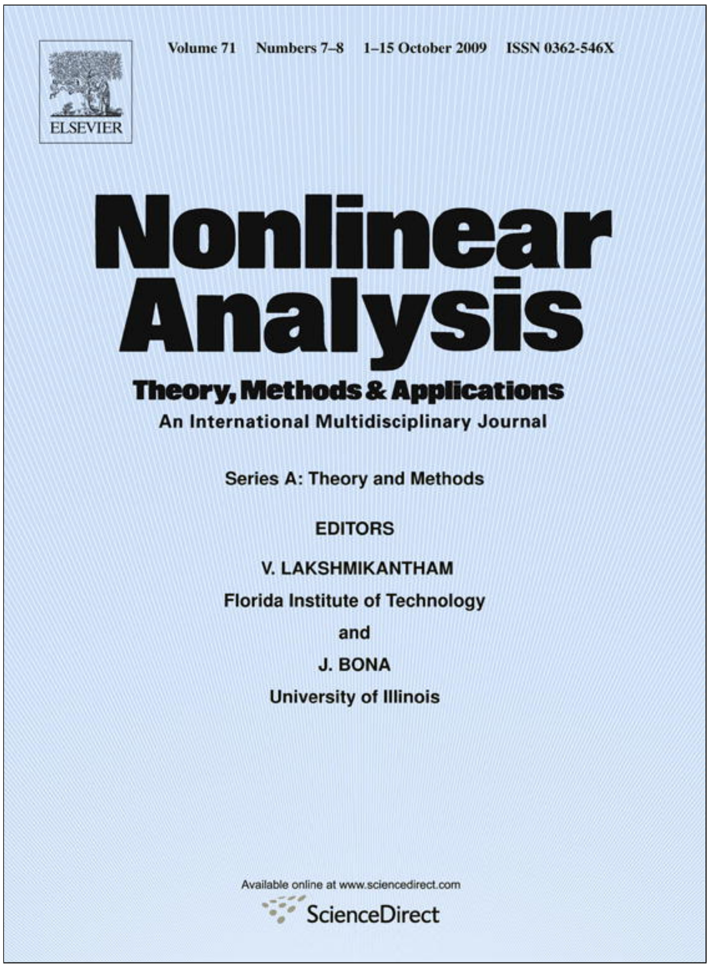

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Fixed point theory of multimaps in abstract convex uniform spaces

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ABSTRACT

This is to establish fixed point theorems for multimaps in abstract convex uniform spaces. Our new results generalize corresponding ones in topological vector spaces (t.v.s.), convex spaces due to Lassonde, *C*-spaces due to Horvath, and *G*-convex spaces due to Park. We show that fixed point theorems on multimaps of the Fan–Browder type, multimaps having ranges of the Zima–Hadžić type, and multimaps whose ranges are Φ-sets or Klee approximable sets can be established in abstract convex spaces or KKM spaces.

1. Introduction

The aim of this paper is to establish fixed point theorems for multimaps in abstract convex uniform spaces. Our new results generalize and unify corresponding ones in topological vector spaces (t.v.s.) due to many authors, convex spaces due to Lassonde, *C*-spaces due to Horvath, and *G*-convex spaces due to Park.

Recall that the celebrated Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM principle) in 1929 [1] was first applied to give a simple proof of the Brouwer fixed point theorem. Since then there has appeared a large number of applications of the principle to fixed point theorems and other results in topological vector spaces. In fact, in our previous works [2,3], we showed that the KKM principle has many equivalent formulations and that the principle implies many fixed point theorems.

The KKM theory, first called by the author in [4,5], is the study of applications of various equivalent formulations of the KKM principle and their generalizations. At the beginning, the theory was mainly devoted to studies on convex subsets of topological vector spaces. Later, it has been extended to convex spaces by Lassonde [6], and to *C*-spaces (or *H*-spaces) by Horvath [7–10] and others. In the last decade, the KKM theory has been extended to generalized convex (*G*-convex) spaces in a sequence of papers of the author; for details, see [11–28] and references therein. Especially, in [24], we established a unified fixed point theory of multimaps in *G*-convex uniform spaces.

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In our recent works [29–31], we introduced a new concept of abstract convex spaces and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ having a certain KKM property. These new spaces and multimap classes are known to be adequate to establish the KKM theory. Especially, in [30], we generalized and simplified known results of the theories on convex spaces, H-spaces, G-convex spaces, and others. Moreover, in [30], we noticed that the class of abstract convex spaces $(E, D; \Gamma)$ satisfying the KKM principle play a major role in the KKM theory. Therefore such spaces shall be called the KKM spaces.

In the present paper, our aim is to establish a unified fixed point theory on multimaps in abstract convex spaces or KKM spaces, as in [24] for G-convex spaces. We study various types of multimaps and of abstract convex uniform spaces. In fact, some fixed point theorems on multimaps of the Fan–Browder type, multimaps having ranges of the Zima–Hadžić type, and multimaps whose ranges are Φ-sets can be extended to KKM spaces. Moreover, for multimaps in the 'better' admissible class $\mathfrak{B}$ defined on abstract convex spaces, we show that they have almost fixed point property whenever their ranges are Klee approximable.

Section 2 deals with preliminaries on abstract convex spaces and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$. In Section 3, certain 'local convexity' of abstract convex uniform spaces are introduced; e.g., $L\Gamma$-spaces or locally Γ-convex spaces. Section 4 deals with subsets of the Zima–Hadžić type in KKM uniform spaces. We obtain almost fixed point theorems on u.s.c. multimaps with convex values having totally bounded ranges of Zima–Hadžić type. In Section 5, we define Fan–Browder type maps and Φ-subsets in abstract convex uniform spaces. We obtain fixed point theorems on $\mathfrak{KC}$-maps whose ranges are Φ-sets. Section 6 deals with generalized forms of the Fan–Browder fixed point theorem and the maximal element theorem on KKM spaces. In Section 7, we examine the 'better' admissible class $\mathfrak{B}$ defined on abstract convex spaces. Finally, Section 8 deals with a new fixed point theorem on $\mathfrak{B}$-maps whose ranges are Klee approximable in abstract convex spaces.

Throughout this paper, multimaps are called simply maps.

2. Abstract convex spaces

In this section, we follow mainly [29–32].

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a map $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the *Γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

[co is reserved for the convex hull in vector spaces.]

A subset X of E is called a *Γ-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be *Γ-convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Examples of abstract convex spaces are given in [29–32].

