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UNIFIED FIXED POINT THEOREMS FOR COMPACT CLOSED MULTIMAPS IN GENERALIZED CONVEX SPACES

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Abstract. This is to establish a new fixed point theorem for compact closed multimaps in the better admissible class defined on subsets of generalized convex uniform spaces and having Klee approximable ranges. Our new theorem unifies a large number of previous results. Some of them are listed in the chronological order.

1. Introduction

Analytical fixed point theory is mainly concerned with applications of fixed point theorems for multimaps in topological vector spaces and their generalizations. In the last two decades, the author has tried to unify and generalize various results in that theory; see the references in the end of this paper.

In fact, in [4,5,11], we obtained very general fixed point theorems for generalized upper hemicontinuous multimaps having convex values defined on convex subsets of topological vector spaces. Moreover, in [8-10,15,17], another general fixed point theorems for compact closed multimaps in the "admissible" or "better admissible" classes defined on subsets of topological vector spaces were deduced as unifications of many previously known results.

On the other hand, it became evident that some fixed point theorems in topological vector spaces can be extended to generalized convex spaces which are abstract convex spaces without any linear structure. In fact, in our recent work [18], for example, we introduced the concept of the Klee approximability of subsets of generalized convex uniform spaces and showed that *any compact closed multimap in the "better admissible" class defined on a generalized convex space into itself with the Klee approximable range has a fixed point.* This theorem contains a large

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number of known results on topological vector spaces or on various subclasses of generalized convex spaces. Such subclasses are those of admissible spaces (in the sense of Klee), Φ-spaces, sets of the Zima-Hadzić type, locally G-convex spaces, and LG-spaces.

The present paper is a continuation of [18]. In this paper, we deduce a new general fixed point theorem for *a compact closed multimap in the "better admissible" class defined on a subset of a generalized convex space into itself with the Klee approximable range.* This theorem unifies many of our previous results in [3-18] and clarify relations among them. We list some of such results in the chronological order.

In Section 2, we introduce multimap classes defined on generalized convex spaces and some mutual relations among them. Section 3 deals with the Klee approximability of the ranges of multimaps and a new fixed point theorem for the compact, closed, and better admissible multimaps defined on a subset of generalized convex spaces. In Section 4, we list some of our previous results which are direct consequences of the main theorem in the chronological order. Consequently, our new theorem unifies and generalize nearly one hundred fixed point theorems appeared in the history of generalizations of the Brouwer theorem; see [9].

For all terminology and notations, we follow [18].

2. Multimap Classes in Generalized Convex Spaces

In this section, we follow mainly [18,20] and references therein.

Definitions. A *generalized convex space* or a *G-convex space* $(E, D; \Gamma)$ consists of a topological space E and a nonempty set D such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exist a subset $\Gamma(A)$ of E and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A := \Gamma(A)$. When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Examples. The following are typical examples of G-convex spaces [18,21]:

(1) Any nonempty convex subset of a t.v.s.

(2) A convex space due to Lassonde.

(3) A C-space (or an H-space) due to Horvath. Hyperconvex metric spaces are very particular ones of C-spaces.

(4) An L-space due to Ben-El-Mechaiekh et al. The so-called FC-spaces are particular ones of L-spaces.

(5) A ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consisting of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ for $A \in \langle D\rangle$ with $|A| = n + 1$, can be made into a G-convex space [21].

Definition. Let $(E, D; \Gamma)$ be a G-convex space, X a nonempty subset of E, and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of multimaps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D\rangle$ with the cardinality $|N| = n + 1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \xrightarrow{F|_{\Gamma_N}} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

There are lots of subclasses of $\mathfrak{B}$; see [18] and references therein. We give some subclasses of $\mathfrak{B}$ as follows:

Examples. For topological spaces X and Y, an *admissible* class $\mathfrak{A}^\kappa_c(X, Y)$ [resp., $\mathfrak{A}^\sigma_c(X, Y)$] of maps $F : X \multimap Y$ is one such that, for each nonempty compact [resp., σ-compact] subset K of X, there exists a map $G \in \mathfrak{A}_c(K, Y)$ satisfying $G(x) \subset F(x)$ for all $x \in K$; where $\mathfrak{A}_c$ consists of finite compositions of maps in a class $\mathfrak{A}$ of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $T \in \mathfrak{A}_c$ is u.s.c. with nonempty compact values; and

(iii) for any polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Here, a polytope P is a homeomorphic image of a standard simplex. There are lots of examples of $\mathfrak{A}$ and $\mathfrak{A}^\kappa_c$; see [9] and references therein.

