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Remarks on KKM Maps and Fixed Point Theorems in Generalized Convex Spaces

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ABSTRACT

Various types of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ are simply G-convex spaces. Various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. As such examples, we obtain KKM type theorems and a very general fixed point theorem on ϕ_A-spaces.

RESUMEN

Varios tipos de ϕ_A-espacios $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ son simplemente espacios G-convexos. Varios tipos de aplicaciones KKM generalizadas sobre ϕ_A-espacios son aplicaciones simplemente KKM sobre espacios G-convexos. Por lo tanto, nuestra teoria de espacios G-Convexos puede ser aplicada a varios tipos de ϕ_A-espacios. Como ejemplo obtenemos teoremas do tipo KKM y un teorema general de punto fijo sobre ϕ_A-espacios.

Key words and phrases: *Abstract convex space, generalized (G-) convex space, ϕ_A-space, L-spaces, FC-spaces, property (H); H-condition.*

Math. Subj. Class.: *47H04, 47H10, 49J27, 49J35, 54H25, 91B50.*

1 Introduction

The KKM theory, first called by the author, is the study on applications of equivalent formulations of the KKM principle due to Knaster, Kuratowski, and Mazurkiewicz. The KKM principle provides the foundations for many of the modern essential results in diverse areas of mathematical sciences.

Since 1993, the author has initiated the study of the KKM theory on generalized convex spaces (or G-convex spaces) $(X, D; \Gamma)$ as a common generalization of various general convexities without linear structures due to other authors. We have established within such a frame the foundations of the KKM theory, as well as fixed point theorems and many other equilibrium results for multimaps. This direction of study has been followed by a number of other authors.

In the last decade, some authors who introduced spaces of the form $(X, \{\varphi_A\})$ having a family $\{\varphi_A\}$ of continuous functions defined on simplices claimed that such spaces generalize G-convex spaces without giving any justifications or proper examples. In fact, a number of modifications or imitations of the G-convex spaces have followed; for example, L-spaces due to Ben-El-Mechaiekh et al. [1], spaces having property (H) due to Huang [10], FC-spaces due to Ding [6,7], convexity structures satisfying the H-condition [22], and others. Some authors also tried to generalize the KKM principle for their own settings. They introduced various types of generalized KKM maps; for example, generalized KKM maps on L-spaces [5,20], generalized R-KKM maps [2,8,9], and many others.

In order to destroy such inadequate concepts and to upgrade the KKM theory, recently, we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [15-18]. Moreover, we noticed that all spaces of the form $(X, \{\varphi_A\})$ can be unified to ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ or spaces having a family $\{\phi_A\}_{A\in\langle D\rangle}$ of singular simplices.

In the present note, we show that various types of ϕ_A-spaces $(X, D;\ \{\phi_A\}_{A\in\langle D\rangle})$ are simply G-convex spaces, and various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. As such examples, we obtain KKM type theorems and a very general fixed point theorem on ϕ_A-spaces.

2 Abstract convex spaces

In this section, we follow mainly [15,16]. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of nonempty sets E, D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

Examples. In [15-17], we gave plenty of examples of abstract convex spaces. Here we give only two classes of them as follows:

1. A *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see [21], where the bibliography lists 283 papers.

2. A *generalized convex space* or a *G-convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is a standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For G-convex spaces; see [11-14,19] and references therein.

From now on, in an abstract convex space $(E, D; \Gamma)$, E is assumed to be a topological space.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.

The following is the origin of the KKM theory; see [11,12].

The KKM Principle. *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map (that is, $\mathrm{co}\, A \subset G(A)$ for each $A \subset D$) with closed [resp., open] values. Then $\bigcap_{z \in D} G(z) \neq \emptyset$.*

3 ϕ_A-spaces

Recently, there have appeared authors in [2,3,6-10,20,22] and others who introduced spaces of the form $(X, \{\varphi_A\})$. Some of them tried to rewrite some results on G-convex spaces by simply replacing $\Gamma(A)$ by $\varphi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples.

