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Generalized convex spaces, L-spaces, and FC-spaces

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Abstract We show that FC-spaces due to Ding are particular types of L-spaces due to Ben-El-Mechaiekh et al., and hence particular types of G-convex spaces. Some counter-examples are given and related matters are also discussed.

Keywords Multimap (map) · Generalized (G-) convex space · KKM principle · L-spaces · FC-spaces · Property (H)

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Since the concept of generalized convex spaces (simply, G-convex spaces) in the KKM theory appeared in 1993 [35–37], a number of modifications or imitations have followed. Such examples are L-spaces due to Ben-El-Mechaiekh et al. [1], spaces having property (H) due to Huang [24], FC-spaces due to Ding [5–16,19], and others. In the present short note, we show that all of such examples are particular forms of G-convex spaces contrary to the routine claim of Ding that the class of FC-spaces contains L-spaces and G-convex spaces as true subclasses. We believe that reputed journals should clarify such incorrect statements in their publications.

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. The following appeared in [27–30,32,38]:

Definition 1 A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

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Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. In certain cases, it is possible to assume $\Gamma(A) = \phi_A(\Delta_n)$. We may write $\Gamma_A := \Gamma(A)$ for each $A \in \langle D \rangle$. In case $X \supset D$, the G-convex space is denoted by $(X \supset D; \Gamma)$.

For details on G-convex spaces, see [18,20,27–32,35–38], where basic theory was extensively developed and lots of examples of G-convex spaces were given.

Examples 1

(1) The original KKM principle [25] is for the triple $(\Delta_n \supset V; \text{co})$, where V is the set of vertices and $\text{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation. This triple can be regarded as $(\Delta_n, N; \Gamma)$, where $N := \{0, 1, \ldots, n\}$ and $\Gamma_A := \text{co}\{e_i \mid i \in A\}$ for each $A \subset N$.
(2) Fan's celebrated KKM lemma [21] is for $(E \supset D; \text{co})$, where D is a nonempty subset of a topological vector space E.

These are the origins of our G-convex space $(X, D; \Gamma)$. Note that any KKM type theorem on $(X; \Gamma)$ can not generalize the KKM principle and the Fan lemma. □

For the definition of a G-convex space, at first in [35–37], we assumed $X \supset D$ and an additional condition that

$(*)$ for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

This monotonicity was removed since 1998 in [27] and the restriction $X \supset D$ since 1999 in [28]; see also [29–33,38]. However, note that most of useful examples of G-convex spaces satisfy $(*)$, but, examples not satisfying $(*)$ seem to be artificial:

Examples 2 Let $\Delta_3 = \text{co}\, V$ where $V = \{e_0, e_1, e_2, e_3\}$.

(1) As in the KKM principle, we have a G-convex space $(\Delta_3, V; \text{co})$ where $\text{co} : \langle V \rangle \multimap \Delta_3$ is the convex hull operator. Here $(*)$ holds.
(2) Let $(\Delta_3, V; \Gamma)$ be a G-convex space given by $\Gamma\{e_0, e_1\} := \text{co}\{e_0, e_1, e_2\}$ and $\Gamma(N) := \text{co}\, N$ for all other $N \in \langle V \rangle$. Then Γ violates the isotonicity $(*)$.

In 1998, Ben-El-Mechaiekh et al.[1] defined an L-space (E, Γ), which is a particular form of our G-convex space $(X, D; \Gamma)$ [without assuming $(*)$] for the case $E = X = D$.

Definition 2 [1]. An L-structure on a topological space E is given by a nonempty set-valued map $\Gamma : \langle E \rangle \to E$ verifying

$(**)$ for each $A \in \langle E \rangle$, say $A = \{x_0, x_1, ..., x_n\}$, there exists a continuous function $f^A : \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0, 1, ..., n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_i, i \in J\})$.

The pair (E, Γ) is then called an *L-space*, and $X \subset E$ is said to be *L-convex* if $\forall A \in \langle X \rangle$, $\Gamma(A) \subset X$.

Then the authors of [1] stated that, in particular, if Γ, as in Definition 2[1] , verifies the additional condition $(*)$, then the pair (E, Γ) is what is called by Park and Kim [36], a *G-convex space*. This does not mean that the class of L-spaces contains G-convex spaces. In fact, the authors of [1] imitated our definition of G-convex spaces and implicitly stated that, under the condition $(*)$, their L-spaces reduce to our original G-convex spaces. From the beginning, any L-space is a G-convex space and not conversely.

