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ScienceDirect
Nonlinear Analysis 69 (2008) 4352–4364
Nonlinear Analysis
www.elsevier.com/locate/na

Equilibrium existence theorems in KKM spaces

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Received 16 April 2007; accepted 29 October 2007

Abstract

An abstract convex space satisfying the KKM principle is called a KKM space. This class of spaces contains G-convex spaces properly. In this work, we show that a large number of results in KKM theory on G-convex spaces also hold on KKM spaces. Examples of such results are theorems of Sperner and Alexandroff–Pasynkoff, Fan–Browder type fixed point theorems, Horvath type fixed point theorems, Ky Fan type minimax inequalities, variational inequalities, von Neumann type minimax theorems, Nash type equilibrium theorems, and Himmelberg type fixed point theorems.

MSC: primary 47H04; 47H10; secondary 46A16; 46A55; 52A07; 54C60; 54H25; 91B50

Keywords: Multimap (map); Abstract convex space; KKM space; G-convex space; Map classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KD}$

1. Introduction

Equilibrium problems (or more generally, generalized quasiequilibrium problems) contain as special cases, for instance, optimization problems, problems of Nash type equilibrium, complementarity problems, fixed point problems, and variational inequalities, as well as many others; see [19]. Precisely, a large number of important existence results on equilibrium problems can be deduced by applying the celebrated Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM principle) in 1929 [12], which is concerned with certain types of multimaps called KKM maps.

KKM theory, at first, was the study of KKM maps and their applications; see [15,16]. Nowadays, it would be better to regard it as the study of applications of various equivalent formulations of the KKM principle and their generalizations. For the early history, see [17]. At the beginning, the theory was devoted to the study of convex subsets of topological vector spaces mainly by Ky Fan [3–5]. Later, it has been extended to convex spaces by Lassonde [13], and to C-spaces (or H-spaces) by Horvath [7–11] and others. In the last decade, the KKM theory was extended to generalized convex (G-convex) spaces in a sequence of papers of the author and others; for details, see [17–23] and the references therein.

In our recent work [24], we introduced a new concept of abstract convex spaces and multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KD}$ having certain KKM property. These new spaces and multimap classes are known to be adequate to establish

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0362-546X/$ - see front matter
doi:10.1016/j.na.2007.10.058

KKM theory; see [24–30]. Especially, in [28], we generalized and simplified the known results of the theory on convex spaces, H-spaces, G-convex spaces, and others. We found there that the class of abstract convex spaces $(E, D; \Gamma)$ satisfying the corresponding KKM principle plays the major role in KKM theory. Therefore it seems to be quite natural to call such spaces the KKM spaces.

In the present paper, we show that a large number of well-known results in KKM theory on G-convex spaces also hold on KKM spaces. Examples of such results are theorems of Sperner and Alexandroff–Pasynkoff, Horvath type fixed point theorems, Fan–Browder type coincidence theorems, Ky Fan type minimax inequalities, variational inequalities, von Neumann type minimax theorems, Nash type equilibrium theorems, and Himmelberg type fixed point theorems. For related results, see [24–30].

Section 2 deals with preliminaries on definitions and examples of abstract convex spaces, KKM spaces, and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$. In Section 3, we deduce some KKM type theorems and theorems of Sperner and Alexandorff–Pasynkoff on KKM spaces. Section 4 deals with Horvath type fixed point theorems on KKM spaces. In Section 5, we deduce Fan–Browder type fixed point theorems and a maximal element theorem on KKM spaces. Section 6 deals with Ky Fan type minimax inequalities on KKM spaces. In Section 7, variational inequalities on compact KKM spaces and an application to best approximation are given. Section 8 deals with von Neumann type minimax theorems on KKM spaces. In Section 9, a Ky Fan type intersection theorem and a Nash type equilibrium theorem are established for compact KKM spaces. Finally, Section 10 deals with generalizations of the Himmelberg fixed point theorem to the KKM spaces.

