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A UNIFIED APPROACH TO $\mathfrak{KC}$-MAPS IN THE KKM THEORY

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Abstract. In the KKM theory on abstract convex spaces, there have appeared multimap classes $\mathfrak{A}_c^\kappa$, KKM, S-KKM, s-KKM, $\mathfrak{B}$, $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$. In this paper, we review mutual relations among them. We show that the multimap class S-KKM is included in the class $\mathfrak{KC}$. We also show that most of known fixed point theorems on s-KKM maps follow from the corresponding ones on $\mathfrak{B}$-maps. Consequently, we can unify all the classes KKM, S-KKM and s-KKM to $\mathfrak{KC}$-maps, and the KKM theory can be upgraded.

1. Introduction

The KKM theory is the study on applications of equivalent formulations or generalizations of the classical KKM theorem. At the beginning, the theory was mainly concerned with convex subsets of topological vector spaces as can be seen in the works of Fan; see [22]. Later, it has been extended to convex spaces by Lassonde [14], and to C-spaces (or H-spaces) by Horvath [7,8]. This line of generalizations of earlier works was followed by the author for generalized convex spaces or G-convex spaces; see [22]. Moreover, there have appeared several variations of such spaces; see [33-36]. Recently, in [28-31,33-35,38-41], the author introduced the concepts of abstract convex spaces and KKM spaces, which seem to be more adequate to establish the KKM theory for various purposes.

Early in 1993, the author [17] introduced the admissible class $\mathfrak{A}_c^\kappa(X, Y)$ of multimaps $X \multimap Y$ between topological spaces and gave lots of examples of multimaps belonging to the class. In the same year, the author initiated the study of generalized (or G-) convex spaces and various subclasses of the admissible class. In 1994, the author [18] showed that fundamental results in the KKM theory can be obtained in generalized forms related to $\mathfrak{A}_c^\kappa(X, Y)$ and that this class has the KKM property when X is a convex space and Y is a Hausdorff space. Since then there have appeared more general multimap classes KKM, the better admissible class $\mathfrak{B}$, S-KKM, s-KKM, $\mathfrak{KC}$, and $\mathfrak{KO}$; see the references in the end of this paper.

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It is known that the multimap with KKM property has S-KKM property. Most studies of the KKM class are generalized to those of s-KKM class with a surjective single-valued map $s : D \to X$; see Chang et al. [3-6], Kuo et al. [12,13], Agarwal and O'Regan [1], and Shahzad [43]. In 2005, H. Kim [10] showed that KKM class is equivalent to s-KKM class with a surjective single-valued map s. Using this, she unified the results about these classes and the class $\mathfrak{B}$.

In the present paper, we investigate mutual relations among the various multimap classes in abstract convex spaces. We show that the class S-KKM is included in the class $\mathfrak{KC}$, and that most of known fixed point theorems on s-KKM maps follows from the corresponding ones on $\mathfrak{B}$-maps. Consequently, we can destroy the classes KKM, S-KKM and s-KKM from the KKM theory.

2. Abstract convex spaces

In this preliminary section, we follow mainly our previous work [41].

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Recall the following in [28-31,37-41]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. The following are typical examples of abstract convex spaces. Others can be seen in [41].

(1) A *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a topological space E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see [44], where the bibliography lists 283 papers. For any subset $X \subset E$, its $\mathcal{C}$-*convex hull* is defined and denoted by $\mathrm{Co}_\mathcal{C} X := \bigcap \{Y \in \mathcal{C} \mid X \subset Y\}$. We say that X is $\mathcal{C}$-*convex* if $X = \mathrm{Co}_\mathcal{C} X$. Now we can consider the map $\Gamma : \langle E \rangle \multimap E$ given by $\Gamma_A := \mathrm{Co}_\mathcal{C} A$ for each $A \in \langle E \rangle$. Then $(E, \mathcal{C})$ becomes our abstract convex space $(E; \Gamma)$.

(2) A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$; see [14]. However he obtained several KKM type theorems w.r.t. $(X, D; \Gamma)$.

