Comments on 2-KKM Maps on Hyperconvex Metric Spaces

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Abstract

Recently, C.-M. Chen et al. [5,7] introduced 2-KKM maps and generalized 2-KKM maps on metric spaces and obtained a KKM type theorem for hyperconvex metric spaces. In the present paper, we show that main results in [5,7] follow from our G-convex space theory. Consequently, we obtain equivalent formulations of the 2-KKM theorem and unify various concepts and results in [5,7]. Some additional results are also given in the last section.

1. Introduction

The notion of hyperconvex metric spaces was introduced by Aronszajn and Panitchpakdi [1] in 1956. Later, in 1979, independently Sine [26] and Soardi [27] proved that a bounded hyperconvex metric space has the fixed point property for nonexpansive maps. Since then many interesting works have appeared for hyperconvex metric spaces. For the literature, see [13].

For a long time, the study of hyperconvex metric spaces was concentrated on the relationship with nonexpansive maps. However, Khamsi [12] established the Knaster-Kuratowski-Mazurkiewicz theorem (in short, KKM theorem) for hyperconvex metric spaces and applied it to obtain a Schauder type fixed point theorem. This line of study was followed by a number of authors. In particular, the present author obtained extensions or equivalent forms of the KKM theorem, a Fan-Browder type fixed point theorem, and other results for hyperconvex spaces in [18,19]. Moreover, Kirk et al. [14] established the KKM theorem, its equivalent formulations, fixed point theorems, and their applications for hyperconvex spaces.

However, most of the above-mentioned works are simple consequences of much more general results. In fact, Horvath [9,10] initiated the study of the KKM theory and fixed point theory for C-spaces, which are meaningful generalizations of convex subsets of topological vector spaces or convex spaces due to Lassonde [16]. In [10], it is known that hyperconvex metric spaces are particular types of C-spaces. Moreover, the present author

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[20-25] investigated generalized convex spaces or G-convex spaces, which properly include the class of C-spaces and a large number of spaces having particular types of abstract convexity.

Recently, C.-M. Chen et al. [5-7] introduced 2-KKM maps and generalized 2-KKM maps on metric spaces and obtained a KKM type theorem for hyperconvex metric spaces. This KKM type theorem was applied to some variational inequality and minimax inequality theorems.

In the present paper, we show that main results in [5,7] follow from our G-convex space theory. Consequently, we obtain equivalent formulations of the 2-KKM theorem and unify various concepts and results in [5,7]. Moreover, we raise a problem, on a hyperconvex metric space, whether there is a 2-KKM map with admissible values, which is not a KKM map. Some additional results are also given in the last section.

2. Generalized convex spaces and the map classes $\mathfrak{KC}$ and $\mathfrak{KO}$

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Definition. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$; see [20-25].

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset X.$$

A subset Y of X is called a Γ-*convex subset* of $(X, D; \Gamma)$ relative to $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset Y$, that is, $\mathrm{co}_\Gamma D' \subset Y$. Then $(Y, D'; \Gamma|_{\langle D'\rangle})$ is called a Γ-*convex subspace* of $(X, D; \Gamma)$.

When $D \subset X$, the space is denoted by $(X \supset D; \Gamma)$. In such case, a subset Y of X is said to be Γ-*convex* if $\mathrm{co}_\Gamma(Y \cap D) \subset Y$; in other words, Y is Γ-convex relative to $D' := Y \cap D$. In case $X = D$, let $(X; \Gamma) := (X, X; \Gamma)$. We use $\Gamma_A := \Gamma(A)$ sometimes.

Examples. 1. A G-convex space $(X, D; \Gamma)$ is called a *C-space* (or an *H-space*) if each Γ_A is ω-connected (that is, n-connected for all $n \geq 0$) and $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D\rangle$; see Horvath [9,10] for particular C-spaces for $X = D$.

The concepts of C-spaces, LC-spaces, and LC-metric spaces were introduced and extensively studied by Horvath in a sequence of papers; see [9,10] and references therein.