Multimaps are simply called maps in this paper.

2. KKM spaces

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

Let $(E, D; \Gamma)$ be an abstract convex space. For any $D' \subset D$, the *Γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

[co is reserved for the convex hull in vector spaces.] A subset X of E is called a *Γ-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D' \rangle})$ itself is an abstract convex space called a *subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such a case, a subset X of E is said to be *Γ-convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In the case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Many examples of abstract convex spaces were given in [24,28].

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A map $F : E \multimap Z$ is said to have the *KKM property* and called *a $\mathfrak{K}$-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Note that if Z is discrete then the three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation $\mathrm{KKM}(E, Z)$ instead of $\mathfrak{KC}(E, Z)$.

Definition. For an abstract convex topological space $(E, D; \Gamma)$, the *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$.

A *KKM space* is an abstract convex topological space satisfying the KKM principle.

Definition. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For details on G-convex spaces; see [17–23] and the references therein.

In our recent work [28], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to KKM spaces.

Examples 2.1. The following are typical examples of G-convex spaces:

1. Any nonempty convex subset of a topological vector space (t.v.s.).
2. Convex spaces due to Lassonde [13].
3. C-spaces (or H-spaces) due to Horvath [7–11]. Hyperconvex metric spaces are particular C-spaces.
4. Hyperbolic spaces due to Reich and Shafrir [34].
5. L-spaces due to Ben-El-Mechaiekh et al. The so-called FC-spaces are particular forms of L-spaces; see [30].
6. A ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consisting of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ for $A \in \langle D \rangle$ with $|A| = n+1$ and $n \in \mathbb{N} \cup \{0\}$, can be made into a G-convex space [30].

Examples 2.2. We give examples of KKM spaces:

1. Every G-convex space is a KKM space; see [18,31].
2. A connected linearly ordered space $(X, \le)$ can be made into an abstract convex topological space $(X \supset D; \Gamma)$ for any nonempty $D \subset X$ by defining $\Gamma_A := [\min A, \max A] = \{x \in X \mid \min A \le x \le \max A\}$ for each $A \in \langle D \rangle$. Further, it is a KKM space; see [27, Theorem 5(i)].
3. The extended long line L^* can be made into a KKM space $(L^* \supset D; \Gamma)$; see [27]. In fact, L^* is constructed from the ordinal space $D := [0, \Omega]$ consisting of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology. Recall that L^* is a generalized arc obtained from $[0, \Omega]$ by placing a copy of the interval $(0, 1)$ between each ordinal α and its successor $\alpha + 1$ and we give L^* the order topology. Now let $\Gamma : \langle D \rangle \multimap L^*$ be the one as in 2. But L^* is not a G-convex space. In fact, since $\Gamma\{0, \Omega\} = L^*$ is not path connected, for $A := \{0, \Omega\} \in \langle L^* \rangle$ and $\Delta_1 := [0, 1]$, there does not exist a continuous function $\phi_A : [0, 1] \to \Gamma_A$ such that $\phi_A\{0\} \subset \Gamma\{0\} = \{0\}$ and $\phi_A\{1\} \subset \Gamma\{\Omega\} = \{\Omega\}$. Therefore $(L^* \supset D; \Gamma)$ is not G-convex.

3. KKM type theorems in KKM spaces

The following is a prototype of KKM type theorems given in [24,28]:

Theorem 3.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F : E \multimap Z$ a map. Then $F \in \mathfrak{K}(E, Z)$ if and only if for any map $G : D \multimap Z$ satisfying*

(1) *$F(\Gamma_N) \subset G(N)$ for any $N \in \langle D \rangle$,*

we have $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.

Remark. If Z is any topological space and if $F \in \mathfrak{KO}(E, Z)$ [resp., $F \in \mathfrak{KC}(E, Z)$], then we have to assume that G is open-valued [resp., closed-valued].