(3) A triple $(X, D; \Gamma)$ is called an H-*space* by Park if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally,

ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma)$ is called a C-space by Horvath [7,8].

(4) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. Recall that examples (2) and (3) are G-convex spaces. For details, see references of [22-27].

(5) A *ϕ_A-space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [33-36].

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } \; A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A map $F : E \multimap Z$ is called a *$\mathfrak{KC}$-map* [resp., a *$\mathfrak{KO}$-map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp, $F \in \mathfrak{KO}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)]$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

Examples of $\mathfrak{KC}$-maps, $\mathfrak{KO}$-maps, and KKM spaces are given in [31,38-42]. Especially, every G-convex space is a KKM space.

In our recent works [38-40], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space

$\Longrightarrow$ Space satisfying the partial KKM principle

$\Longrightarrow$ Abstract convex space.

It is not known yet whether there is a space satisfying the partial KKM principle which is not a KKM space.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space, X a nonempty subset of E, and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N)^p \longrightarrow \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

This concept extends the corresponding one for G-convex spaces appeared in [27], where lots of examples are given. The above definition also works for ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$.

3. The rise and fall of S-KKM maps

In this section, we show that all of the so-called S-KKM maps can be made into $\mathfrak{KC}$-maps on certain abstract convex spaces.

In early 1990's, the author introduced the admissible class $\mathfrak{A}_c^\kappa(X, Y)$ of multimaps $X \multimap Y$ between topological spaces. In 1994, the author [18] showed that fundamental results in the KKM theory for convex spaces can be obtained in far-reaching generalized forms related to $\mathfrak{A}_c^\kappa(X, Y)$ and that this class has the KKM property when X is a convex space and Y is a Hausdorff space as follows:

Proposition 3.1. [18, Corollary 2] *Let (X, D) be a convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, and $G : D \multimap Y$ such that, for any $N \in \langle D\rangle$, $F(\mathrm{co}\, N) \subset G(N)$. Then the family $\{\overline{Gx} \mid x \in D\}$ has the finite intersection property.*

Motivated by this, Chang and Yen [5] in 1996 defined the KKM class of maps on convex subsets of vector spaces as follows:

Definition. [5] Let X be a convex subset of a vector space and Y a topological space. A multimap $F : X \multimap Y$ is said to have *the KKM property* if, for any map $G : X \multimap Y$ satisfying

$$F(\mathrm{co}\, N) \subset G(N) \qquad \text{for all} \quad N \in \langle X\rangle,$$

the family $\{\overline{Gx}\}_{x\in X}$ has the finite intersection property. Let

$$\mathrm{KKM}(X, Y) := \{F : X \multimap Y \mid F \text{ has the KKM property}\}.$$

Note that, in our terminology in Section 2, *for the convex space $(X; \mathrm{co})$, this implies $F \in \mathfrak{KC}(X, Y)$.*

Naturally, their KKM class contains $\mathfrak{A}_c^\kappa$-class from convex spaces into Hausdorff spaces, but significant proper examples of the former not in the latter were hard to find.

In the same year, Chang and Yen [6] introduced the concept of S-KKM map. Since then, a number of authors followed their way and tried to rewrite many known results in the KKM theory using this concept or its modifications like S-KKM class.

On the other hand, in 1997, the author [19] extended the $\mathfrak{A}_c^\kappa$-class to the 'better' admissible $\mathfrak{B}$-class on convex spaces, supplied a large number of examples, and showed that, *in the class of compact closed multimaps from convex spaces into Hausdorff spaces, two subclasses* $\mathfrak{B}$ *and* $\mathfrak{KC}$ *coincide* [19].

For G-convex spaces, such multimap classes are extended and investigated by a number of authors. For references, see [27].

In 1998, the S-KKM map in [6] was generalized as follows:

Definition. [15] Let X be a nonempty set, $(Y;\Gamma)$ be a G-convex space, $S,T : X \multimap Y$. Then T is a *generalized* S-KKM *map* if, for each $A \in \langle X\rangle$, $\mathrm{co}_\Gamma S(A) \subset T(A)$.