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \quad \text{for all } A \in \langle D\rangle,$$

then G is called a KKM *map* with respect to F. A KKM *map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A map $F : E \multimap Z$ is said to have the KKM *property* and called *a $\mathfrak{K}$-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{is a } \mathfrak{K}\text{−map}\}.$$

Similarly, a *$\mathfrak{KC}$-map* is defined for closed-valued maps G, and a *$\mathfrak{KO}$-map* for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors [33] use the notation KKM(E, Z) instead of $\mathfrak{KC}(E, Z)$.

Definition. For an abstract convex space $(E, D; \Gamma)$, the KKM *principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$.

A KKM *space* is an abstract convex space satisfying the KKM principle.

In our recent work [30], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed that most of important results therein, that is, Theorems 1–11 and 18–23 in [30], are related to KKM spaces.

Definition. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

We have established a large number of literature on G-convex spaces; see [11–18,20–28] and references therein.

Example 2.1. The following are typical examples of G-convex spaces:

1. Any nonempty convex subset of a t.v.s.
2. Convex spaces due to Lassonde [6].
3. C-spaces (or H-spaces) due to Horvath [7–10]. Hyperconvex metric spaces are particular ones of C-spaces.
4. Hyperbolic spaces due to Reich and Shafrir [34].
5. L-spaces due to Ben-El-Mechaiekh et al. [35,36]. The so-called FC-spaces are particular forms of L-spaces; see [32].
6. A ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consisting of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ for $A \in \langle D\rangle$ with $|A| = n + 1$ and $n \in \mathbb{N} \cup \{0\}$, can be made into a G-convex space [32].

Example 2.2. We give examples of KKM spaces:

1. Recall that $1_X \in \mathfrak{KC}(X, X) \cap \mathfrak{KO}(X, X)$ in a G-convex space $(X, D; \Gamma)$; e.g., see [15,16,21]. Therefore every G-convex space is a KKM space.
2. A connected linearly ordered space is a KKM space; see [31, Theorem 5(i)].
3. The extended long line L^* is a KKM space but not a G-convex space; see [31].

The following are given in [29,30]:

Proposition 2.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, and $F : E \multimap Z$ a map. Then $F \in \mathfrak{K}(E, Z)$ if and only if for any map $G : D \multimap Z$ satisfying*

$$F(\Gamma_N) \subset G(N) \quad \text{for each } N \in \langle D\rangle,$$

we have $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$ for each $N \in \langle D\rangle$.

Remark. If $F \in \mathfrak{KO}(E, Z)$ [resp., $F \in \mathfrak{KC}(E, Z)$], then we have to assume G is open-valued [resp., closed-valued].

Under an additional requirement, we have the whole intersection property for the map-values of a KKM map:

Proposition 2.2. *Let $(E, D; \Gamma)$ be a KKM space, and $G : D \multimap E$ a map satisfying*

(1) *G has closed [resp., open] values,*
(2) *G is a KKM map, and*
(3) *$\bigcap_{z\in M} \overline{G(z)}$ is compact for some $M \in \langle D\rangle$.*

Then we have

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Theorem 2.3. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $S : D \multimap Z$, $T : E \multimap Z$ maps, and $F \in \mathfrak{K}(E, Z)$. Suppose that*

(1) *for each $z \in F(E)$, $\text{co}_\Gamma S^-(z) \subset T^-(z)$; and*
(2) *$F(E) \subset S(N)$ for some $N \in \langle D\rangle$.*

Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Remark. Note that if S has open [resp., closed] values, then we can assume $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KO}(E, Z)$] in Theorem 2.3.

3. Locally convex abstract convex uniform spaces

We need the following:

Definition. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is an abstract convex space with a basis $\mathcal{U}$ of a uniform structure of E.

In this section, we introduce particular subclasses or subsets of abstract convex uniform spaces.

Definition. An abstract convex uniform space $(E \supset D; \Gamma; \mathcal{U})$ is called an *$L\Gamma$-space* if D is dense in E and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] := \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset E$ is Γ-convex.

In particular, for G-convex spaces or C-spaces $(E \supset D; \Gamma; \mathcal{U})$, we can define LG-spaces or LC-spaces ($l.c.$-spaces), resp.

Remark. 1. The LG-spaces are introduced in [20].
2. A singleton is not necessarily Γ-convex in an $L\Gamma$-space.

Example 3.1. For a C-space $(X; \Gamma)$, an $L\Gamma$-space reduces to an LC-space [9,10] (or a locally C-convex space [37]). Any nonempty convex subset X of a locally convex t.v.s. E is an obvious example of an LC-space $(X; \Gamma)$ with $\Gamma_A = \operatorname{co} A$ for $A \in \langle X \rangle$. For other examples, see [9,37].

Example 3.2. A G-convex space $(X \supset D; \Gamma)$ is called an *LG-metric space* if X is equipped with a metric d such that (1) D is dense in X, (2) for any $\varepsilon > 0$, the set $\{x \in X \mid d(x, C) < \varepsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, and (3) open balls are Γ-convex. This concept generalizes that of LC-metric spaces due to Horvath [9].

Example 3.3 (*Horvath [10]*). Any hyperconvex metric space (H, d) is a complete metric LC-space $(H; \Gamma)$. For fixed point theorems on hyperconvex metric spaces, see [38] and references therein.

Definition. An abstract convex uniform space $(E \supset D; \Gamma; \mathcal{U})$ is said to be *locally Γ-convex* if D is dense in E and, for each $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \subset U$ and, for each $x \in E$, $\operatorname{co}_\Gamma(V[x] \cap D) \subset U[x]$, that is,

$$N \in \langle V[x] \cap D \rangle \Rightarrow \Gamma_N \subset U[x].$$

Remark. 1. A particular type of local Γ-convexity was considered by Ben-El-Mechaiekh et al. [36].
2. In particular, if E is Hausdorff and if the U-ball $U[x]$ itself is Γ-convex for each $x \in X$, then $(E \supset D; \Gamma; \mathcal{U})$ is locally Γ-convex. In such case, every singleton is Γ-convex since E is Hausdorff, $\{x\} = \bigcap_{U \in \mathcal{U}} U[x]$ and the intersection of Γ-convex subsets is Γ-convex.

Example 3.4. Any convex subset of a locally convex t.v.s. is a locally Γ-convex space. Note that the concept of local Γ-convexity does not generalize that of local convexity of a subset of a t.v.s.