Subclasses of the admissible class $\mathfrak{A}^\kappa_c$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}_c^\sigma$ of Lassonde, the class $\mathbb{V}_c^\sigma$ of Park et al., approximable maps of Ben-El-Mechaiekh and Idizk, and many others.

Note that for a G-convex space $(X, D; \Gamma)$ and any space Y, an admissible class $\mathfrak{A}_c^\kappa(X, Y)$ is a subclass of $\mathfrak{B}(X, Y)$. Some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ were known. Note that the connectivity map due to Nash and Girolo is such an example.

Definition. Let $(E, D; \Gamma)$ be a G-convex space and Z a set. A map $F : E \multimap Z$ is said to have *the KKM property* and is called a $\mathfrak{K}$*-map* if, for any map $G : D \multimap Z$ satisfying

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ has the KKM property}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$*-map* is defined for closed-valued maps G, and a $\mathfrak{KO}$*-map* for open-valued maps G. Some authors use the notation KKM(E, Z) instead of $\mathfrak{KC}(E, Z)$.

Examples.

(1) Every G-convex space $(E, D; \Gamma)$ has a map $F \in \mathfrak{K}(E, Z)$ for any nonempty set Z. For a trivial example, choose $F(x) := Z$ for all $x \in E$. If $1_E \in \mathfrak{K}(E, E)$, then $f \in \mathfrak{K}(E, E)$ for any function $f : E \to E$.

(2) It is known that for a G-convex space $(E, D; \Gamma)$, we have the identity map $1_X \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$; see [2,20]. Moreover, for any topological space Y, if $F : E \to Y$ is a continuous single-valued map or if $F : E \multimap Y$ has a continuous selection, then it is easy to check that $F \in \mathfrak{KC}(E, Y) \cap \mathfrak{KO}(E, Y)$. Note that there are many known selection theorems due to Michael and others; see [19].

(3) In early 1990's, the author introduced the admissible class $\mathfrak{A}_c^\kappa(X, Y)$ of multimaps $X \multimap Y$ between topological spaces and showed that this class has the KKM property when X is a convex space and Y is a Hausdorff space. Motivated by this, Chang and Yen [1] defined the KKM class of maps on convex subsets of topological vector spaces. Naturally, their $\mathfrak{KC}$ class contains $\mathfrak{A}_c^\kappa$-class on convex spaces, but significant proper examples of the former not in the latter are hard to find. Further, Chang et al. extended the KKM-class to S-KKM class. On the other hand, the author extended the $\mathfrak{A}_c^\kappa$-class to the 'better' admissible $\mathfrak{B}$-class on convex spaces, supplied a large number of examples, and showed that, in the class of compact closed multimaps from

convex spaces into Hausdorff spaces, two subclasses $\mathfrak{B}$ and $\mathfrak{KC}$ coincide [7]. Moreover, recently H. Kim showed that two classes KKM and s-KKM of multimaps from a convex space into a topological space are identical whenever s is surjective [this is the only case s-KKM is meaningful]. For G-convex spaces, such multimap classes are extended and investigated by a number of authors. For references, see [20].

The following is known [20], where $\mathbb{C}$ is the class of continuous functions:

Proposition 2.1. *Let $(E, D; \Gamma)$ be a G-convex space and Z a topological space. Then*

(1) $\mathbb{C}(E, Z) \subset \mathfrak{A}_c^\kappa(E, Z) \subset \mathfrak{B}(E, Z)$;
(2) $\mathbb{C}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z)$; *and*
(3) [2] $\mathfrak{A}_c^\kappa(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z)$ *if Z is Hausdorff.*

Consider the following condition for a G-convex space $(E \supset D; \Gamma)$:

$(*)$ $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$; and, for each $N \in \langle D \rangle$ with the cardinality $|N| = n+1$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that $\phi_N(\Delta_n) = \Gamma_N$ and that $J \in \langle N \rangle$ implies $\phi_N(\Delta_J) = \Gamma_J$.