When $X = Y$ is a convex subset of a vector space, if we let $\mathrm{co}_\Gamma(A)$=co A for any $A \in \langle X\rangle$, then T is an S-KKM map, implying T is a generalized S-KKM map. However, note that, *for the H-space* $(Y,X;\Lambda)$, $T : X \multimap Y$ *becomes simply a KKM map, where* $\Lambda_A :=\mathrm{co}_\Gamma S(A)$ *for each* $A \in \langle X\rangle$.

The so-called S-KKM maps in [6] was extended in 1999 [4] as follows:

Definition. [4] Let X be a nonempty set, Y a nonempty convex set of a vector space, and Z a topological space. If $S : X \multimap Y$, $F : Y \multimap Z$, and $G : X \multimap Z$ are three multimaps satisfying

$$F(\mathrm{co}\, S(A)) \subset G(A)$$

for any $A \in \langle X\rangle$, then G is called a generalized S-KKM map with respect to F. If the map F satisfies the requirement that for any generalized S-KKM map G with respect to F the family $\{\overline{Gx} \mid x \in X\}$ has the finite intersection property, then F is said to have the S-KKM property. The class S-KKM(X,Y,Z) is defined to be the set $\{F : Y \multimap Z \mid F$ has the S-KKM property$\}$.

In the case where $X = Y$ and S is the identity function 1_X, then S-KKM(X,Y,Z) = KKM(X,Z).

Note that, by letting $\Gamma_A :=$ co $S(A)$ for any $A \in \langle X\rangle$, *we have an abstract convex space* $(Y,X;\Gamma)$, *for which* G *is simply KKM with respect to*

$$F \in \mathfrak{KC}(X,Z) := S\text{-KKM}(X,Y,Z).$$

For a single-valued map $s : X \to Y$, we can consider a class s-KKM(X,Y,Z).

In 2000, Huang and Jeng [9] obtained a fixed point result for S-KKM maps and corrected a result due to Park [20].

In 2001, Chang et al. [3] followed Park's previous works and claimed to extend one of Park's theorem to s-KKM maps. Later it was known that their new result is a simple reformulation of known one.

For G-convex spaces, such multimap classes are extended and investigated by a number of authors. In 2003, the author [25] introduced the classes $\mathfrak{K}$, $\mathfrak{KC}$ and $\mathfrak{KO}$ of multimaps from a G-convex space into a topological space. There the KKM property is generalized to G-convex spaces as follows:

Definition. [25] Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. A multimap $F : X \multimap Y$ is said to have *the KKM property* if, for any map $G : D \multimap Y$ satisfying

$$F(\Gamma_N) \subset G(N) \qquad \text{for all} \quad N \in \langle D \rangle,$$

the family $\{G(x)\}_{x \in D}$ has the finite intersection property. We denote $\mathfrak{K}(X, Y) := \{F : X \multimap Y \mid F \text{ has the KKM property}\}$. Let $\mathfrak{KC}$ denote the class $\mathfrak{K}$ for closed-valued maps G, and $\mathfrak{KO}$ for open-valued maps G.

This is followed by H. Kim and the author [11].

In 2005, Kim [10] showed that the S-KKM property can be extended to G-convex spaces as follows:

Definition. [10] If a map $S : D \multimap X$ and multimaps $F : X \multimap Y$ and $G : D \multimap Y$ are satisfying

$$F(\Gamma_{S(A)}) \subset G(A) \qquad \text{for all} \quad A \in \langle D \rangle,$$

the family $\{\overline{G(x)}\}_{x \in D}$ has the finite intersection property. The class S-KKM(D, X, Y) is defined to be the set $\{F : X \multimap Y \mid F \text{ has the } S\text{-KKM property}\}$.