Note that most of the examples of $L\Gamma$-spaces are locally Γ-convex spaces. Moreover, we have

Proposition 3.1. *Every $L\Gamma$-space $(E \supset D; \Gamma; \mathcal{U})$ is locally Γ-convex if every singleton is Γ-convex.*

Proof. For each symmetric entourage $U \in \mathcal{U}$ and any $x \in E$,

$$U[x] = \{x' \in E \mid (x, x') \in U\} = \{x' \in E \mid x \in U^-[x']\} = \{x' \in E \mid \{x\} \cap U^-[x'] \neq \emptyset\}.$$

Since $\{x\}$ is Γ-convex and $(E \supset D; \Gamma; \mathcal{U})$ is an $L\Gamma$-space, $U[x]$ is Γ-convex. Therefore, $(E \supset D; \Gamma; \mathcal{U})$ is locally Γ-convex. □

4. KKM uniform spaces of the Zima–Hadžić type

We introduce a particular type of subset of abstract convex uniform spaces:

Definition. For an abstract convex uniform space $(E \supset D; \Gamma; \mathcal{U})$, a subset X of E is said to be *of the Zima type* or *of the Zima–Hadžić type* if $D \cap X$ is dense in X and for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that, for each $N \in \langle D \cap X \rangle$ and any Γ-convex subset A of X, we have

$$A \cap V[z] \neq \emptyset \quad \forall z \in N \Rightarrow A \cap U[x] \neq \emptyset \quad \forall x \in \Gamma_N.$$

Example 4.1. (1) Hadžić [39] defined that a nonempty subset K of a t.v.s. E is of the Zima type whenever for any $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ satisfying $\operatorname{co}(V \cap (K - K)) \subset U$, where $\mathcal{V}$ is a neighborhood system of the origin of E.

Note that any nonempty subset of a locally convex t.v.s. is of the Zima type, and that there exists a subset of the Zima type in a non-locally convex t.v.s.; see Hadžić [40,41].

(2) For a C-space, our definition reduces to that of Hadžić [41].

Example 4.2. Motivated by a well-known work of Idzik [42] on convexly totally bounded (c.t.b. for short) sets, Weber [43, 44] defined the following:

A subset K of a t.v.s. E is said to be *strongly convexly totally bounded* (s.c.t.b.) if for every neighborhood V of $0 \in E$ there exist a convex subset C of V and a finite subset N of K such that $K \subset N + C$.

Proposition 4.1 (*Weber [43, Corollary 2.8]*). *Let K be a compact convex subset of a t.v.s. (E, τ) and $F = \text{span}\, K$. Then the following conditions are equivalent:*

(1) *K is s.c.t.b.*
(2) *K is of Zima type.*
(3) *K is locally convex.*
(4) *K is affinely embeddable in a locally convex t.v.s.*
(5) *E admits a Hausdorff locally convex linear topology $\sigma = \sigma(E, E')$, which induces on F a finer topology than τ such that $\sigma|_K = \tau|_K$.*

Proposition 4.2. *For an $L\Gamma$-space $(E \supset D; \Gamma; \mathcal{U})$, any nonempty subset X of E is of the Zima–Hadžić type.*

Proof. Consider the case $U = V \in \mathcal{U}$ in the definition of the Zima–Hadžić type. For every $N \in \langle D \rangle$ and every Γ-convex subset A of X, suppose that $A \cap U[z] \neq \emptyset$ for every $z \in N$. Since

$$N \subset \{x \in D \mid A \cap U[x] \neq \emptyset\}$$

and $\{x \in E \mid A \cap U[x] \neq \emptyset\}$ is Γ-convex by the definition of $L\Gamma$-spaces, we have

$$\Gamma_N \subset \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

and hence $A \cap U[u] \neq \emptyset$ for every $u \in \Gamma_N$. □

For a selfmap of an abstract convex uniform space, we consider the almost fixed point property:

Definition. For an abstract convex uniform space $(E, D; \Gamma; \mathcal{U})$, a point $x \in E$ is called a *U-fixed point* of a map $F : E \multimap E$ if $F(x) \cap U[x] \neq \emptyset$. The map F is said to have the *almost fixed point property* whenever it has a U-fixed point for any $U \in \mathcal{U}$.

We show that a map defined on a KKM uniform space with totally bounded range of Zima type has the almost fixed point property:

Theorem 4.3. *Let $(X \supset D; \Gamma; \mathcal{U})$ be a KKM uniform space and K a totally bounded subset of X such that $D \cap K$ is dense in X. Let $T : X \multimap X$ be a u.s.c. [resp., an l.s.c.] map such that $T(x)$ is Γ-convex and $T(x) \cap K \neq \emptyset$ for each $x \in X$. If $T(X)$ is of the Zima type, then for each $U \in \mathcal{U}$, T has a U-fixed point $x_* \in X$; that is, $T(x_*) \cap U[x_*] \neq \emptyset$.*

Proof. Let $U \in \mathcal{U}$ be open [resp., closed]. Then there exists an open [resp., a closed] member V of $\mathcal{U}$ satisfying the definition of the Zima type. Note that for each $x \in X$, $V[x]$ is a neighborhood of x. Since K is totally bounded and $D \cap K$ is dense in K, there exists an $M := \{y_1, \ldots, y_n\} \in \langle D \cap K \rangle$ such that $K \subset \bigcup_{y \in M} V[y]$.

For each $y_i \in M$, let $F(y_i) := \{x \in X \mid T(x) \cap V[y_i] = \emptyset\}$. Since T is u.s.c. [resp., l.s.c.], each $F(y_i)$ is open [resp., closed]. Moreover, since $T(X) \cap K \subset \bigcup_{i=1}^n V[y_i]$, we have

$$\bigcap_{i=1}^n F(y_i) \subset \left\{ x \in X \mid T(x) \cap \bigcup_{i=1}^n V[y_i] = \emptyset \right\} = \emptyset.$$

Note that $(X \supset M; \Gamma)$ is a KKM space and that $F : M \multimap X$ is not a KKM map. Hence there exist an $N \in \langle M \rangle$ and an $x_U \in \Gamma_N$ such that $x_U \notin F(N) = \bigcup_{y \in N} F(y)$. Hence $T(x_U) \cap V[y] \neq \emptyset$ for all $y \in N$, and

$$N \subset \{y \in D \mid T(x_U) \cap V[y] \neq \emptyset\}.$$

Then

$$x_U \in \Gamma_N \subset \{y \in X \mid T(x_U) \cap U[y] \neq \emptyset\}.$$

Hence $T(x_U) \cap U[x_U] \neq \emptyset$. This completes our proof. □

Remark. 1. Note that in the above proof, if $\Gamma_A \subset D$ for each $A \in \langle D \rangle$, then $x_U \in \Gamma_A \subset D$; and hence it is sufficient to assume that T has Γ-convex values on D, not necessarily on the whole X.