Motivated by this fact, we are concerned with a reformulation of the class of G-convex spaces as follows [17]:

Definition. A ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

Any G-convex space is a ϕ_A-space. The converse also holds:

Theorem 1. *A ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ can be made into a G-convex space $(X, D; \Gamma)$.*

Proof. This can be done in two ways.

(1) For each $A \in \langle D\rangle$, by putting $\Gamma_A := X$, we obtain a trivial G-convex space $(X, D; \Gamma)$.

(2) Let $\{\Gamma^\alpha\}_\alpha$ be the family of maps $\Gamma^\alpha : \langle D\rangle \multimap X$ giving a G-convex space $(X, D; \Gamma^\alpha)$. Note that, by (1), this family is not empty. Then, for each α and each $A \in \langle D\rangle$ with $|A| = n + 1$, we have

$$\phi_A(\Delta_n) \subset \Gamma^\alpha_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma^\alpha_J \quad \text{for} \quad J \subset A.$$

Let $\Gamma := \bigcap_\alpha \Gamma^\alpha$, that is, $\Gamma_A := \bigcap_\alpha \Gamma^\alpha_A$ for each $A \in \langle D\rangle$. Then

$$\phi_A(\Delta_n) \subset \Gamma_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma_J \quad \text{for} \quad J \subset A.$$

Therefore, $(X, D; \Gamma)$ is a G-convex space.

Consequently, G-convex spaces and ϕ_A-spaces are essentially the same.

Definition. For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map*.

Theorem 2. (1) *A KKM map $G : D \multimap X$ on a G-convex space $(X, D; \Gamma)$ is a KKM map on the corresponding ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$.*

(2) *A KKM map $T : D \multimap X$ on a ϕ_A-space $(X, D; \{\phi_A\})$ is a KKM map on a new G-convex space $(X, D; \Gamma)$.*

Proof. (1) This is clear from the definition of a KKM map on a G-convex space.

(2) Define $\Gamma : \langle D\rangle \multimap X$ by $\Gamma_A := T(A)$ for each $A \in \langle D\rangle$. Then $(X, D; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n+1$, we have a continuous function $\phi_A : \Delta_n \to T(A) =: \Gamma(A)$

such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle D \rangle$ and hence $T : D \multimap X$ is a KKM map on a G-convex space $(X, D; \Gamma)$.

The following is a KKM theorem for ϕ_A-spaces. The proof is just a simple modification of the corresponding one in [12,13,19]:

Theorem 3. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, let $G : D \multimap X$ be a KKM map with closed [resp., open] values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D \rangle$ with $|N| = n + 1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$).*

Further, if

(3) *$\bigcap_{z\in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

Proof. Let $N = \{z_0, z_1, \dots, z_n\}$. Since G is a KKM map, for each vertex e_i of Δ_n, we have $\phi_N(e_i) \in G(z_i)$ for $0 \le i \le n$. Then $e_i \mapsto \phi_N^{-1} G(z_i)$ is a closed [resp., open] valued map such that $\Delta_k = \text{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1} G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the original KKM principle, $\Delta_n \supset \bigcap_{i=0}^{n} \phi_N^{-1} G(z_i) \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \left(\bigcap_{z\in N} G(z)\right) \neq \emptyset$.

The second conclusion is clear.

Remarks. (1) We may assume that, for each $a \in D$ and $N \in \langle D \rangle$, $G(a) \cap \phi_N(\Delta_n)$ is closed [resp., open] in $\phi_N(\Delta_n)$. This is said by some authors that G has finitely closed [resp., open] values. However, by replacing the topology of X by its finitely generated extension, we can eliminate "finitely"; see [13].

(2) For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \text{co}$, the convex hull, Theorem 3 reduces to the original KKM principle and its open version; see [11,12].

(3) If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 3 extends Fan's KKM lemma; see [11,12].

(4) Note that any KKM theorem on spaces of the form $(X, \{\varphi_A\})$ can not generalize the original KKM principle or Fan's KKM lemma.