Note that, *a map $F : X \multimap Y$ having the S-KKM property has the KKM property for the abstract convex space $(X, D; \Lambda)$, where $\Lambda_A :=\mathrm{co}_\Gamma S(A)$ for each $A \in \langle X \rangle$. Note that $(X, D; \Lambda)$ is a G-convex space if S is a single-valued function s.*

Moreover, Kim [10] showed that two classes KKM and s-KKM of multimaps from a convex space into a topological space are identical whenever s is surjective [this is the only case S-KKM is meaningful].

In 2007, imitating the original definition of S-KKM maps of Chang et al. [3], Amini et al. [2] defined the S-KKM class for a classical convexity space $(X, \mathcal{C})$ with a nonempty set Z and a topological space Y as follows:

Definition. [2] If $S : Z \multimap X$, $F : X \multimap Y$, and $G : Z \multimap Y$ are three multimaps satisfying

$$F(\mathrm{Co}_\mathcal{C}(S(A)) \subset G(A) \quad \text{for each} \quad A \in \langle Z \rangle,$$

then G is called a $\mathcal{C}$-S-KKM map with respect to F. If the map $F : X \multimap Y$ satisfies the requirement that for any $\mathcal{C}$-S-KKM map G with respect to F, the family $\{\overline{G(z)} \mid z \in Z\}$ has the finite intersection property, then F is said to have the S-KKM property with respect to $\mathcal{C}$. Let

$$S\text{-KKM}_\mathcal{C}(Z, X, Y) := \{F : X \multimap Y \mid F \text{ has the } S\text{-KKM property with respect to } \mathcal{C}\}.$$

It should be noted that, *by putting $\Gamma_A := \mathrm{Co}_\mathcal{C}(S(A))$ for each $A \in \langle Z \rangle$, we have an abstract convex space $(X, Z; \Gamma)$ and S-KKM$_\mathcal{C}(Z, X, Y)$ becomes simply $\mathfrak{KC}(X, Y)$.*

In 2006, Kuo et al. [12] stated a coincidence theorem in S-KKM class and some corollaries.

In 2007, Kuo et al. [13] introduced the strict KKM property of multimaps in generalized convex spaces and investigated the fixed point problem for multimaps having this property on almost Γ-convex subsets of locally G-convex uniform spaces.

Recently, in [27], we observed that any S-KKM class on abstract convex spaces is included in the $\mathfrak{KC}$ class. Therefore, it seems to be the proper time to eliminate such S-KKM class.

The aim of introducing such S-KKM class was to properly extend known fixed point theorems (mainly due to the present author) to maps in such class. However, the map classes adequate to establish such theorems are known to belong to the 'better' admissible class $\mathfrak{B}$, and actually the so-called S-KKM classes are not adequate and have no practical examples; see [26,30,31].

The following is known [38], where $\mathbb{C}$ is the class of continuous functions:

Proposition 3.2. [38, Lemma 6] *Let* $(E, D; \Gamma)$ *be a* G*-convex space and* Z *a topological space. Then*
(1) $\mathbb{C}(E, Z) \subset \mathfrak{A}_c^\kappa(E, Z) \subset \mathfrak{B}(E, Z)$;
(2) $\mathbb{C}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z)$; *and*
(3) [11] $\mathfrak{A}_c^\kappa(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z)$ *if* Z *is Hausdorff.*

Consider the following condition for a G-convex space $(E \supset D; \Gamma)$:

$(*)$ $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$; and, for each $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that $\phi_N(\Delta_n) = \Gamma_N$ and that $J \in \langle N \rangle$ implies $\phi_N(\Delta_J) = \Gamma_J$.

Note that every convex space satisfies condition $(*)$.

Theorem 3.3. [38, Theorem 16] *Let* $(E, D; \Gamma)$ *be a* G*-convex space and* Z *a topological space.*
(1) *If* Z *is a Hausdorff space, then every compact map* $F \in \mathfrak{B}(E, Z)$ *belongs to* $\mathfrak{KC}(E, Z)$.
(2) *If* $F : E \multimap Z$ *is a closed map such that* $F\phi_N \in \mathfrak{KC}(\Delta_n, Z)$ *for any* $N \in \langle D \rangle$ *with the cardinality* $|N| = n + 1$*, then* $F \in \mathfrak{B}(E, Z)$.
(3) *In the class of closed maps defined on a* G*-convex space* $(E \supset D; \Gamma)$ *satisfying condition* $(*)$ *into a space* Z*, a map* $F \in \mathfrak{KC}(E, Z)$ *belongs to* $\mathfrak{B}(E, Z)$.