2. Hadžić [39, Theorem 1] is a particular case of our Theorem 4.3.

Corollary 4.4. *Let $(X \supset D; \Gamma; \mathcal{U})$ be a KKM uniform space with a Hausdorff space X and $T : X \multimap X$ a compact u.s.c. map with nonempty closed Γ-convex values. If $T(X)$ is of the Zima–Hadžić type, then T has a fixed point $x_* \in T(x_*)$.*

Proof. By Theorem 4.3, for each $U \in \mathcal{U}$, there exists $x_U, y_U \in X$ such that $y_U \in T(x_U)$ and $y_U \in U[x_U]$. Since $T(X)$ is relatively compact, we may assume that y_U converges to some $x_* \in \overline{T(X)}$. Since X is Hausdorff, x_U also converges to x_*. Since $T(X)$ is regular and T is u.s.c. with closed values, the graph of T is closed in $X \times T(X)$, and hence we have $x_* \in T(x_*)$. This completes our proof. □

Remark. If $(X \supset D; \Gamma; \mathcal{U})$ is an $L\Gamma$-space, then Corollary 4.4 reduces to the following generalization of [20, Theorem 2].

Corollary 4.5. *Let $(X \supset D; \Gamma; \mathcal{U})$ be a Hausdorff KKM $L\Gamma$-space and $T : X \multimap X$ a compact u.s.c. map with nonempty closed Γ-convex values. Then T has a fixed point.*

Proof. Note that $T(X)$ is of the Zima–Hadžić type by Proposition 4.2. Now, the conclusion follows from Corollary 4.4. □

In order to give an example of Theorem 4.3, we introduce a notion due to Himmelberg [45].

A nonempty subset Y of a t.v.s. E is said to be *almost convex* if for any $V \in \mathcal{V}$, where $\mathcal{V}$ is a neighborhood system of the origin 0 in E, and for any finite set $\{y_1, y_2, \dots, y_n\} \subset Y$, there exists a finite set $\{z_1, z_2, \dots, z_n\} \subset Y$ such that $z_i - y_i \in V$ for each $i = 1, 2, \dots, n$ and $\text{co}\{z_1, z_2, \dots, z_n\} \subset Y$.

We give a new example of G-convex spaces:

Lemma 4.6. *Let X be a subset of a t.v.s. E. If X has an almost convex subset Y, then X can be made into a G-convex space.*

Proof. Let $V \in \mathcal{V}$. For any $A = \{y_1, y_2, \dots, y_n\} \in \langle Y \rangle$, there exists a $B = \{z_1, z_2, \dots, z_n\} \in \langle Y \rangle$ such that $z_i - y_i \in V$ for all $i = 1, 2, \dots, n$ and $\text{co}\, B \subset Y \subset X$. Choose one such B for each A. Let $\Gamma : \langle Y \rangle \multimap X$ be defined by $\Gamma_A := \text{co}\, B$ as above. Then $(X \supset Y; \Gamma)$ becomes a G-convex space. □

From Theorem 4.3 with its Remark 1 and Lemma 4.6, we have the following:

Theorem 4.7. *Let X be a subset of a t.v.s. E, Y an almost convex dense subset of X, and K a totally bounded subset of X. Let $T : X \multimap X$ be an l.s.c. [resp., a u.s.c.] map such that $T(y)$ is convex for each $y \in Y$ and $T(x) \cap K \neq \emptyset$ for each $x \in X$. If $T(X)$ is of the Zima–Hadžić type, then for each $V \in \mathcal{V}$, T has a V-almost fixed point $x_* \in X$; that is, $T(x_*) \cap (x_* + V) \neq \emptyset$.*

Proof. From Lemma 4.6, we have a G-convex space $(X \supset Y; \Gamma; \mathcal{U})$ with the uniformity $\mathcal{U}$ such that Y is dense in X. Therefore, Theorem 4.7 follows from Theorem 4.3. □

From Theorem 4.7, we can deduce several fixed point theorems as in [46].

5. Fixed point theorems on Φ-sets

Motivated by Horvath [9,10], we define the following:

Definition. For a given abstract convex space $(E, D; \Gamma)$ and a topological space X, a map $H : X \multimap E$ is called a *Φ-map* (or a *Fan–Browder map*) if there exists a map $G : X \multimap D$ such that

(i) for each $x \in X$, $\text{co}_\Gamma G(x) \subset H(x)$ [that is, $H(x)$ is Γ-convex relative to $G(x)$]; and

(ii) $X = \bigcup\{\text{Int}\, G^-(y) \mid y \in D\}$.

Definition. In $(E, D; \Gamma; \mathcal{U})$, a subset Z of E is called a *Φ-set* if, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : Z \multimap E$ such that $\text{Gr}(H) \subset U$. If E itself is a Φ-set, then it is called a *Φ-space*.

We give examples of Φ-sets as follows:

Example 5.1. Any locally convex subset of a t.v.s. E is a Φ set in E (for example, every nonempty subset of a locally convex t.v.s. and every nonempty subset of a locally convex set). For other nontrivial examples of convex and locally convex subsets, see Hadžić [40]. Moreover, there is an example of a nonconvex, admissible, locally convex subset of a non-locally convex t.v.s.; see Hahn [47].

Example 5.2. Horvath [9] gave examples of Φ-spaces in the class of his C-spaces as follows:

(1) A particular type of uniform space including locally convex t.v.s.

(2) Convex metric spaces in the sense of Takahashi with a metric satisfying certain property.

Example 5.3. A metric G-convex space $(X \supset D; \Gamma)$ is a Φ-space whenever (1) D is dense in X and (2) every open ball is Γ-convex.