4 Examples of ϕ_A-spaces

In this section, we give some examples of spaces of the form $(X, \{\varphi_A\})$ given by other authors:

(I) In 1998, Ben-El-Mechaiekh et al. [1] defined an L-space (E, Γ), which is a particular form of our G-convex space $(X, D; \Gamma)$ for the case $E = X = D$. Some authors incorrectly claimed that the class of L-spaces contains our class of G-convex spaces; for example, [4,5], which contain a number of particular results (with certain defects) of known ones.

(II) In 2003, the authors of [20] considered the L-space.

(III) [10] A topological space Y is said to have property (H) if, for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$.

(IV) [6,7] $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. This definition appears in a large number of papers of the same author and his followers. Note that for each N, there should be infinitely many φ_N's.

The author of [6,7] wrote in more than one dozen papers that: "It is easy to see that the class of FC-spaces includes the classes of convex sets in topological vector spaces, C-spaces (or H-spaces) [20], G-convex spaces, L-convex spaces [1], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces." There he failed to give any justification or any proper example of his space which is not G-convex. One wonders how could a pair $(Y, \{\varphi_N\})$ generalize a triple $(X, D; \Gamma)$.

(V) In [22], a pair $(Y, \mathcal{C})$ is introduced, where Y is a topological space and $\mathcal{C}$ is a family of subsets of Y such that $(Y, \mathcal{C})$ is similar to the convexity space in the classical sense.

A pair $(X, \mathcal{C})$ is said to have the selection property with respect to a topological space S if every multimap $F : S \multimap X$ admits a single-valued continuous selection whenever F is lower semicontinuous and nonempty closed convex valued.

A pair $(Y, \mathcal{C})$ is said to satisfy H-condition if $\mathcal{C}$ has the following property:

(H) For each finite subset $\{y_0, \dots, y_n\} \subset Y$, there exists a continuous mapping $f : \Delta_N \to \overline{\text{conv}}\{y_0, \dots, y_n\}$, where Δ_N is the standard n-simplex, such that $f(\Delta_J) \subset \overline{\text{conv}}\{y_j : j \in J\}$ for each nonempty subset $J \subset N = \{0, 1, \dots, n\}$, where $\overline{\text{conv}}$ denotes the closed convex hull.

For these definitions, we note the following remarks:

(i) A pair $(Y, \mathcal{C})$ is a particular form of our abstract convex space $(E, D; \Gamma)$ with $Y = E = D$ and $\Gamma_A := \text{conv}(A) = \bigcap\{B \in \mathcal{C} \mid A \subset B\}$ for $A \in \langle Y \rangle$. Then $(Y, \mathcal{C})$ becomes our abstract convex space $(Y; \Gamma)$.

(ii) The selection property would be better to call the Michael selection property.

(iii) A pair $(Y, \mathcal{C})$ satisfying the H-condition is a particular form of our G-convex space $(X, D; \Gamma)$ with $Y = X = D$ such that Γ is closed-valued.

The following new result gives an example of ϕ_A-spaces:

Theorem 4. *If an abstract convex space $(E, D; \Gamma)$ has the Michael selection property with respect to a simplex and if Γ is closed-valued, then we have a ϕ_A-space*

$$(E, D; \{\phi_A\}_{A \in \langle D \rangle}).$$

Corollary 4.1. [22, Theorem 1] *If a pair $(Y, \mathcal{C})$ has the selection property with respect to any simplex, then the pair satisfies the H-condition.*

In fact, just following the proof of [22, Theorem 1], we can easily deduce the more general Theorem 4.

Moreover, in [22], several results on the pairs $(Y, \mathcal{C})$ satisfying the H-condition are obtained. Some of such results are particular forms of known results on G-convex spaces.

5 Various KKM maps

A number of authors tried to generalize the concept of KKM maps on particular forms of ϕ_A-spaces. In this section, we show that all of them are particular forms of our KKM maps.

(I) In 2003 [20, Definition 2], for an L-space (X, Γ) and a topological space Y, a correspondence $G : Y \multimap X$ is called a *generalized KKM-correspondence*, if for all $A = \{y_0, y_1, \ldots, y_n\} \in \langle Y \rangle$, there exists a subset $B = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, such that for all $J \subseteq \{0, 1, \ldots, n\}$, it is satisfied that $\phi_B(\Delta_J) \subseteq \bigcup_{j \in J} G(y_j)$.