Remark. In (2), note that for any map $F \in \mathfrak{A}_c^\kappa(E, Z)$, we have $F\phi_N \in \mathfrak{A}_c^\kappa(\Delta_n, Z) \subset \mathfrak{KC}(\Delta_n, Z) \cap \mathfrak{KO}(\Delta_n, Z)$ when Z is Hausdorff; see [11].

Corollary 3.4. [38, Corollary 16.1] *In the class of compact closed maps defined on a* G*-convex space* $(E \supset D; \Gamma)$ *satisfying condition* $(*)$ *into a Hausdorff space* Z*, two subclasses* $\mathfrak{KC}(E, Z)$ *and* $\mathfrak{B}(E, Z)$ *are identical.*

Corollary 3.5. [38, Corollary 16.2] *In the class of compact closed maps defined on a convex space* (X, D) *into a Hausdorff space* Z*, two subclasses* $\mathfrak{KC}(X, Z)$ *and* $\mathfrak{B}(X, Z)$ *are identical.*

Remarks. 1. This is noted in [19] by a different method. In view of Corollary 3.5, the class $\mathfrak{B}$ is favorable to use for convex spaces since it has already plenty of examples and is much easy to find examples.
2. Proposition 3.2, Theorem 3.3, Corollaries 3.4 and 3.5 hold also for ϕ_A-spaces $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D \rangle$ with $|A| = n + 1$.

Corollary 3.6. *Let X be a subset of a Hausdorff t.v.s., I a nonempty set, $s : I \to X$ a map such that* $\operatorname{co} s(I) \subset X$, *and* $T \in s$*-KKM*(I, X, X). *If T is closed and compact, then* $T \in \mathfrak{B}(X, X)$.

Proof. Note that $(X, s(I))$ is a convex space and the class s-KKM(I, X, X) is $\mathfrak{KC}(X, X)$. The conclusion follows from Corollary 3.5.

In 2004, the author [26] showed that a compact closed s-KKM map from a convex subset of a topological vector space into itself belongs to $\mathfrak{B}$ whenever $s : I \to X$ is a surjection.

Corollary 3.7. *Let X be a subset of a Hausdorff t.v.s., I a nonempty set, $s : I \to X$ a map such that* $\operatorname{co} s(I) \subset X$, *and Y a Hausdorff space. Then, in the class of closed compact maps, four classes* $\mathfrak{KC}(X, Y)$, KKM(X, Y), s-KKM(I, X, Y), *and* $\mathfrak{B}(X, Y)$ *coincide.*

Proof. For the convex space $(X, s(I))$, we have $\mathfrak{KC}(X, Y) = \mathrm{KKM}(X, Y) = \mathfrak{B}(X, Y)$ by Theorem 3.3(1) and (3). Note that KKM$(X, Y) = s$-KKM(I, X, Y) by following the proof of [10, Proposition 2.3].

In view of Corollary 3.7, all fixed point theorems on s-KKM maps on t.v.s. are consequences of corresponding ones on $\mathfrak{B}$-maps.

Since we introduced the multimap class $\mathfrak{A}_c^\kappa$, $\mathfrak{B}$, $\mathfrak{KC}$, and $\mathfrak{KD}$, many authors or printers mistook $\mathfrak{A}$ for $\mathcal{U}$ or $\mathbb{U}$, $\mathfrak{B}$ for $\mathcal{B}$ or $\mathbb{B}$, and $\mathfrak{KD}$ for $\mathcal{KD}$. *The author cordially asks his followers to keep the original notations.*

4. Admissible abstract convex spaces

In our work [42], we obtained very general fixed point theorems and their various particular cases. In this section, we introduce the main results of [42].