Note that every convex space satisfies condition $(*)$.

Theorem 2.2. ([20]). *Let $(E, D; \Gamma)$ be a G-convex space and Z a topological space.*

(1) *If Z is a Hausdorff space, then every compact map $F \in \mathfrak{B}(E, Z)$ belongs to $\mathfrak{KC}(E, Z)$.*
(2) *If $F : E \multimap Z$ is a closed map such that $F\phi_N \in \mathfrak{KC}(\Delta_n, Z)$ for any $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, then $F \in \mathfrak{B}(E, Z)$.*
(3) *In the class of closed maps defined on a G-convex space $(E \supset D; \Gamma)$ satisfying condition $(*)$ into a space Z, a map $F \in \mathfrak{KC}(E, Z)$ belongs to $\mathfrak{B}(E, Z)$.*

Remark. In (2), note that for any map $F \in \mathfrak{A}_c^\kappa(E, Z)$, we have $F\phi_N \in \mathfrak{A}_c^\kappa(\Delta_n, Z) \subset \mathfrak{KC}(\Delta_n, Z) \cap \mathfrak{KO}(\Delta_n, Z)$ when Z is Hausdorff; see [2].

Corollary 2.3. *In the class of compact closed maps defined on a G-convex space $(E \supset D; \Gamma)$ satisfying condition $(*)$ into a Hausdorff space Z, two subclasses $\mathfrak{KC}(E, Z)$ and $\mathfrak{B}(E, Z)$ are identical.*

Corollary 2.4. *In the class of compact closed maps defined on a convex space (X, D) into a Hausdorff space Z, two subclasses $\mathfrak{KC}(X, Z)$ and $\mathfrak{B}(X, Z)$ are identical.*

Remark. This is noted in [7] by a different method. In view of Corollary 2.4, the class $\mathfrak{B}$ is favorable to use for convex spaces since it has already plenty of examples and is much easier to find examples.

3. A Unified Fixed Point Theorem

In this section, we begin with a generalized version of our previous definition of the Klee approximability of ranges as in [18].

Definition. A *generalized convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is a generalized convex space such that $(E, \mathcal{U})$ is a uniform space with a basis $\mathcal{U}$ of the uniformity consisting of symmetric entourages. For each $U \in \mathcal{U}$, let

$$U[x] = \{x' \in X \mid (x, x') \in U\}$$

be the U-*ball* around a given element $x \in E$.

We introduce particular types of subsets of generalized convex uniform spaces adequate to establish our fixed point theory:

Definition. For a generalized convex uniform space $(E, D; \Gamma; \mathcal{U})$, a subset X of E is said to be *admissible* (in the sense of Klee) if, for each nonempty compact subset K of X and for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to X$ satisfying

(1) $(x, h(x)) \in U$ for all $x \in K$;
(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$; and
(3) there exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ with $|N| = n + 1$ such that $h = \phi_N \circ p$.

For more general purposes, we introduce a generalized version of our previous definition of the Klee approximability of ranges of multimaps which was used instead of the admissibility of domains:

Definition. Let $(E, D; \Gamma; \mathcal{U})$ be a generalized convex uniform space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to E$ satisfying conditions (1)-(3) in the preceding definition. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K) \subset \Gamma_N \subset X$ for some $N \in \langle D \rangle$ in condition (2).

In the category of topological vector spaces or C-spaces, the concepts of locally convex spaces, LC-spaces, Φ-spaces, subsets of the Zima-Hadzić type, admissible subsets, and Klee approximable sets are quite well-known. They were introduced in order to generalize known fixed point theorems.