Note that a generalized KKM-correspondence becomes simply our KKM map on a ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ by putting $D := Y$ and, for any $A \in \langle D \rangle$, by defining $\phi_A(\Delta_{|A|-1}) := \phi_B(\Delta_{|B|-1})$ for $B \in \langle X \rangle$ corresponding to A.

(II) In 2003 [2, Definition 2.1], for a nonempty set X and a topological space Y, $T : X \to 2^Y$ is said to be generalized relatively KKM (R-KKM) mapping if for any $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, there exists a continuous mapping $\phi_N : \Delta_n \to Y$ such that, for each $e_{i_0}, e_{i_1}, \cdots, e_{i_k}$,

$$\phi_N(\Delta_k) \subset \bigcup_{j=0}^{k} T x_{i_j},$$

where Δ_k is a standard k-simplex of Δ_n with vertices $e_{i_0}, e_{i_1}, \cdots, e_{i_k}$.

For a ϕ_A-space $(Y, X; \{\phi_N\}_{N \in \langle X \rangle})$, $T : X \to 2^Y$ is simply a KKM map.

(III) Let X be a nonempty set and Y be a topological space with property (H). In 2005 [10], $T : X \to 2^Y$ is said to be a generalized R-KKM mapping if for each $\{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ such that

$$\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

for all $\{i_0, \dots, i_k\} \subset \{0, \dots, n\}$.

Similarly to (II), a generalized R-KKM map $T : X \to 2^Y$ is simply a KKM map for the ϕ_A-space $(Y, X; \{\phi_A\}_{A \in \langle X \rangle})$.

The author of [8] claimed as follows: "The above class of generalized *R-KKM* mappings includes those classes of *KKM* mappings, *H-KKM* mappings, *G-KKM* mappings, generalized *G-KKM* mappings, generalized *S-KKM* mappings, *GLKKM* mappings and *GMKKM* mappings defined in topological vector spaces, H-spaces, G-convex spaces, *G-H*-spaces, L-convex spaces and hyperconvex metric spaces, respectively, as true subclasses." This is partially incorrect.

In view of this claim and Theorem 2, so many variants of KKM type theorems in [2-10,20,22] and a large number of other papers can be reduced to the ones in our G-convex space theory. We should recognize that, in the KKM theory on G-convex spaces, every argument is related to the finite intersection property of functional values of KKM maps having closed [resp., open] values, in other words, related to some $N \in \langle D \rangle$ in $(X, D; \Gamma)$.

(IV) Motivated by a large number of recent works on generalized KKM maps, we introduced the following definition in [19]: Let (X, D, Γ) be a G-convex space and I a nonempty set. A map $F : I \multimap X$ is called a *generalized KKM map* provided that for each $N \in \langle I \rangle$, there exists a function $\sigma : N \to D$ such that $\Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle N \rangle$.

In [19], a unified account on results for such maps was given; for example, the KKM type theorem, characterizations of such maps, an equilibrium theorem implying minimax inequalities, variational inequalities, and so on.

A little later than [19], similar results appeared in [4,5], which has trivial defects in certain aspects.

6 Various KKM type theorems

For particular forms of G-convex spaces, some authors obtained KKM type theorems or equivalents which can not be applicable even to the KKM principle for $(\Delta_n, V; \mathrm{co})$ or to the Ky Fan lemma for $(X \supset D; \mathrm{co})$, where X is a topological vector space.

In this section, we give two KKM type theorems which improve corresponding ones in [2,20]:

Theorem 5. *Let X be a topological space, D a nonempty set, and $G : D \multimap X$ a map such that*

(1) *G is transfer closed-valued [that is, $\bigcap_{z\in D}\overline{G(z)} = \bigcap_{z\in D} G(z)$];*

(2) *there exists $a^* \in Y$ with $\overline{G(a^*)}$ compact.*

Then, there exists a G-convex space $(X, D; \Gamma)$ such that G is a KKM map if and only if $\bigcap_{z\in D} G(z) \neq \emptyset$.