2. A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept for $X = D$ reduces to the one due to Lassonde [16].

Definition. Let $(X, D; \Gamma)$ be a G-convex space and Z a topological space. For a multimap $F : X \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap X$ is a KKM map with respect to the identity map $1_X : X \to X$.

A multimap $F : X \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp., a $\mathfrak{KO}$*-map*] [21] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{KC}(X, Z) := \{F : X \multimap Z \mid F \text{ is a } \mathfrak{KC}\text{-map}\}.$$

Similarly, $\mathfrak{KO}(X, Z)$ is defined. Some authors use the notation $\mathrm{KKM}(X, Z)$ instead of $\mathfrak{KC}(X, Z)$.

If $1_X \in \mathfrak{KC}(X, X)$, then $f \in \mathfrak{KC}(X, Z)$ for any continuous function $f : X \to Z$. This also holds for $\mathfrak{KO}$. Moreover, if $F : X \to Z$ is a continuous single-valued map or if $F : X \multimap Z$ has a continuous selection, then $F \in \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$.

The following is well-known; see [15,17]:

The KKM Principle. *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map (that is, $\operatorname{co} A \subset G(A)$ for each $A \subset D$) with closed [resp., open] values. Then $\bigcap_{z \in D} G(z) \neq \emptyset$.*

The following well-known KKM theorem for a G-convex space $(X, D; \Gamma)$ shows that the identity map $1_X \in \mathfrak{KC}(X, X) \cap \mathfrak{KO}(X, X)$; see [20,25]:

Theorem 2.1. *Let $(X, D; \Gamma)$ be a G-convex space and $G : D \multimap X$ a map such that*

(1) G has closed [resp., open] values; and

(2) G is a KKM map.

Then $\{F(z)\}_{z \in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D \rangle$ with $|N| = n + 1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z \in N} G(z) \neq \emptyset$.)

Further, if

(3) $\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,

then we have $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Corollary 2.2. *Let $(X, D; \Gamma)$ be a G-convex space, and $A \in \langle D \rangle$, $\{C_a\}_{a \in A}$ a family of closed [resp., open] subsets of X such that for each nonempty subset $B \subset A$, we have $\Gamma_B \subset \bigcup_{b \in B} C_b$. Then $\bigcap_{a \in A} C_a \neq \emptyset$.*

In the KKM theory, it is well-known that the KKM Theorem 2.1 or the whole intersection property of functional values of a closed-valued KKM map is equivalent to or implies a large number of statements; for examples, the Fan type matching property, the geometric or section properties, the Fan-Browder type fixed point theorem, existence of maximal elements, analytic alternatives (a basis of various equilibrium problems), the Fan type minimax inequality, variational inequalities, best approximation theorems, the von Neumann type minimax theorem, the von Neumann type intersection theorem, the Nash type equilibrium theorem, and others. See [17,20,24].

3. KKM maps in metric spaces

Let (M, d) be a metric space. Motivated by [12] and others, for a bounded subset $A \subset M$, we set

$\mathrm{ad}(A) := \bigcap\{B \mid B \text{ is a closed ball such that } A \subset B\}$.

$\mathcal{A}(M) := \{A \subset M \mid A = \mathrm{ad}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.

For $x \in M$ and $\varepsilon > 0$, let

$$B(x, \varepsilon) := \{y \in M \mid d(x, y) \le \varepsilon\} \quad \text{and} \quad N(x, \varepsilon) := \{y \in M \mid d(x, y) < \varepsilon\}.$$

We introduce new definitions:

Definition. A triple $(M, D; \Gamma)$ is called simply a *metric space* if (M, d) is a metric space, D is a nonempty set, and $\Gamma : \langle D \rangle \to \mathcal{A}(M)$ is a map having admissible values.

A Γ-convex subset of $(M, D; \Gamma)$ relative to some $D' \subset D$ is said to be *subadmissible* by some authors.