In our previous work [18], we extended those concepts to G-convex uniform spaces and established the mutual relations among them as follows:

Theorem 3.1. *In the class of G-convex uniform spaces, the following hold:*

(1) *Any LG-space is of the Zima-Hadzić type.*

(2) *Every LG-space is locally G-convex whenever every singleton is* Γ*-convex.*

(3) *Any nonempty subset of a locally G-convex space is a* Φ*-set.*

(4) *Any Zima-Hadzić type subset of a G-convex uniform space such that every singleton is* Γ*-convex is a* Φ*-set.*

(5) *Every* Φ*-space is admissible. More generally, every nonempty compact* Φ*-subset is Klee approximable.*

We have the following main fixed point result in this paper:

Theorem 3.2. *Let* $(E, D; \Gamma; \mathcal{U})$ *be a G-convex uniform space,* $X \subset Y$ *subsets of* E*, and* $F : Y \multimap Y$ *a multimap such that* $F|_X \in \mathfrak{B}(X, Y)$ *and* $F(X)$ *is Klee approximable into X. Then F has the almost fixed point property (that is, for each* $U \in \mathcal{U}$*, F has a U-fixed point* $x_U \in Y$ *satisfying* $F(x_U) \cap U[x_U] \neq \emptyset$*).*

Further if E is Hausdorff, F is closed, and $\overline{F(X)}$ *is compact in* Y*, then F has a fixed point* $x_0 \in Y$ *(that is,* $x_0 \in F(x_0)$*).*

Proof. Since $K := F(X)$ is Klee approximable into X, for each symmetric entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to X$ satisfying conditions (1) - (3) of the definition of Klee approximable subsets, and we have

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \xrightarrow{F|_{\Gamma_N}} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

for some $N \in \langle D \rangle$ with $|N| = n + 1$ and $\Gamma_N \subset X$. Let $p' := p|_{F(\Gamma_N)}$. Since $F|_X \in \mathfrak{B}(X, Y)$, the composition $p' \circ (F|_{\Gamma_N}) \circ \phi_N : \Delta_n \multimap \Delta_n$ has a fixed point $a_U \in \Delta_n$. Let $x_U := \phi_N(a_U)$. Then

$$a_U \in (p' \circ F \circ \phi_N)(a_U) = (p' \circ F)(x_U)$$

and hence

$$x_U = \phi_N(a_U) \in (\phi_N \circ p' \circ F)(x_U).$$

Since $h = \phi_N \circ p$ by definition, we have

$$x_U = h(y_U) \quad \text{for some} \quad y_U \in (F|_{\Gamma_N})(x_U).$$

Therefore, for each entourage $U \in \mathcal{U}$, there exist points $x_U \in X$ and $y_U \in F(x_U)$ such that $(x_U, y_U) = (h(y_U), y_U) \in U$. So, for each U, there exist $x_U, y_U \in X$ such that $y_U \in F(x_U)$ and $y_U \in U[x_U]$.

Now suppose that F is closed and $\overline{F(X)}$ is compact. Since $F(X)$ is relatively compact, we may assume that the net y_U in $F(X)$ converges to some $x_0 \in \overline{F(X)}$. Since $(x_U, y_U) \in U$ for each $U \in \mathcal{U}$, by the Hausdorffness of E, the net x_U also converges to x_0. Since the graph of F is closed in $Y \times Y$ and $(x_U, y_U) \in \mathrm{Gr}(F)$, we have $(x_0, y_0) \in \mathrm{Gr}(F)$ and hence we have $x_0 \in F(x_0)$. This completes our proof.

Note that, by choosing particular subclass of multimaps or particular types of G-convex spaces, we can deduce a large number of known or new fixed point theorems from Theorem 3.2.

4. Various Consequences of the Main Theorem

In this section, we list some of our previous results which are direct consequences of Theorem 3.2 in the chronological order. For simplicity, all topological spaces are assumed to be Hausdorff unless explicitly stated otherwise.

In 1992, the celebrated Himmelberg fixed point theorem was extended to acyclic maps in [3]:

Theorem 4.1. *Let X be a convex subset of a locally convex t.v.s. Then every compact acyclic map $T : X \multimap X$ has a fixed point.*

Theorem 4.1 was extended to the class $\mathfrak{A}_c$ (1993, [5]), $\mathbb{V}_c^\sigma$ (1994), $\mathfrak{A}_c^\sigma$ (1994, [6]), and $\mathfrak{B}$ (1997, [7]).