Proof. (Necessity) Follows from Theorem 3.

(Sufficiency) Choose an $x^* \in \bigcap_{z\in D} G(z) \neq \emptyset$. Define a map $\Gamma : \langle D\rangle \multimap X$ given by the constant function $\Gamma(A) = \{x^*\}$ for all $A \in \langle D\rangle$ with $|A| = n+1$, and a function $\phi_A : \Delta_n \to \Gamma(A)$ by $\phi_A(\lambda) = x^*$ for all $\lambda \in \Delta_n$. Then it is easy to verify that, with this G-convex space $(X, D; \Gamma)$, G is a KKM map.

Corollary 5.1. [20, Theorem 1] *Let X and Y be topological spaces and $\Gamma : Y \to X$ a transfer closed-valued correspondence on Y such that there exists $y^* \in Y$ with $cl[\Gamma(y^*)]$ compact. Then, there exists an L-structure on X such that Γ is a generalized KKM-correspondence if and only if $\bigcap_{y\in Y}\Gamma(y) \neq \emptyset$.*

Recall that several generalizations of [20, Theorem 1] already appeared in [19].

Theorem 6. *For a ϕ_A-space $(Y, D; \{\phi_N\}_{N\in\langle D\rangle})$, let $T : D \to 2^Y$ be a map such that $T(z)$ is nonempty and closed for each $z \in D$.*

(i) *If T is a KKM map, then for each $N \in \langle D\rangle$ with $|N| = n+1$,*

$$\phi_N(\Delta_n) \cap \bigcap_{x\in N} T(x) \neq \emptyset.$$

(ii) *If the family $\{T(z) \mid z \in Z\}$ has finite intersection property, then T is a KKM map.*

Proof. (i) Apply Theorem 3.

(ii) Just follow the sufficiency part of Theorem 5.

The following is the key result in [2] with almost a page proof:

Corollary 6.1. [2, Theorem 3.1] *Let X be a nonempty set and Y be a topological space. Let $T : X \to 2^Y$ be a set-valued mapping such that $T(x)$ is nonempty and compactly closed in Y for each $x \in X$.*

(i) *If T is a generalized R-KKM mapping, then for each $N = \{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$,*

$$\phi_N(\Delta_n) \cap \left(\bigcap_{x\in N} Tx\right) \neq \emptyset,$$

where ϕ_N is the continuous mapping in touch with N in definition of a generalized R-KKM map-

ping.

(ii) *If the family $\{T(x) \mid x \in X\}$ has finite intersection property, then T is a generalized R-KKM mapping.*

Proof. Switch the topology of Y to its compactly generated extension [13]. Then we can eliminate 'compactly' and apply Theorem 6.

Remark. In [2], its authors used the partition of unity subordinated to a cover of $\phi_{N_0}(\Delta_n)$ which should be assumed Hausdorff. They claim that, applying their Theorem 3.1, they obtained new theorems which unify and extend many known results in recent literature. However, theirs are all disguised forms of known results and their practical applicability is doubtful.

7 Fixed points of $\mathfrak{B}$-maps

In this section, the well-known better admissible class $\mathfrak{B}$ on G-convex spaces [14] can be introduced on ϕ_A-spaces.

A ϕ_A-space $(E, D; \{\phi_A\}_{A\in\langle D\rangle})$ is an abstract convex space $(E, D; \Gamma)$, where $\Gamma_N := \phi_N(\Delta_n)$ for each $N \in \langle D\rangle$ with $|N| = n+1$, and hence there is a continuous function $\phi_N : \Delta_n \to \Gamma_N$. This $(E, D; \Gamma)$ is not necessarily a G-convex space.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space, X a nonempty subset of E and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \Longleftrightarrow F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D\rangle$ with the cardinality $|N| = n+1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that $\phi_N(\Delta_n)$ is a compact subset of X.