Examples. We give examples of metric spaces $(M, D; \Gamma)$ and KKM maps on them.

(1) $(M \supset X; \Gamma)$ where $\Gamma_A := \mathrm{ad}(A)$; see [12]. A map $G : X \multimap M$ is called a KKM map if $\Gamma_A \subset G(A)$ for each $A \in \langle X \rangle$.

(2) For each $A := \{a_0, a_1, \ldots, a_n\} \in \langle D \rangle$, choose a subset $B := \{x_0, x_1, \ldots, x_n\} \in \langle M \rangle$ and define $\Gamma_A := \mathrm{ad}(B)$. Then $(M, D; \Gamma)$ becomes a metric space. For this metric space, the so-called generalized $gKKM$ mapping in [7] simply becomes a KKM map.

4. Hyperconvex metric spaces

The following originates from [1].

Definition. A metric space (H, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$.

Examples. It is known that the space $\mathbb{C}(E)$ of all continuous real functions on a Stonian space E (that is, extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space $\mathbb{C}(E)$ for some Stonian space E. Therefore, $(\mathbf{R}^n, \|\cdot\|_\infty)$, l^∞, and L^∞ are concrete examples of hyperconvex metric spaces.

Results of Aronszajn and Panitchpakti [1, Theorem 1′] and Isbell [11, Theorem 1.1] are combined in the following:

Lemma 4.1. *A hyperconvex metric space is complete and (freely) contractible.*

The following is easy to prove:

Lemma 4.2. *An admissible subset of a hyperconvex metric space is hyperconvex.*

We introduce the following concept:

Definition. An abstract convex space $(H, D; \Gamma)$ is called simply a *hyperconvex metric space* if (H, d) is a hyperconvex metric space, D is a nonempty set, and $\Gamma : \langle D \rangle \to \mathcal{A}(H)$ is a map having admissible values such that

$$A, B \in \langle D \rangle,\ A \subset B \text{ implies } \Gamma_A \subset \Gamma_B.$$

There should be no confusion between a hyperconvex metric space $H = (H, d)$ and $(H, D; \Gamma)$.

Theorem 4.3. *Any hyperconvex metric space $(H, D; \Gamma)$ is a G-convex space.*

Proof. For each $A \in \langle D \rangle$, Γ_A is hyperconvex by Lemma 4.2 and hence contractible by Lemma 4.1. Therefore, $(H, D; \Gamma)$ is an H-space and hence a G-convex space.

From Theorems 2.1 and 4.3, we have the following:

Theorem 4.4. *Let $(H, D; \Gamma)$ be a hyperconvex metric space and $G : D \multimap H$ a map such that*

(1) *G has closed [resp., open] values; and*

(2) *G is a KKM map.*

Then $\{G(z)\}_{z \in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D \rangle$ with $|N| = n + 1$, we have $\Gamma_N \cap \bigcap_{z \in N} G(z) \neq \emptyset$.)

Further, if

(3) *$\bigcap_{z \in A} \overline{G(z)}$ is compact for some $A \in \langle D \rangle$,*

then we have $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Examples. 1. As a consequence of Theorem 4.4, we obtain Khamsi's KKM theorem for a particular Γ and for particular KKM maps with finitely closed values; see [9]. In fact, by replacing the original topology of H by its finitely generated extension, we can eliminate "finitely".

2. From Theorem 4.4, we obtain another particular forms in [5, Theorems 2 and 3].

It is well-known that any family of closed balls in a hyperconvex metric space has nonempty intersection whenever each two members of the family intersects. More precisely, we have the following due to Penot [13, p.406]::

Lemma 4.5. *Let (H, d) be a hyperconvex metric space and $\{A_\alpha\}_{\alpha \in I} \subset \mathcal{A}(H)$. If for each $\alpha, \beta \in I$, $A_\alpha \cap A_\beta \neq \emptyset$, then $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$.*

Note that Lemma 4.5 simply tells that, for a family of admissible subsets of a hyperconvex metric space, the finite intersection property implies the whole intersection property.

