for FC-spaces.

Remarks. (1) Since we do not know the proper existence of any partial KKM space which is not a KKM space, we are still not easy to use the term.

(2) Finally, for the more historical backgrounds or details on the results in this paper, see the references of [29,37-39] and the literature therein.

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