From Theorem 3.1 with $F = 1_E$, we immediately have the following KKM theorem for KKM spaces:

Theorem 3.2. *An abstract convex space $(E, D; \Gamma)$ is a KKM space if and only if for any map $G : D \multimap E$ satisfying*

(1) *G has closed [resp., open] values, and*

(2) *G is a KKM map,*

the family $\{G(y)\}_{y\in D}$ *has the finite intersection property.*

Moreover, if

(3) $\bigcap_{y\in M} \overline{G(y)}$ *is compact for some* $M \in \langle D\rangle$,

then we have

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Remark. The reader might prefer to assume that each $\overline{G(y)}$ is compactly closed, but this does not generalize anything.

For G-convex spaces, we have the following [31]:

Theorem 3.3. *Let* $(X, D; \Gamma)$ *be a* G*-convex space and* $G : D \multimap X$ *a KKM map with closed [resp., open] values. Then* $\{G(y)\}_{y\in D}$ *has the finite intersection property. More precisely, for each* $N \in \langle D\rangle$ *with* $|N| = n + 1$*, we have*

$$\phi_N(\Delta_n) \cap \bigcap_{y\in N} G(y) \neq \emptyset.$$

Note that the first part of Theorem 3.3 follows from Theorem 3.2 But the precise part does not.

Recall that, at first, a G-convex space was defined under the additional isotonicity condition:

$$\text{if } M, N \in \langle D\rangle \text{ and } M \subset N, \quad \text{then } \Gamma_M \subset \Gamma_N. \tag{$*$}$$

Condition $(*)$ holds for convex spaces due to Lassonde [13] or C-spaces due to Horvath [7–11], but not for G-convex spaces in general. Later, it was known that the isotonicity was superfluous (see [18–23]) in most applications.

However, for any G-convex space $(X, D; \Gamma)$, when $D = A$ is finite, by putting $\Gamma_J = \phi_A(\Delta_J)$ for each $J \subset A$, we may assume the isotonicity condition $(*)$.

From Theorem 3.1, as in [29], we can deduce KKM space versions of theorems due to Sperner [35] and Alexandroff–Pasynkoff [1], resp., as follows:

Theorem 3.4. *Let* $(E, D; \Gamma)$ *be a KKM space with* $D = \{a_0, a_1, \ldots, a_n\}$ *and* $S : D \multimap E$ *a map with nonempty closed [resp., open] values such that*

(i) $E = S(D)$ *and*

(ii) *for each* i, $0 \le i \le n$, $S(a_i)$ *is disjoint from* $\Gamma(\{a_0, \ldots, \widehat{a_i}, \ldots, a_n\})$.

If the isotonicity condition $(*)$ *holds, then* $\bigcap_{i=0}^n S(a_i) \neq \emptyset$.

Proof. It suffices to show that S is a KKM map. Let $N \in \langle D\rangle$. If $N = D$, then $\Gamma_N \subset E = S(N)$ by (i). Suppose that $N \subsetneq D$. Then there exists an index j, $0 \le j \le n$, such that $a_j \notin N$. By (ii) and condition $(*)$, we have

$$S(a_j) \cap \Gamma_N \subset S(a_j) \cap \Gamma(\{a_0, \ldots, \widehat{a_j}, \ldots, a_n\}) = \emptyset.$$

However, we have $\Gamma_N \subset E = S(D) = \bigcup_{i=0}^n S(a_i)$ and hence

$$\Gamma_N \subset \bigcup \{S(a_i) \mid a_i \in N\} = S(N).$$

Now the conclusion follows from Theorem 3.2 □

For $D = \{a_0, a_1, \ldots, a_n\}$, we denote as follows:

$$D_0 := \{a_0, \ldots, a_{n-1}\} \quad \text{and} \quad D_i := \{a_0, \ldots, \widehat{a_{i-1}}, \ldots, a_n\}$$

for $1 \le i \le n$.