As was known by the celebrated Fan lemma [8], this fact is usually shown under the certain compactness assumption.

5. 2-KKM maps in metric spaces

We introduce the following new concepts:

Definition. A *γ-metric space* $(M, D; \gamma)$ consists of a metric space (M, d), a nonempty set D, and a map $\gamma : D \times D \to \mathcal{A}(M)$ such that $\gamma(a, b) = \gamma(b, a)$ for $a, b \in D$.

Sometimes, we use the conventions $(a, b) = \{a, b\}$ and $\{a\} = a$.

Examples. 1. Any metric space (M, d) can be made into a γ-metric space $(M, M; \gamma)$ by defining $\gamma(x, y) := \mathrm{ad}\{x, y\}$ for $x, y \in M$.

2. Any metric space $(M, D; \Gamma)$ can be made into a γ-metric space $(M, D; \gamma)$ by defining $\gamma := \Gamma|_{D\times D} : D \times D \to \mathcal{A}(M)$.

Definition. For a γ-metric space $(M, D; \gamma)$, a map $G : D \multimap M$ is called a 2-*KKM map* if

$$\gamma(a, a) \subset G(a) \quad \text{and} \quad \gamma(a, b) \subset G(a) \cup G(b) \quad \text{for each } a, b \in D.$$

Examples. Our definition of 2-KKM maps unifies various similar ones in [5,7].

1. [5] Let M be a metric space and $X \subset M$. A map $G : X \multimap M$ is called a 2-KKM map if for each $x_1, x_2 \in X$,

$$x_1 \in G(x_1), \quad x_2 \in G(x_2), \quad \text{and} \quad \mathrm{ad}(\{x_1, x_2\}) \subset G(x_1) \cup G(x_2).$$

This reduces to our definition by putting $D := X$ and $\gamma(x, y) := \mathrm{ad}(\{x, y\})$.

2. [5] Let X be a nonempty set and Y a metric space. A map $G : X \multimap Y$ is called a generalized 2-KKM map if for each $x_1, x_2 \in X$, there exists $y_1, y_2 \in Y$ such that

$$y_1 \in G(x_1), \quad y_2 \in G(x_2), \quad \text{and} \quad \mathrm{ad}(\{y_1, y_2\}) \subset G(x_1) \cup G(x_2).$$

This reduces to our 2-KKM map by putting $D := X$, $M := Y$, and $\gamma(x_1, x_2) := \mathrm{ad}(\{y_1, y_2\})$.

3. [7] Let Z be a nonempty set and Y a metric space. A map $G : Z \multimap Y$ is called a generalized 2-$gKKM$ map if for each $z_1, z_2 \in Z$, there exists $y_1, y_2 \in Y$ such that

$$y_1 \in G(z_1), \quad y_2 \in G(z_2), \quad \text{and} \quad \mathrm{ad}(\{y_1, y_2\}) \subset G(z_1) \cup G(z_2).$$

This definition is same to that of the preceding one.

The following is a 2-KKM theorem:

Theorem 5.1. *Let $(H, D; \Gamma)$ be a hyperconvex metric space. If $G : D \to \mathcal{A}(H)$ is a 2-KKM map, then $\bigcap_{z\in D} G(z) \neq \emptyset$.*

Proof 1. Recall that $(H, D; \Gamma)$ is a G-convex space and we apply Corollary 2.2. For any $z_1, z_2 \in D$, let $A := \{z_1, z_2\}$ and $C_1 := G(z_1), C_2 := G(z_1)$. Since G is a 2-KKM map, by

Corollary 2.2, $C_1 \cap C_2 = G(z_1) \cap G(z_2) \neq \emptyset$. Since each $G(z) \in \mathcal{A}(H)$, by Lemma 4.5, we have the conclusion.