In order to give another justification of necessity of using the triple $(X, D; \Gamma)$ instead of the pair (E, Γ), we give examples:

Examples 3

(1) The well-known Sperner theorem and Alexandorff-Pasynkoff's theorem on $n+1$ closed sets covering the n-simplex can be derived by applying the KKM principle to the triple $(\Delta_n, V; \mathrm{co})$; see [34] and references therein. No other proof of these theorems using a pair (E, Γ) appeared yet.

(2) In Shapley's generalization of the KKM principle, a triple $(\Delta_n, N; \Gamma)$ appears, where $N := \{0, 1, \ldots, n\}$ and $\Gamma_S := \Delta^S = \mathrm{co}\{e_i \mid i \in S\}$ for each $S \in \langle N \rangle$; see [28] and references therein.

(3) Let $\mathcal{C} := \mathcal{C}[0, 1]$ be the class of all real continuous functions on $[0, 1]$ and $\mathcal{P} := \mathcal{P}[0, 1]$ the subclass of all polynomials $p(x)$ on $x \in [0, 1]$ with real coefficients. Let $\varepsilon > 0$. For each $f \in \mathcal{C}$, choose a fixed $p_f \in \mathcal{P}$ which is ε-near to f, that is, $\max_{x\in[0,1]} |f(x) - p_f(x)| < \varepsilon$. Let $\Gamma : \langle \mathcal{C} \rangle \multimap \mathcal{P}$ be defined by $\Gamma_A := \mathrm{co}\{p_{f_i}\}_{i=0}^n \in \mathcal{P}$ for each $A = \{f_i\}_{i=0}^n \in \langle \mathcal{C} \rangle$. Moreover, let $\phi_A : \Delta_n \to \Gamma_A$ be a linear map such that $e_i \mapsto p_{f_i}$. Then $(X, D; \Gamma) := (\mathcal{P}, \mathcal{C}; \Gamma)$ is a G-convex space satisfying condition $(*)$ and $X \subsetneq D$.

(4) Similarly, by choosing a proper subset D of $\mathcal{C}$, we can obtain G-convex spaces $(X, D; \Gamma)$ satisfying $X \not\subseteq D$ or $X \not\supseteq D$. This is why we assumed X and D are not comparable in general.

(5) Since there are various forms of the Stone-Weierstrass approximation theorem, we can construct a large number of examples similar to the ones in (3) or (4). □

Under the misconception that the class of L-spaces contains our G-convex spaces, a number of careless authors [3,4,17,26,39] restated a number of particular results (with certain defects) on L-spaces which are already known for G-convex spaces. All of these authors failed to give any proper example justifying their misconception.

In [24], a topological space Y is said to have property (H) if, for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. Then the following is introduced:

Definition 3 [24]. Let X be a nonempty set and Y be a topological space with property (H). $T : X \to 2^Y$ is said to be a generalized R-KKM mapping if for each $\{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ such that

$$\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} T x_{i_j},$$

for all $\{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$.

Adopting these concepts, in [24], its author obtained modifications of some known results in the G-convex space theory in which we supplied a large number of examples of such spaces. It is noteworthy that the authors of [2,8,17,19,24] adopted R-KKM maps and claimed to obtain generalizations of known results without giving any justifications or any proper examples.

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$.

We should recognize that, in the KKM theory on G-convex spaces, every argument is related to the finite intersection property of functional values of KKM maps, in other words, related to some $N \in \langle D \rangle$ in $(X, D; \Gamma)$. Therefore, the works in [2,8,17,19,24] can be reduced to the ones in our G-convex space theory as follows:

Proposition 1 *Every argument on KKM maps on a space having property* (*H*) *can be switched to the one for the G-convex space* $(Y, N; \Gamma')$ *for some* $N \in \langle Y \rangle$ *where*

$$\Gamma'_J = \Gamma'(J) := \varphi_N(\Delta_J) \quad \textit{for all} \quad J \in \langle N \rangle.$$

Moreover, we have

Proposition 2 *A generalized R-KKM map $T : X \to 2^Y$ is simply a KKM map for a G-convex space $(Y, X; \Gamma)$ satisfying the monotonicity* $(*)$.

Proof Let $A \in \langle X \rangle$ with $|A| = n+1$. Then there corresponds an $N \in \langle Y \rangle$ with $|N| = n+1$. Define $\Gamma : \langle X \rangle \multimap Y$ by $\Gamma_A := T(A)$ for each $A \in \langle X \rangle$. Then $(Y, X; \Gamma)$ becomes a G-convex space since, for each A with $|A| = n+1$, we have a continuous function $\phi_A := \varphi_N : \Delta_n \to T(A) =: \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle X \rangle$ and hence $T : X \multimap Y$ is a KKM map on a G-convex space $(Y, X; \Gamma)$. □

Contrary to Proposition 2, Ding in [8] claimed as follows: "The above class of generalized *R-KKM* mappings include those classes of *KKM* mappings, *H-KKM* mappings, *G-KKM* mappings, generalized *G-KKM* mappings, generalized *S-KKM* mappings, *GLKKM* mappings and *GMKKM* mappings defined in topological vector spaces, H-spaces, G-convex spaces, G-H-spaces, L-convex spaces and hyperconvex metric spaces, respectively, as true subclasses."