Definition. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is an abstract convex space such that $(E, \mathcal{U})$ is a uniform space with a basis $\mathcal{U}$ of the uniformity consisting of symmetric entourages. For each $U \in \mathcal{U}$, let

$$U[x] = \{x' \in X \mid (x, x') \in U\}$$

be the *U-ball* around a given element $x \in E$.

We introduce particular types of subsets of abstract convex uniform spaces adequate to establish our fixed point theory:

Definition. For an abstract convex uniform space $(E, D; \Gamma; \mathcal{U})$, a subset X of E is said to be *admissible* (in the sense of Klee) if, for each nonempty compact subset K of X and for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to X$ satisfying

(1) $(x, h(x)) \in U$ for all $x \in K$;

(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$; and

(3) there exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ with $|N| = n + 1$ such that $h = \phi_N \circ p$.

Example. A nonempty subset X of a t.v.s. E is said to be admissible (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous function $h : K \to X$

such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

For more general purposes, we introduce a generalized version of our previous definition of the admissibility of domains of maps by switching it to the Klee approximability of their ranges, as follows:

Definition. Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to E$ satisfying conditions (1)-(3) in the preceding definition. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K) \subset \Gamma_N \subset X$ for some $N \in \langle D \rangle$ in condition (2).

Examples of Klee approximable sets were given in [42].

The following in [42] summarizes the mutual relations among the various subclasses of abstract convex uniform spaces:

Theorem 4.1. *In the class of abstract convex uniform spaces* $(X, D; \Gamma; \mathcal{U})$, *the following hold:*

(1) *Any* $L\Gamma$*-space is of the Zima-Hadžić type.*

(2) *Every nonempty subset of an* $L\Gamma$*-space is locally* Γ*-convex whenever every singleton is* Γ*-convex.*

(3) *Any nonempty subset of a locally* Γ*-convex space is a* Φ*-set.*

(4) *Any Zima-Hadžić type subset of an abstract convex uniform space such that every singleton is* Γ*-convex is a* Φ*-set.*

(5) *Every* G*-convex* Φ*-space is admissible. More generally, every nonempty compact* Φ*-subset of a* G*-convex space is Klee approximable.*

We have the following main result in [42]:

Theorem 4.2. *Let* $(E, D; \Gamma; \mathcal{U})$ *be an abstract convex uniform space,* $X \subset Y$ *subsets of* E, *and* $F : Y \multimap Y$ *a map such that* $F|_X \in \mathfrak{B}(X, Y)$ *and* $F(X)$ *is Klee approximable into X. Then F has the almost fixed point property (that is, for any* $V \in \mathcal{U}$, $F|_X$ *has a* V*-fixed point* $x_V \in X$ *satisfying* $F(x_V) \cap V[x_V] \neq \emptyset$).

Further if $(E, \mathcal{U})$ *is Hausdorff,* F *is closed, and* $\overline{F(X)}$ *is compact in* Y, *then* F *has a fixed point* $x_0 \in Y$ (*that is,* $x_0 \in F(x_0)$).

We immediately have the following:

Corollary 4.3. *Let* $(E \supset D; \Gamma; \mathcal{U})$ *be an abstract convex Hausdorff uniform space, and* X *an admissible* Γ*-convex subset of* E. *Then every closed compact map* $F \in \mathfrak{B}(X, X)$ *has a fixed point.*

Corollary 4.4. *Let* $(E, D; \Gamma; \mathcal{U})$ *be a locally* Γ*-convex Hasdorff uniform space and* X *a* Γ*-convex subset of* E. *Then every closed compact map* $F \in \mathfrak{B}(X, X)$ *has a fixed point.*

5. Various fixed point theorems for $\mathfrak{B}$-maps

In [27,42], we gave some of our previous results which are direct consequences of Theorem 4.2.

In this section, we give some history of analytical fixed point theory related to $\mathfrak{B}$-maps. For simplicity, all topological spaces are assumed to be Hausdorff unless explicitly stated otherwise.