Proof 2. Since any element of $\mathcal{A}(H)$ is contratible by Lemma 4.2 and hence connected. Suppose that $G(a) \cap G(b) = \emptyset$ for some $a, b \in D$. Since $a \in G(a),\ b \in G(b)$, $G(a)$ and $G(b)$ are closed and $\gamma(a,b) \in \mathcal{A}(H)$ is connected, it is impossible that $\gamma(a,b) \subset G(a) \cup G(b)$. Now we apply Lemma 4.5.

Note that Theorem 5.1 unifies the main results in [5, Theorem 4], [7, Theorem 5].

Theorem 5.2. *Let X be an admissible subset of a hyperconvex metric space $(H \supset X; \Gamma)$. If $G : X \to \mathcal{A}(X)$ is a 2-KKM map, then G has a fixed point.*

Proof. By Lemma 4.2, X itself is a hyperconvex metric space. Hence $\bigcap_{x \in X} G(x) \neq \emptyset$ by Theorem 5.1. Then any point in the intersection is fixed under G.

Remark. Any KKM map on a metric space $(M, D; \Gamma)$ is 2-KKM. The converse does not hold; see [5].

Question. For a hyperconvex metric space $(H, D; \Gamma)$, is there any $\mathcal{A}(H)$-valued 2-KKM map $G : D \to \mathcal{A}(H)$, which is not a KKM map?

We have a partial solution to this problem:

Theorem 5.3. *Let $(H, D; \Gamma)$ be a hyperconvex metric space and $G : D \to \mathcal{A}(H)$ a 2-KKM map. Then there is a hyperconvex metric space $(H, D; \Gamma')$ for which G is a KKM map.*

Proof. Since G is a 2-KKM map, by Theorem 5.1, there exists an $x_* \in H$ such that $x_* \in \bigcap_{z \in D} G(z)$. Define a map $\Gamma' : \langle D \rangle \to \mathcal{A}(H)$ by $\Gamma'_A := \{x_*\}$ for each $A \in \langle D \rangle$. Then $\Gamma'_A \subset G(A)$ and hence G is a KKM map on the hyperconvex metric space $(H, D; \Gamma')$.

Note that Theorem 5.1 follows from Theorems 4.4 and 5.3. Consequently, main results in [5,7] follow from Theorem 2.1 and hence from our G-convex space theory. Other results in [5,7] can be deduced by routine ways in the KKM theory.

6. 2-KKM maps in topological spaces

Some other concepts in [5] can be upgraded as follows:

Definition. Let $(X, D; \gamma)$ be a triple consisting of a topological space X, a nonempty set D, and a multimap $\gamma : D \times D \multimap X$ such that for each $a, b \in D$, $\gamma(a,b)$ is a nonempty subset of X.

Let Y be a topological space and $F : X \multimap Y$ be a map. A map $G : D \multimap Y$ is called a 2-*KKM map* with respect to F if

$$F(\gamma(a,a)) \subset G(a) \quad \text{and} \quad F(\gamma(a,b)) \subset G(a) \cup G(b) \quad \text{for each } a, b \in D.$$

A 2-*KKM map* $G : D \multimap X$ is a 2-KKM map with respect to the identity map 1_X.

A multimap $F : X \multimap Y$ is called a 2-$\mathfrak{KC}$-*map* [resp., 2-$\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] 2-KKM map $G : D \multimap Y$ with respect to F, the family

$\{G(a)\}_{a\in D}$ has the 2-valued intersection property, that is, $G(a)\cap G(b)\neq\emptyset$ for any $a,b\in D$. We denote

$$2-\mathfrak{KC}(X,Y):=\{F:X\multimap Y\mid F \text{ is a } 2-\mathfrak{KC}\text{-map}\}.$$

Similarly, 2-$\mathfrak{KO}(X,Y)$ is defined. Some authors use the notation KKM(X,Y) instead of $\mathfrak{KC}(X,Y)$.