Therefore, all of the KKM type theorems on such variants are simple consequences of our G-convex space theory. Consequently, all results in [8] are artificial disguised forms of known ones having no proper examples.

In 2005, Ding [5] introduced the following notion of "a finitely continuous" topological space (in short, FC-space):

Definition 4 [5]. $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset D of $(Y, \{\varphi_N\})$ is said to be a FC-subspace of Y if for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$ where $\Delta_k = \mathrm{co}\{e_{i_j} : j = 0, \ldots, k\}$.

Then Ding [5] wrote that "it is clear that the class of G-convex spaces $(Y; \Gamma)$ is a true subclass of FC-spaces," with no justification.

In 2006, Ding [6] added the following to Definition 4 [5]:

Definition 5 [6]. If A and B are two subsets of Y, B is said to be a FC-subspace of Y relative to A if for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset A \cap N$, $\varphi_N(\Delta_k) \subset B$ where $\Delta_k = \mathrm{co}\{e_{i_j} : j = 0, \ldots, k\}$. If $A = B$, then B is called a FC-subspace of Y.

Then Ding [6] wrote: "It is easy to see that the class of FC-spaces includes the classes of convex sets in topological vector spaces, C-spaces (or H-spaces) [23], G-convex spaces [36], L-convex spaces [1], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces." Here again he failed to give any justification or any proper example of his space which is not G-convex. One wonders that how could a pair $(Y, \{\varphi_N\})$ generalize a triple $(X, D; \Gamma)$ in [36,37].

The above definition and Ding's claim have appeared also in [7–16,22,40,41], and possibly more. One dozen of such papers on FC-spaces have appeared within 2 years! In these papers, known results in KKM theory on G-convex spaces are restated or modified for the so-called FC-spaces. In order to prevent such unnecessary efforts, something has to be done.

Note that L-spaces, spaces having property (H), and FC-spaces have a family $\{\phi_N\}$ of continuous functions. For such spaces we have the following:

Proposition 3 *A triple $(X, D; \{\phi_N\})$ consisting of a topological space X, a nonempty set D, and a family of continuous functions $\phi_N : \Delta_n \to X$ for $N \in \langle D\rangle$ with the cardinality $|N| = n + 1$, can be made into a G-convex space $(X, D; \Gamma)$.*

Proof This can be done at least in three ways.

(1) For each $A \in \langle D\rangle$, by putting $\Gamma_A := X$, we obtain a trivial G-convex space $(X, D; \Gamma)$.
(2) Let $\{\Gamma^\alpha\}_\alpha$ be the family of maps $\Gamma^\alpha : \langle D\rangle \multimap X$ giving a G-convex space $(X, D; \Gamma^\alpha)$. Note that, by (1), this family is not empty. Then, for each α and each $A \in \langle D\rangle$ with $|A| = n + 1$, we have

$$\phi_A(\Delta_n) \subset \Gamma^\alpha_A \text{ and } \phi_A(\Delta_J) \subset \Gamma^\alpha_J \quad \text{for } J \subset A.$$

Let $\Gamma := \bigcap_\alpha \Gamma^\alpha$, that is, $\Gamma_A = \bigcap_\alpha \Gamma^\alpha_A$ for each $A \in \langle D\rangle$. Then

$$\phi_A(\Delta_n) \subset \Gamma_A \text{ and } \phi_A(\Delta_J) \subset \Gamma_J \quad \text{for } J \subset A.$$

Therefore,$(X, D; \Gamma)$ is a G-convex space.
(3) Let $N \in \langle D\rangle$ with $|N| = n+1$. For each $M \in \langle D\rangle$ with $N \subset M$, $M = \{a_0, \dots, a_m\}$ and $N = \{a_{i_0}, \dots, a_{i_n}\}$, there exists a subset $\phi_M(\Delta^M_n)$ of X such that $\Delta^M_n := \mathrm{co}\{e_{i_j} \mid j = 0, \dots, n\} \subset \Delta_m$. Now let

$$\Gamma_N = \Gamma(N) := \bigcup_{M \supset N} \phi_M(\Delta^M_n).$$

Then $\Gamma : \langle D\rangle \multimap X$ is well-defined and $(X, D; \Gamma)$ becomes a G-convex space: For each $A \in \langle D\rangle$ with $|A| = n + 1$, there exists a continuous map $\phi_A : \Delta_n \multimap \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. □