Since 1992, we have introduced and supplied a lot of examples of the map class $\mathfrak{A}_c^\kappa$ in [18,19,38]. Up to now, many authors used the class $\mathfrak{A}_c^\kappa$, but no one could find any new example of maps in that class.

In 1993 [17] and 1994 [18], we obtained the following with different methods:

(I) *Let X be a compact convex subset of a t.v.s. E on which its dual E^* separates points. Then any map $F \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

(II) *Let X be a convex subset of a locally convex t.v.s. E and $F \in \mathfrak{A}_c^\sigma(X,X)$. If F is compact, then it has a fixed point.*

In 1996, Chang and Yen [5] introduced the KKM class (see Section 3) and obtained the following:

(III) [5] *Let X be a convex subset of a locally convex t.v.s. E. Then every compact closed map $T \in \mathrm{KKM}(X,X)$ has a fixed point.*

In 1997, the author introduced the 'better' admissible class $\mathfrak{B}$ of multimaps. We noticed that $\mathfrak{A}_c^\kappa \subset \mathfrak{B}$ and that, in the class of compact closed maps, two subclasses $\mathfrak{B}$ and KKM coincide. Recall that some variants of (III) were given in [19]. One of the simplest results is the following restatement of (III):

(IV) [19] *Let X be a convex subset of a locally convex t.v.s. E. Then every compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

Note that (I)-(IV) follow from Corollary 4.4.

In 1997, we obtained the following consequence of Corollary 4.3:

(V) [21] *Let E be a t.v.s. and X an admissible* (*in the sense of Klee*) *convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

In 1998, this was restated as follows:

(VI) [20] *Let X be an admissible convex subset of a t.v.s. Then any compact closed map $T \in \mathfrak{KC}(X,X)$ has a fixed point.*

Moreover, in [21], we listed more than sixty papers in chronological order, from which we could deduce particular forms of (V) and (VI). In 1999, Lin and Yu [16] showed that (VI) holds.

Since then formal generalizations of KKM class to S-KKM or s-KKM classes followed. One of the main targets of such works was to try to generalize (III) and (VI).

In fact, studies on KKM class are generalized to those on s-KKM class with a surjective map $s : I \to X$; see Chang et al. [3,6]. And new results on s-KKM maps assumed also that $s : I \to X$ is surjective, see also Agarwal and O'Regan [1], Amini et al. [2], Huang and Jeng [9], Kuo et al. [11,12], and Shahzad [43]. But we showed that if T has the s-KKM property for a surjective map s, then T has the KKM property. Therefore (V) and (VI) are equivalent.

Moreover, in 2004 , we obtained the following for the s-KKM class:

(VII) [26] *Let X be a convex subset of a t.v.s., I a nonempty set, $s : I \to X$ a surjection, and $T \in s$-KKM(I, X, X). If T is closed and compact, then $T \in \mathfrak{B}(X, X)$.*

From (VII) and (V), we have the following:

(VIII) [26] *Let E be a t.v.s. and X an admissible convex subset of E, I a nonempty set, $s : I \to X$ a surjection, and $T \in s$-KKM(I, X, X). If T is closed and compact, then T has a fixed point.*

Note that if I is a nonempty subset of X, then (VIII) reduces to Chang, Huang, and Jeng [3, Theorem 3.1].

It should be noticed that the main fixed point theorems in [3,5,16] and others are disguised forms of our (V). Most of other results in those papers are also formally generalized (but not practical) or disguised forms of earlier works of the author on the classes $\mathfrak{A}_c^\kappa$ or $\mathfrak{B}$ of multimaps.

Theorem 4.2 for $X = Y$ reduces the following form of the main theorem of [26] in 2004:

(IX) [26] *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

For $X = Y = E$, Theorem 4.2 reduces to the following main theorem of [27] in 2007:

(X) [27] *Let $(X, D; \Gamma; \mathcal{U})$ be a G-convex uniform space and $F \in \mathfrak{B}(X, X)$ a multimap such that $F(X)$ is Klee approximable. Then F has the almost fixed point property.*

Further if F is closed and compact, then F has a fixed point $x_0 \in X$ (that is, $x_0 \in F(x_0)$).