Examples. We give examples of triples $(X,D;\gamma)$ and some related maps:

1. A γ-metric space $(M,D;\gamma)$.

2. [28] An interval space X is a topological space with a map $[\cdot,\cdot]:X\times X\multimap X$ such that for any $x_1,x_2\in X$, $[x_1,x_2]=[x_2,x_1]$ is a connected set containing x_1,x_2. Define $\gamma(x_1,x_2):=[x_1,x_2]$.

3. [2,4] Let X be a Hausdorff topological space and $\{C_A\}$ a family of nonempty connected subsets of X indexed by finite subsets A of X such that $A\subset C_A$, then we call $(X,\{C_A\})$ a W-space. Define $\gamma(x,y):=C_{\{x,y\}}$ for each $x,y\in X$.

4. [3] A topological space X is called a generalized interval space if there exists a mapping $\Gamma:X\times X\to\mathcal{A}(X)$, where $\mathcal{A}(X)$ is a family of nonempty connected subsets of X. Define $\gamma(x,y):=\Gamma(x,y)$ for each $x,y\in X$.

5. The generalized 2-$gKKM$ map in [7, Definition 5] can be made into a 2-$\mathfrak{KC}$-*map* in our sense.

The following due to Thompson and Yuan [29, Theorems 1 and 2] gives us another example:

Lemma 6.1. *Let X and Y be two topological spaces and $\Phi:X\multimap Y$ be an u.s.c. or a l.s.c. map with nonempty closed [resp., open] connected values. If X is an interval space such that for each $x_1,x_2\in X$, we have*

$$\Phi([x_1,x_2])\subset\Phi(x_1)\cup\Phi(x_2),$$

then Φ has the 2-value intersection property, that is, $\Phi(x_1)\cap\Phi(x_2)\neq\emptyset$ for each $x_1,x_2\in X$.

Note that Φ is a 2-KKM map with respect to itself. Moreover, in [29, Theorems 3 and 4], Lemma 6.1 is applied to obtain the finite intersection property of the family $\{\Phi(x)\mid x\in X\}$. It is routine that, under some additional assumption, we can deduce the whole intersection property of the family and its equivalents or consequences as for the KKM Theorem 2.1. A part of this kind of works was done in [2-4 and some others] and we will not repeat here. But, we give a new result on hyperconvex metric space.

From Lemmas 4.5 and 6.1, we immediately have the following whole intersection property.

Theorem 6.2. *Let X be a topological space, H a hyperconvex metric space and $\Phi:X\multimap H$ be an u.s.c. or a l.s.c. map with admissible values. If X is an interval space such that for each $x_1,x_2\in X$, we have*

$$\Phi([x_1,x_2])\subset\Phi(x_1)\cup\Phi(x_2),$$

then $\bigcap_{x\in X}\Phi(x)\neq\emptyset$.

Proof. Note that any admissible subset is closed and contractible by Lemmas 4.1 and 4.2, and hence, connected. Therefore, by Lemma 6.1, Φ has the 2-value intersection property. Now, by Lemma 4.5, the conclusion follows.

Corollary 6.3. *Let H be a hyperconvex metric space and $\Phi : H \multimap H$ be an u.s.c. or a l.s.c. map with admissible values such that for each $x_1, x_2 \in H$, we have*

$$\Phi(\mathrm{ad}\{x_1, x_2\}) \subset \Phi(x_1) \cup \Phi(x_2).$$

Then

(i) Φ *has a fixed point; and*

(ii) *any map $F : H \multimap H$ satisfying $F(x) \supset \Phi(x)$ for each $x \in H$ is a 2-KKM map with respect to Φ.*

Example. [7] If X is a metric space and Y is a topological space, any continuous map $f : X \to Y$ or any multimap $F : X \multimap Y$ having a continuous selection belong to $\mathfrak{KC}(X, Y) \subset 2\text{-}\mathfrak{KC}(X, Y)$.

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