Recall that for a ϕ_A-space $(E, D; \{\phi_A\}_{A\in\langle D\rangle})$, by letting $\Gamma_N := \phi_N(\Delta_n)$, the above definition works. There are a large number of examples of $\mathfrak{B}$-maps; see [14] and references therein.

We introduce particular types of subsets of abstract convex uniform spaces adequate to establish our fixed point theory. In fact, as in [14], we introduce the Klee approximability of ranges of maps:

Definition. Let $(E, D; \{\phi_A\}_{A\in\langle D\rangle}; \mathcal{U})$ be a uniform ϕ_A-space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to E$ satisfying

(1) $(x, h(x)) \in U$ for all $x \in K$;

(2) $h(K) \subset \phi_N(\Delta_n)$ for some $N \in \langle D \rangle$ with $|N| = n+1$; and

(3) there exist a continuous function $p : K \to \Delta_n$ such that $h = \phi_N \circ p$.

Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K) \subset \phi_N(\Delta_n) \subset X$ for some $N \in \langle D \rangle$ in condition (2).

We have given a lot of examples of Klee approximable subsets in [14]. Now we have the following generalizations of the main result of [14]:

Theorem 7. *Let $(E, D; \{\phi_A\}_{A \in \langle D \rangle}; \mathcal{U})$ be a uniform ϕ_A-space, $X \subset Y$ subsets of E, and $F : Y \multimap Y$ a map such that $F|_X \in \mathfrak{B}(X, Y)$ and $F(X)$ is Klee approximable into X. Then F has the almost fixed point property (that is, for any $U \in \mathcal{U}$, there exist $x_U \in X$ such that $F(x_U) \cap U[x_U] \neq \emptyset$).*

Further if $(E,\mathcal{U})$ is Hausdorff, F is closed, and $\overline{F(X)}$ is compact in Y, then F has a fixed point $x_0 \in Y$ (that is, $x_0 \in F(x_0)$).

Proof. Since $K := F(X)$ is a Klee approximable into X, for each symmetric entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to X$ satisfying conditions (1) - (3) of the definition of Klee approximable subsets, and we have

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} K \xrightarrow{p} \Delta_n$$

for some $N \in \langle D \rangle$ with $|N| = n+1$ and $\Gamma_N := \phi_N(\Delta_n) \subset X$. Let $p' := p|_{F(\Gamma_N)}$. Since $F|_X \in \mathfrak{B}(X, Y)$, the composition $p' \circ (F|_{\Gamma_N}) \circ \phi_N : \Delta_n \multimap \Delta_n$ has a fixed point $a_U \in \Delta_n$. Let $x_U := \phi_N(a_U)$. Then

$$a_U \in (p' \circ F \circ \phi_N)(a_U) = (p' \circ F)(x_U)$$

and hence

$$x_U = \phi_N(a_U) \in (\phi_N \circ p' \circ F)(x_U).$$

Since $h = \phi_N \circ p$ by definition, we have

$$x_U = h(y_U) \quad \text{for some} \quad y_U \in (F|_{\Gamma_N})(x_U).$$

Therefore, for each entourage $U \in \mathcal{U}$, there exist points $x_U \in X$ and $y_U \in F(x_U)$ such that $(x_U, y_U) = (h(y_U), y_U) \in U$. So, for each U, there exist $x_U, y_U \in X$ such that $y_U \in F(x_U)$ and $y_U \in U[x_U]$.

Now suppose that F is closed and $\overline{F(X)}$ is compact. Since $F(X)$ is relatively compact, we may assume that the net y_U in $F(X)$ converges to some $x_0 \in \overline{F(X)}$. Since $(x_U, y_U) \in U$ for each $U \in \mathcal{U}$, by the Hausdorffness of E, the net x_U also converges to x_0. Since the graph of F is closed in $Y \times Y$ and $(x_U, y_U) \in \mathrm{Gr}(F)$, we have $(x_0, y_0) \in \mathrm{Gr}(F)$ and hence we have $x_0 \in F(x_0)$. This completes our proof.

Note that, by choosing particular subclass of multimaps or particular types of ϕ_A-spaces, we can deduce a large number of known or new fixed point theorems from Theorem 7.

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