In 1993 [5], we obtained the following with a different method:

Theorem 4.2. *Let X be a compact convex subset of a t.v.s. E on which its dual E^* separates points. Then any map $F \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

In 1998 [8], Theorem 4.1 was extended to a non-locally convex t.v.s. as follows:

Theorem 4.3. *Let X be an admissible convex subset of a t.v.s. E. Then any compact map $F \in \mathfrak{A}_c(X, X)$ has a fixed point.*

In 2000 [10] and others, a particular form of Theorem 4.3 for $F \in \mathbb{V}_c(X, X)$ (that is, a finite composition of acyclic maps) was applied to a Simons type cyclic coincidence theorem for acyclic maps, the von Neumann type intersection theorems for graphs of compact compositions of acyclic maps, the Nash type equilibrium theorems, saddle point or minimax theorems, quasi-equilibrium problems, and quasi-variational inequalities, where most of related convexity were replaced by acyclicity.

Since an admissible convex subset of a t.v.s. is an admissible G-convex space, we have the following from Theorem 3.2:

Theorem 4.4. *Let X be an admissible convex subset of a t.v.s. E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Theorem 4.4 was given in 1998 [8], where we listed more than sixty papers in chronological order, from which we could deduce particular forms of Theorem 4.4.

For $X = Y$, Theorem 3.2 reduces the following form of the main theorem of [15] in 2004:

Theorem 4.5. *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

For $X = Y = E$, Theorem 3.2 reduces to the following main theorem of [18] in 2007:

Theorem 4.6. *Let $(X, D; \Gamma; \mathcal{U})$ be a G-convex uniform space and $F \in \mathfrak{B}(X, X)$ a multimap such that $F(X)$ is Klee approximable. Then F has the almost fixed point property.*

Further if F is closed and compact, then F has a fixed point $x_0 \in X$ (that is, $x_0 \in F(x_0)$).

This theorem contains a large number of known results on topological vector spaces or on various subclasses of the class of admissible G-convex spaces. Such subclasses are those of admissible spaces, Φ-spaces, sets of the Zima-Hadzić type, locally G-convex spaces, and LG-spaces; see [18]. Mutual relations among those subclasses and some related results on approximable maps, Kakutani maps, acyclic maps, Φ-maps, and others are investigated in [18].

Corollary 4.7. *Let $(X, D; \Gamma; \mathcal{U})$ be an admissible G-convex space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Corollary 4.8. *Let $(X, D; \Gamma; \mathcal{U})$ be a compact admissible G-convex space. Then any map $F \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

The following is a consequence of Theorem 3.2:

Theorem 4.9. *Let X and Y be subsets of a t.v.s. E such that $X \subset Y$ and $F : Y \multimap Y$ a map.*

(1) *If* $F|_X \in \mathfrak{B}(X,Y)$ *and* $F(X)$ *is Klee approximable into* X, *then* $F|_X$ *has the almost fixed point property* (*that is, for any* $V \in \mathcal{V}$, $F|_X$ *has a* V*-fixed point* $x_V \in X$ *satisfying* $F(x_V) \cap (x_V + V) \neq \emptyset$).

(2) *Further if* F *is closed and* $F|_X$ *is compact, then* F *has a fixed point.*

Note that, in (1), E is not necessarily Hausdorff. Theorem 4.9 would be better than [17, Theorem 2.2]. In [17], it should be $\mathfrak{B} = \mathfrak{B}^p$.

5. Applications

Earlier fixed point theorems in Section 4 were applied to the following problems in the author's works in 1991-2007; see [9] and MATHSCINET.

Best approximations, variational inequalities, quasi-variational inequalities, the Leray-Schauder type alternatives, existence of maximal elements, minimax inequalities, the Walras excess demand theorems, generalized equilibrium problems, generalized complementarity problems, condensing maps, openness of multimaps, the Birkhoff-Kellogg type theorems, saddle points in nonconvex sets, acyclic or other versions of the Nash equilibrium theorems, quasi-equilibrium theorems, extensions of monotone sets, eigenvector problems, the KKM theory, and others.

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