We give further examples of Φ-sets:

Proposition 5.1. *For a locally Γ-convex space $(E \supset D; \Gamma; \mathcal{U})$, any nonempty subset X of E is a Φ-set. A locally Γ-convex space $(E \supset D; \Gamma; \mathcal{U})$ is a Φ-space.*

Proof. Let $U \in \mathcal{U}$ and $V \subset U$ as in the definition of locally Γ-convex spaces. We may assume V is open. Define maps $S : X \multimap D$ and $T : X \multimap E$ by

$$S(x) := \{z \in D \mid (x, z) \in V\} \quad \text{and} \quad T(x) := \{y \in E \mid (x, y) \in U\}$$

for $x \in X$. Since D is dense in E, we have $\emptyset \neq S(x) \subset T(x)$ for $x \in X$. For each $x \in X$, $N \in \langle V[x] \cap D\rangle = \langle S(x)\rangle$ implies $\Gamma_N \subset U[x] = T(x)$. Moreover, $S^-(z)$ is open for each $z \in D$ and $X = \bigcup\{S^-(z) \mid z \in D\}$ since D is dense in X. Note that $\mathrm{Gr}(T) \subset U$. Therefore, X is a Φ-set. □

Corollary 5.2. *For an $L\Gamma$-space $(E \supset D;\ \Gamma;\ \mathcal{U})$ such that every singleton is Γ-convex, any nonempty subset X of E is a Φ-set. A locally Γ-convex space $(E \supset D;\ \Gamma;\ \mathcal{U})$ is a Φ-space.*

We show that certain set of the Zima–Hadžić type is a Φ-set as follows:

Proposition 5.3. *Let $(E \supset D;\ \Gamma;\ \mathcal{U})$ be an abstract convex uniform space such that every singleton is Γ-convex. Then any subset X of the Zima–Hadžić type in E is a Φ-set.*

Proof. For each $U \in \mathcal{U}$, let $V \in \mathcal{U}$ be the one satisfying definition of the Zima–Hadžić type. Define $S : X \multimap D$ and $T : X \multimap E$ by

$$S(x) := \{z \in D \mid (z, x) \in V\} \quad \text{and} \quad T(x) := \{y \in E \mid (y, x) \in U\}$$

for $x \in X$. Since $D \cap X$ is dense in X, $S(x)$ is not empty.

We show that $N \in \langle S(x)\rangle$ implies $\Gamma_N \subset T(x)$ for each $x \in X$. In fact, for each $z \in N$,

$$z \in S(x) \Rightarrow x \in V[z] \Rightarrow \{x\} \cap V[z] \neq \emptyset,$$

which implies, for all $y \in \Gamma_N$,

$$\{x\} \cap U[y] \neq \emptyset \Rightarrow x \in U[y] \Rightarrow y \in T(x)$$

since $\{x\}$ is Γ-convex.

Further, since $S(x) \neq \emptyset$ for $x \in X$ and $x \in S^-(z) = V[z]$ for some $z \in D$, we have $X = \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in D\}$.

Therefore, we have a Φ-map $T : X \multimap E$ such that $\mathrm{Gr}(T) \subset U$. This shows that Y is a Φ-set. □

Corollary 5.4. *Any subset of the Zima–Hadžić type in a KKM uniform space $(E \supset D;\ \Gamma;\ \mathcal{U})$ such that every singleton is Γ-convex is a Φ-set.*

Since we have many examples of Φ-sets, the following results in [30] is useful:

Theorem 5.5. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space and $F \in \mathfrak{KC}(E, E)$ a compact map. If $\overline{F(E)}$ is a Φ-set, then F has the almost fixed point property.*

Corollary 5.6. *Under the hypothesis of Theorem 5.5, further if $(E, \mathcal{U})$ is Hausdorff and if F is closed, then it has a fixed point.*

From Corollary 5.6 as in [30], we have the following generalization of the Schauder–Tychonoff–Hukuhara fixed point theorem:

Corollary 5.7. *Let $(E, D; \Gamma; \mathcal{U})$ be a Hausdorff abstract convex uniform space. If $f : E \to E$ is a continuous function such that $\overline{f(E)}$ is a compact Φ-set in E, then f has a fixed point.*

Note that Theorem 5.5 and Corollaries 5.6 and 5.7 contain a large number of known fixed point theorems since there are so many Φ-sets.

6. The Fan–Browder theorems on KKM spaces

In our previous works [15,19], we gave some fixed point theorems for Φ-maps on G-convex spaces. More generally, from Theorem 2.3, we have the following prototype of the Fan–Browder fixed point theorem as in [29,30]:

Theorem 6.1. *Let $(E, D; \Gamma)$ be a KKM space, and $G : E \multimap D$, $H : E \multimap E$ maps satisfying*

(1) *for each $x \in E$, $\mathrm{co}_\Gamma G(x) \subset H(x)$;*
(2) *$E = G^-(N)$ for some $N \in \langle D\rangle$; and*
(3) *G^- has open [resp., closed] values.*

Then H has a fixed point.

Proof. In Theorem 2.3, let $Z := E$, $S := G^-$, $T := H^-$, and $F := 1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$. □

In [19], the G-convex space version of the above theorem is applied to obtain various forms of known Fan–Browder type theorems, the Ky Fan intersection theorem, and the Nash equilibrium theorem.

From Theorem 6.1, we have the following as in [19]:

Theorem 6.2. *Let $(E \supset D; \Gamma)$ be a KKM space and $A : E \multimap E$ be a map such that $A(x)$ is Γ-convex for each $x \in E$. If there exist $z_1, z_2, \ldots, z_n \in D$ and nonempty open [resp., closed] subsets $G_i \subset A^-(z_i)$ for $i = 1, 2, \ldots, n$ such that $E = \bigcup_{i=1}^n G_i$, then A has a fixed point.*

Theorem 6.3. *Let $(E, D; \Gamma)$ be a KKM space and $S : E \multimap D,\ T : E \multimap E$ maps such that*

(1) *for each $x \in E$, $\operatorname{co}_\Gamma S(x) \subset T(x)$; and*
(2) $X = \bigcup\{\operatorname{Int} S^-(z) \mid z \in N\}$ *for some $N \in \langle D \rangle$.*

Then T has a fixed point.

From Theorems 6.1–6.3, most of popular variations or generalizatons of the Fan–Browder theorem (in the forms of the compact or the so-called non-compact versions) can be deduced.

Recall that, from Theorem 6.3, we have the following generalization of the Fan–Browder fixed point theorem:

Theorem 6.4. *Let $(E, D; \Gamma)$ be a compact KKM space (that is, E is compact). Then any Φ-map $T : E \multimap E$ has a fixed point.*

Remark. 1. Note that, in Theorems 6.1–6.4, E is not necessarily Hausdorff.

2. For a compact convex subset $E = D$ of a t.v.s., if $S = T$ and T^- is open-valued, then Theorem 6.4 reduces to the result of Browder in [48].