Examples 4 Let $(\Delta_3, V; \{\phi_N\})$ be a triple where $\phi_N(\Delta_n) = \Gamma(N)$ as in the preceding Examples 2(2). Then

$$\phi_{\{e_0,e_1\}}(\Delta_1) = \phi_{\{e_0,e_1\}}(\mathrm{co}\,\{e_0, e_1\}) = \Gamma\{e_0, e_1\} = \mathrm{co}\,\{e_0, e_1, e_2\},$$

where we may assume $\phi_{\{e_0,e_1\}}$ is a surjective space-filling curve such that $\phi_{\{e_0,e_1\}}(e_0) = e_0$ and $\phi_{\{e_0,e_1\}}(e_1) = e_1$. Then it is easily checked that Γ itself is the one in the proof (3) of Proposition 3 corresponding to $\{\phi_N\}$. □

A nonempty subset Y of a topological vector space E is said to be *almost convex* if for any neighborhood V of the origin 0 of E and for any finite subset $\{y_1, y_2, \dots, y_n\}$ of Y, there exists a finite subset $\{z_1, z_2, \dots, z_n\}$ of Y, such that $z_i - y_i \in V$ for each $i = 1, \dots, n$, and $\mathrm{co}\{z_1, z_2, \dots, z_n\} \subset Y$.

Examples 5 Let Y be an almost convex dense subset of a subset D of E. Let V be a given neighborhood of 0. For each $A := \{x_0, x_1, \dots, x_n\} \in \langle D\rangle$, choose a subset $B := \{y_0, y_1, \dots, y_n\} \in \langle Y\rangle$ such that $y_i - x_i \in V$ for each $i = 0, 1, \dots, n$ and $\mathrm{co}\, B \subset Y$. Define a continuous function $\phi_A : \Delta_n \to \mathrm{co}\, B$ given by

$$\phi_A : u = \sum_{i=0}^{n} \lambda_i(u) e_i \mapsto \phi_A(u) := \sum_{i=0}^{n} \lambda_i(u) y_i$$

for $u \in \Delta_n$. Then $(Y, D; \{\phi_A\}_{A\in\langle D\rangle})$ can be made into a G-convex space. Note that $Y \subset D$. □

From Proposition 3, contrary to Ding's claim, we have the following:

Proposition 4 *An FC-space* $(Y, \{\varphi_N\})$ *can be made into an L-space* $(Y; \Gamma)$*, a particular type of G-convex spaces* $(Y, D; \Gamma)$.

Proof In Definition 4 [5], we can eliminate the case where some elements in N may be same. Then we can define a map $\Gamma : \langle Y\rangle \to Y$ as in the proof of Proposition 3. Therefore, the so-called FC-spaces are L-spaces and hence very particular forms of our G-convex spaces. □

Recall that our G-convex space $(X, D; \Gamma)$ is originated from the KKM principle [25] and the celebrated Ky Fan lemma [21] from the beginning. The case $X = D$ is not applicable to them and this is the most serious defect of L-spaces or FC-spaces. Hence, they are inadequate for the KKM theory.

Now Ding's FC-subspace relative to some subset A in Definition 5 [6] can be extended as follows:

Definition 6 Let $(E, D; \Gamma)$ be a G-convex space and $X \subset E$, $D' \subset D$. Then X is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$.

Recall that, for a G-convex space $(E \supset D; \Gamma)$, we used to say that a subset X of E is Γ-*convex* if, for any $N \in \langle X \cap D\rangle$, we have $\Gamma_N \subset X$. This is now saying that X is Γ-convex relative to $D' := X \cap D$.

Therefore, instead of using the concept of an FC-subspace of $(Y, \{\varphi_N\})$ relative to A as in Definition 5 [6], we may use a Γ-convex subset of the G-convex space $(Y, D; \Gamma)$ relative to $A \subset D$. Any interested reader can check this matter in all of [5–16,22,40,41].

For a topological space $(X, \mathcal{T})$, the compactly generated extension (or the k-extension) $\mathcal{T}_k$ of the original topology $\mathcal{T}$ is a new topology of X finer than $\mathcal{T}$ such that $\mathcal{T}_k$ is the collection of all compactly open [resp., compactly closed] subsets of $(X, \mathcal{T})$. Note that the artificial terminology of compact interior, compact closure, etc., are not practical and can be eliminated by switching the original topology of the underlying space to its compactly generated extension; see [30].

Such inadequate artificial terminology was used in [3–6,10,13,16,22,41], but disappeared or withdrawn in [7,11,12,14,15,19,40].

Finally, in the recent study on abstract convex spaces in [32,33], basic theorems on G-convex spaces are further generalized.

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