This theorem contains a large number of known results on t.v.s. or on various subclasses of the class of admissible G-convex spaces. Such subclasses are those of admissible t.v.s., Φ-spaces, sets of the Zima-Hadžić type, locally G-convex spaces, and LG-spaces; see [27]. Mutual relations among those subclasses and some related results on approximable maps, Kakutani maps, acyclic maps, Φ-maps, and others are investigated in [27].

The following two results are simple consequences of (X):

(XI) *Let $(X, D; \Gamma; \mathcal{U})$ be an admissible G-convex space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

(XII) *Let $(X, D; \Gamma; \mathcal{U})$ be a compact admissible G-convex space. Then any map $F \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

The following is a consequence of Theorem 4.2:

(XIII) *Let X and Y be subsets of a t.v.s. E such that $X \subset Y$ and $F : Y \multimap Y$ a map.*

(1) *If* $F|_X \in \mathfrak{B}(X, Y)$ *and* $F(X)$ *is Klee approximable into* X*, then* $F|_X$ *has the almost fixed point property* (*that is, for any* $V \in \mathcal{V}$*,* $F|_X$ *has a* V*-fixed point* $x_V \in X$ *satisfying* $F(x_V) \cap (x_V + V) \neq \emptyset$).

(2) *Further if* F *is closed and* $F|_X$ *is compact, then* F *has a fixed point.*

Note that, in (1), E is not necessarily Hausdorff and that (XIII) would be better than [32, Theorem 2.2], where it should be $\mathfrak{B} = \mathfrak{B}^p$.

6. Some related results

In 2004, we obtained the following:

Theorem 6.1. [26] *Let* X *be a compact convex subset of a t.v.s.* E*,* I *a nonempty set,* $s : I \to X$*, and* $T \in s$-KKM(I, X, X) *a closed-valued map such that* $T(X) \cap s(I)$ *is dense in* $T(X)$*. If* T *satisfies condition*

$$\bigcap_{U \in \mathcal{V}} \{x \in X \mid x \in T(x) + U\} = \bigcap_{U \in \mathcal{V}} \mathrm{cl}\{x \in X \mid x \in T(x) + \mathrm{co}\, U\},$$

then T *has a fixed point.*

Note that if $X = I$ and $\overline{T(X)} \subset s(I)$, then Theorem 6.1 reduces to Huang and Jeng [9, Theorem 2.2], and if $X = I$ and $s = 1_X$, then Theorem 6.1 originates from Park [20, Theorem 1]. If $T : X \multimap X$ is upper semicontinuous with convex values, then $T \in$ KKM$(X, X) \subset s$-KKM(X, X, X); see Huang and Jeng [9]. Therefore, Theorem 6.1 also generalizes Huang and Jeng [9, Corollary 2.6].

From Theorem 6.1 and [26, Corollary 2.4], we have the following:

Theorem 6.2. [26] *Let* X *be a convex subset of a locally convex t.v.s.,* I *a nonempty set,* $s : I \to X$*, and* $T \in s$-KKM(I, X, X) *a compact closed map such that* $T(X) \cap s(I)$ *is dense in* $T(X)$*. Then* T *has a fixed point.*

Note that the main results in [4,5,9] are generalized and unified by Theorem 6.2. In fact, if $X = I$ and $\overline{T(X)} \subset s(I)$, then Theorem 6.2 reduces to Chang et al. [4, Theorem 3.2] and further if $s = 1_X$, then to Chang and Yen [5, Theorem 2], and if X itself is compact, then to Huang and Jeng [9, Corollary 2.4].

Finally, in view of Corollary 3.7, $T \in s$-KKM(I, X, X) in Theorems 6.1 and 6.2 can be replaced by $T \in \mathfrak{B}(X, X)$.

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