3. Note that Browder's result is a reformulation of Fan's geometric lemma [49] in the form of a fixed point theorem and its proof was based on the Brouwer fixed point theorem and the partition of unity argument. Since then it is known as the Fan–Browder fixed point theorem.

4. Browder [48] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems.

5. For further developments on generalizations and applications of the Fan–Browder theorem, we refer to Park [11,12,19].

Any binary relation R in a set X can be regarded as a map $T : X \multimap X$ and conversely by the following obvious way:

$$y \in T(x) \quad \text{if and only if } (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a map T if $T(x_0) = \emptyset$.

The Fan–Browder type fixed point theorem is used by Borglin and Keiding [50] and Yannelis and Prabhakar [51] to the existence of maximal elements in mathematical economics.

From Theorem 6.1, we have the following maximal element theorem:

Theorem 6.5. *Let $(E, D; \Gamma)$ be a KKM space and $G : E \multimap D,\ H : E \multimap E$ maps satisfying*

(1) *for each $x \in E$, $\operatorname{co}_\Gamma G(x) \subset H(x)$;*
(2) *$H^-(E) \subset G^-(N)$ for some $N \in \langle D \rangle$;*
(3) *G^- has open [resp., closed] values; and*
(4) *$x \notin H(x)$ for all $x \in E$.*

Then H has a maximal element $\bar{x} \in E$, that is, $H(\bar{x}) = \emptyset$.

Proof. Suppose that $H(x) \neq \emptyset$ for each $x \in E$. Then $E = H^-(E) = \bigcup\{H^-(y) \mid y \in E\}$. By (2), condition (2) of Theorem 6.1 holds. Therefore, by Theorem 6.1, H has a fixed point. This violates (4). $\square$

For a G-convex space, Theorem 6.5 generalizes results in [15].

7. The class $\mathfrak{B}$ of multimaps

Let $(E, D; \Gamma)$ be an abstract convex space, X a nonempty subset of E, and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

Remark. 1. Until now, the class $\mathfrak{B}$ is defined for G-convex spaces $(E, D; \Gamma)$, where, for each $N \in \langle D\rangle$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$.

2. A ϕ_A-space $(E, D; \{\phi_A\}_{A\in\langle D\rangle})$ is an abstract convex space $(E, D; \Gamma)$, where $\Gamma_N := \phi_N(\Delta_n)$ for each $N \in \langle D\rangle$ with $|N| = n + 1$, and hence there is a continuous function $\phi_N : \Delta_n \to \Gamma_N$. This $(E, D; \Gamma)$ is not necessarily a G-convex space.

We give some subclasses of $\mathfrak{B}$ as follows:

Example 7.1. For topological spaces X and Y, an *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $F : X \multimap Y$ is one such that, for each nonempty compact subset K of X, there exists a map $G \in \mathfrak{A}_c(K, Y)$ satisfying $G(x) \subset F(x)$ for all $x \in K$; where $\mathfrak{A}_c$ consists of finite compositions of maps in a class $\mathfrak{A}$ of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $T \in \mathfrak{A}_c$ is u.s.c. with nonempty compact values; and

(iii) for any polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Here, a polytope P is a homeomorphic image of a standard simplex. There are lots of examples of $\mathfrak{A}$ and $\mathfrak{A}_c^\kappa$; see [5,52–54,11,14,15,25–28].

Subclasses of the admissible class $\mathfrak{A}_c^\kappa$ are classes of continuous functions $\mathbb{C}$, Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), Aronszajn maps $\mathbb{M}$ (with R_δ values), acyclic maps $\mathbb{V}$ (with acyclic values), Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), Fan–Browder maps (Φ-maps), locally selectionable maps $\mathbb{L}$ having convex values, u.s.c. approachable maps $\mathbb{A}$ (whose domains are uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, Simons maps $\mathbb{K}_c$, the class $\mathbb{K}_c^+$ of Lassonde, the class $\mathbb{V}_c^+$ of Park et al., approximable maps of Ben-El-Mechaiekh and Idzik, and many others.

Note that for a G-convex space $(X, D; \Gamma)$ and any space Y, an admissible class $\mathfrak{A}_c^\kappa(X, Y)$ is a subclass of $\mathfrak{B}(X, Y)$ with some possible exceptions such as Kakutani maps. Some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ were known [53]. Note that the connectivity map due to Nash and Girolo is also such an example; see [54].

Example 7.2. For a convex space $(X \supset D; \Gamma)$, where Γ = co and ϕ_N is a homeomorphism, the class $\mathfrak{B}(X, Y)$ is originally given in [52] and investigated in [54].

Example 7.3. For a convex space X and a topological space Y, motivated by [5], Chang and Yen [33] defined the class of maps $T : X \multimap Y$ having the KKM property as follows:

$T \in \mathrm{KKM}(X, Y) \iff$ the family $\{S(x) \mid x \in X\}$ has the finite intersection property whenever $S : X \multimap Y$ has closed values and $T(\mathrm{co}\, N) \subset S(N)$ for each $N \in \langle X\rangle$.

For a convex space X and a Hausdorff space Y, it is known that $\mathfrak{A}_c^\kappa(X, Y) \subset \mathrm{KKM}(X, Y)$ [5] and we observed that two subclasses $\mathfrak{B}$ and KKM coincide in the class of all compact closed maps $T : X \multimap Y$ [52].

Generalizations of the class KKM to G-convex spaces are possible; see [55].

Note that those are particular examples of the class $\mathfrak{KC}$.

Example 7.4. Ben-El-Mechaiekh et al. [35,36] introduced the class $\mathbb{A}$ of approachable maps as follows:

Let X and Y be uniform spaces (with respective bases $\mathcal{U}$ and $\mathcal{V}$ of symmetric entourages). A map $T : X \multimap Y$ is said to be *approachable* whenever T admits a continuous W-approximative selection $s : X \to Y$ for each W in the basis $\mathcal{W}$ of the product uniformity on $X \times Y$; that is, $\mathrm{Gr}(s) \subset W[\mathrm{Gr}(F)]$, where

$$W[A] := \bigcup_{z\in A} W[z] = \{z' \in X \times Y \mid W[z'] \cap A \neq \emptyset\}$$

for any $A \subset X \times Y$, and

$$W[z] := \{z' \in X \times Y \mid (z, z') \in W]$$

for $z \in X \times Y$.

A map $T : X \multimap Y$ is said to be *approximable* if its restriction $T|_K$ to any compact subset K of X is approachable.

It is known that if $(X \supset D; \Gamma)$ is a G-convex uniform space and Y is a uniform space, then any compact closed approachable map $F : X \multimap Y$ belongs to $\mathfrak{B}(X, Y)$; see [18, Lemma 3].

We give some examples of approachable maps $T : X \to Y$ as follows:

(1) Any selectionable map is approximable.

(2) A locally selectionable map T with convex values is approximable whenever Y is a convex subset of a t.v.s.

(3) A u.s.c. map T with nonempty convex values is approachable whenever X is paracompact and Y is a convex subset of a locally convex t.v.s.

(4) A u.s.c. map T with compact contractible values is approachable whenever X is a finite polyhedron.

(5) A u.s.c. map T with nonempty compact values having a trivial shape (that is, contractible in each neighborhood in Y) is approachable whenever X is a finite polyhedron.

For (1) and (2), see [23]; and for (3)–(5), see [35]. The following is recently known from (5):

Example 7.5. Let $(X, D; \Gamma)$ be a G-convex space and $F : X \multimap X$ a u.s.c. map. If F has nonempty compact values having a trivial shape, then $F \in \mathfrak{B}(X, X)$.

Example 7.6. An important subclass of $\mathfrak{B}$ is the class of Φ-maps (or Fan–Browder maps) as follows:

For Φ-maps on G-convex spaces, we obtain the following selection theorem in view of [9,10]; [12, Lemma 1]:

Lemma 7.1. *Let Y be a normal space, $(X, D; \Gamma)$ a G-convex space, and $S : Y \multimap D$ a map such that $Y = \bigcup\{\operatorname{Int} S^-(z) \mid z \in N\}$ for some $N \in \langle D\rangle$. Then there exists a continuous function $s : Y \to \Gamma_N$ such that $s(y) \in \Gamma(N \cap S(y))$ for all $y \in Y$. In fact, if $|N| = n + 1$, then $s = \phi_N \circ p$, where $\phi_N : \Delta_n \to \Gamma_N$ and $p : Y \to \Delta_n$ are continuous functions.*

Note that Lemma 7.1 sharpens the compact case of [12, Theorem 3.2] and shows that every Φ-map $T : Y \multimap X$ belongs to $\mathbb{C}_c^\kappa(Y, X) \subset \mathfrak{A}_c^\kappa(Y, X)$. Therefore, if $X = Y$, then a Φ-map $T : X \multimap X$ belongs to $\mathfrak{B}(X, X)$.

8. Admissible abstract convex spaces

We introduce particular types of subsets of abstract convex uniform spaces adequate to establish our fixed point theory:

Definition. For an abstract convex uniform space $(E, D; \Gamma; \mathcal{U})$, a subset X of E is said to be *admissible* (in the sense of Klee) if, for each nonempty compact subset K of X and for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to X$ satisfying

(1) $(x, h(x)) \in U$ for all $x \in K$;
(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D\rangle$; and
(3) there exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ with $|N| = n + 1$ such that $h = \phi_N \circ p$.

Example 8.1. A nonempty subset X of a t.v.s. E is said to be admissible (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p and $L^p(0, 1)$ for $0 < p < 1$, the space $S(0, 1)$ of equivalence classes of measurable function on $[0, 1]$, the Hardy spaces H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarreled t.v.s. admitting Schauder basis, and others. Note also that any locally convex subset of an F-normable t.v.s. or any locally convex subset which is a finite union of closed convex subsets of a t.v.s. is admissible. For details, see Hadžić [40], Weber [43,44], Hahn [47] and references therein.

For more general purposes, we introduce a generalized version of our previous definition of the admissibility of domains of maps by switching it to the Klee approximability of their ranges, as follows:

Definition. Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to E$ satisfying conditions (1)–(3) in the preceding definition. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K) \subset \Gamma_N \subset X$ for some $N \in \langle D\rangle$ in condition (2).

Example 8.2. In a t.v.s. E, we give some examples of Klee approximable sets as follows:

(1) A subset X of E is admissible (in the sense of Klee) iff every compact subset K of X is Klee approximable into E.
(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.
(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.
(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.
(5) Any compact subset K of an admissible almost convex subset X of a t.v.s. is Klee approximable into X.
(6) Any compact subset of a Φ-space in a t.v.s. is Klee approximable.
(7) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.

Note that (7) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3).

We give another example of subclasses of admissible abstract convex spaces:

Proposition 8.1. *Every G-convex Φ-space $(X, D; \Gamma; \mathcal{U})$ is admissible. More precisely, for a G-convex uniform space $(X, D; \Gamma; \mathcal{U})$, every nonempty compact Φ-subset K of X is Klee approximable.*

Proof. Since K is a Φ-set, for each $U \in \mathcal{U}$, there exist multimaps $S : K \multimap D$ and $T : K \multimap X$ such that

(i) for each $y \in K$, $M \in \langle S(y)\rangle$ implies $\Gamma_M \subset T(y)$;
(ii) $K = \bigcup\{\operatorname{Int} S^-(z) \mid z \in D\}$; and
(iii) $\operatorname{Gr}(T) \subset U$.

Since K is compact, it follows from Lemma 7.1 that T has a continuous selection $h : K \to X$ such that
(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$ with $|N| = n + 1$; and
(3) there exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ such that $h = \phi_N \circ p$.
Moreover, $h(x) \in T(x)$ for all $x \in K$ implies
(1) $(x, h(x)) \in \mathrm{Gr}(T) \subset U$ for all $x \in K$.
Therefore, K is Klee approximable and hence $(X, D; \Gamma; \mathcal{U})$ is admissible. □

The following summarizes the mutual relations among the various subclasses of abstract convex uniform spaces:

Theorem 8.2. *In the class of abstract convex uniform spaces $(X, D; \Gamma; \mathcal{U})$, the following hold:*

(1) *Any $L\Gamma$-space is of the Zima–Hadžić type.*
(2) *Every nonempty subset of an $L\Gamma$-space is locally Γ-convex whenever every singleton is Γ-convex.*
(3) *Any nonempty subset of a locally Γ-convex space is a Φ-set.*
(4) *Any Zima–Hadžić type subset of an abstract convex uniform space such that every singleton is Γ-convex is a Φ-set.*
(5) *Every G-convex Φ-space is admissible. More generally, every nonempty compact Φ-subset of a G-convex space is Klee approximable.*

Recall that (1)–(4) follow from Propositions 4.2, 3.1, 5.2 and 5.4, respectively. (5) is a restatement of Proposition 8.1.
We have the following main result in this section:

Theorem 8.3. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space, $X \subset Y$ subsets of E, and $F : Y \multimap Y$ a map such that $F|_X \in \mathfrak{B}(X, Y)$ and $F(X)$ is Klee approximable into X. Then F has the almost fixed point property.*
Further if $(E, \mathcal{U})$ is Hausdorff, F is closed, and $\overline{F(X)}$ is compact in Y, then F has a fixed point $x_0 \in Y$ (that is, $x_0 \in F(x_0)$).

Proof. Since $K := F(X)$ is a Klee approximable into X, for each symmetric entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to X$ satisfying conditions (1) - (3) of the definition of Klee approximable subsets, and we have

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} K \xrightarrow{p} \Delta_n$$

for some $N \in \langle D \rangle$ with $|N| = n + 1$ and $\Gamma_N \subset X$. Let $p' := p|_{F(\Gamma_N)}$. Since $F|_X \in \mathfrak{B}(X, Y)$, the composition $p' \circ (F|_{\Gamma_N}) \circ \phi_N : \Delta_n \multimap \Delta_n$ has a fixed point $a_U \in \Delta_n$. Let $x_U := \phi_N(a_U)$. Then

$$a_U \in (p' \circ F \circ \phi_N)(a_U) = (p' \circ F)(x_U)$$

and hence

$$x_U = \phi_N(a_U) \in (\phi_N \circ p' \circ F)(x_U).$$

Since $h = \phi_N \circ p$ by definition, we have

$$x_U = h(y_U) \quad \text{for some } y_U \in (F|_{\Gamma_N})(x_U).$$

Therefore, for each entourage $U \in \mathcal{U}$, there exist points $x_U \in X$ and $y_U \in F(x_U)$ such that $(x_U, y_U) = (h(y_U), y_U) \in U$. So, for each U, there exist $x_U, y_U \in X$ such that $y_U \in F(x_U)$ and $y_U \in U[x_U]$.

Now suppose that F is closed and $\overline{F(X)}$ is compact. Since $F(X)$ is relatively compact, we may assume that the net y_U in $F(X)$ converges to some $x_0 \in \overline{F(X)}$. Since $(x_U, y_U) \in U$ for each $U \in \mathcal{U}$, by the Hausdorffness of E, the net x_U also converges to x_0. Since the graph of F is closed in $Y \times Y$ and $(x_U, y_U) \in \mathrm{Gr}(F)$, we have $(x_0, y_0) \in \mathrm{Gr}(F)$ and hence we have $x_0 \in F(x_0)$. This completes our proof. □

Note that, by choosing particular subclass of multimaps or particular types of abstract convex space, we can deduce a large number of known or new fixed point theorems from Theorem 8.3.

For $X = E$, Theorem 8.3 reduces to the following generalization of the main results of [22,24] for G-convex spaces:

Theorem 8.4. *Let $(X, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space and $F \in \mathfrak{B}(X, X)$ a multimap such that $F(X)$ is Klee approximable. Then F has the almost fixed point property.*
Further if F is closed and compact, then F has a fixed point $x_0 \in X$.

Theorem 8.5. *Let $(X, D; \Gamma; \mathcal{U})$ be an admissible abstract convex uniform space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Proof. Note that $\overline{F(X)}$ is a compact subset of an admissible KKM space, and hence is Klee approximable. Therefore, Theorem 8.4 works. □

Corollary 8.6. *Let $(X, D; \Gamma; \mathcal{U})$ be a compact admissible abstract convex uniform space. Then any map $F \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

Since an admissible convex subset of a t.v.s. is an admissible abstract convex space, we have the following from Theorem 4.3:

Corollary 8.7. *Let X be an admissible convex subset of a t.v.s. E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Corollary 8.7 was given in [54], where we listed more than sixty papers in chronological order, from which we could deduce particular forms of Corollary 8.7. Especially, from Corollary 8.7, we obtain

Corollary 8.8. *Let X be an admissible convex subset of a t.v.s. E. Then any compact map $F \in \mathbb{V}_c(X, X)$ (that is, a finite composition of acyclic maps) has a fixed point.*

Corollary 8.8 was given in [56,57] and applied to a Simons type cyclic coincidence theorems for acyclic maps, the von Neumann type intersection theorems for graphs of compact compositions of acyclic maps, the Nash type equilibrium theorems, saddle point or minimax theorems, quasi-equilibrium problems, and quasi-variational inequalities, where most of related convexity were replaced by acyclicity.

9. Historical remarks

A large number of fixed point theorems in t.v.s. and in G-convex spaces are unified and generalized in the present paper and [54,58,19,23,24]. For the references, see mainly [4,54,11,58].

1. The celebrated Brouwer fixed point theorem in 1912 was generalized by Schauder (1930), Tychonov (1935), Hukuhara (1950), and Fan (1964). The Kakutani fixed point theorem in 1941 was generalized by Bohnenblust and Karlin (1950), Fan (1952), Glicksberg (1952), Himmelberg (1972), Granas and Liu (1986), and Park (1988). These are all for compact Kakutani maps on convex subsets of particular types of t.v.s., and particular forms of our Corollary 8.8.

2. Motivated by a work of Zima (1977) on paranormed spaces (not necessarily locally convex), Rzepecki (1979) obtained a theorem for a compact continuous self-function whose range is locally convex. Later Hadzic (1981-87) introduced sets of the Zima type and obtained several fixed point theorems for compact Kakutani maps whose ranges are of the Zima type; see [59,39,60,40,61,41]. These are generalized by Corollary 4.4.

3. The Fan–Browder fixed point theorem (1968) in [48] has numerous generalizations and applications in the KKM theory and equilibria theory. Theorems 6.1–6.5 and others in [19] are examples of most general forms of the Fan–Browder theorem.

4. Fixed point theorems for maps in t.v.s. were further generalized by Corollary 8.7, which appeared first in [54] and includes earlier results due to O'Neill (1957), Schaefer (1959), Nikaido (1959), Klee (1960), Powers (1970), Hahn and Pötter (1974), Krauthausen (1976), and others; see [54].

5. Since the class $\mathfrak{B}$ contains a large number of subclasses of maps, Theorem 8.4 extends numerous results in Sections 1 and 4, and the first half of Section 2. Moreover, certain results on C-spaces due to Horvath [7,8] and Hadžić [59,39,60,40,61, 41], on L-spaces due to Ben-El-Mechaiekh et al. [35,36], and others are included in Theorem 8.5.

6. Most of the results in this paper reduce to the corresponding ones in [24] for G-convex spaces.

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