Theorem 3.5. *Let* $(E, D; \Gamma)$ *be a KKM space with* $D = \{a_0, a_1, \ldots, a_n\}$ *and* $T : D \multimap E$ *a map with nonempty closed [resp., open] values such that*

(i) $E = T(D)$ *and*

(ii) $\Gamma_{D_i} \subset T(a_i)$ *for* $0 \le i \le n$.

If the isotonicity condition $(*)$ *holds, then* $\bigcap_{i=0}^n T(a_i) \neq \emptyset$.

Proof. We show that T is a KKM map. Let $N \in \langle D \rangle$. If $N = D$, then $\Gamma_N \subset E = T(N)$ by (i). Suppose that $N \subsetneq D$. Then, by (ii) and condition (∗),

$$\Gamma_N \subset \Gamma_{D_i} \subset T(a_i) \quad \text{for some } a_i \in N,$$

and hence

$$\Gamma_N \subset \bigcup \{T(a_i) \mid a_i \in N\} = T(N).$$

Now the conclusion follows from Theorem 3.2 □

4. Horvath type fixed point theorems

In this section, we generalize fixed point theorems in Section 4 of Horvath [10].

We follow the notations in [10]: Let $A : X \multimap Y$ be a map between sets. Let $A^-(y) := \{x \in X \mid y \in A(x)\}$ for $y \in Y$. Define the map $A^* : Y \multimap X$ by $A^*(y) := X \setminus A^-(y)$. Obviously $A^{**} = A$; $x \in A^*(y)$ if and only if $y \notin A(x)$; and if $B : X \multimap Y$ is another map such that $B \subset A$ then $A^* \subset B^*$.

For $Y = D$, the following reduces to Horvath's amusing lemma [10, Section 4, Lemma 1]:

Lemma 4.1. *Let $(Y \supset D; \Gamma)$ be an abstract convex space and $T : Y \multimap Y$ a map such that*
(1) *for each $y \in Y$, $y \in T(y)$; and*
(2) *for each $y \in Y$, $T^*(y)$ is Γ-convex.*
Then T is a KKM map, that is, for each $J \in \langle D \rangle$, $\Gamma(J) \subset T(J)$.

Proof. Suppose that $u \notin T(J)$. Then, for each $y \in J$,

$$u \notin T(y) \Leftrightarrow y \notin T^-(u) \Leftrightarrow y \in Y \setminus T^-(u) = T^*(u).$$

Hence, by (2), $\Gamma(J) \subset T^*(u)$. On the other hand, since $u \in T(u)$ by (1), we have $u \notin T^*(u)$. Therefore $u \notin \Gamma(J)$. □

Lemma 4.1 can be reformulated as a purely set-theoretical fixed point result, as in [10, Section 4, Lemma 1']:

Lemma 4.2. *Let $(Y \supset D; \Gamma)$ be an abstract convex space and $S : Y \multimap Y$ a map such that*
(1) *there exists a $J_0 \in \langle D \rangle$ such that $\Gamma(J_0) \cap \bigcap_{y \in J_0} S^-(y) \neq \emptyset$; and*
(2) *for each $y \in Y$, $S(y)$ is Γ-convex.*
Then there exists a $\overline{y} \in Y$ such that $\overline{y} \in S(\overline{y})$.

Proof. Let $T^* = S$ or $T = S^*$. Then the negation of the conclusion of Lemma 4.1 is condition (1). Since condition (2) of Lemmas 4.1 and 4.2 are identical, we obtain the negation of condition (1) of Lemma 4.1. Hence $\overline{y} \notin T(\overline{y})$ for some $\overline{y} \in Y$. Hence $\overline{y} \in T^*(\overline{y}) = S(\overline{y})$. □

Theorem 4.3. *Let $(E \supset D; \Gamma)$ be a KKM space and $R, S : E \multimap E$ be two maps such that*
(1) *for each $x \in E$, $R(x)$ is closed [resp., open] and $x \in S(x) \subset R(x)$;*
(2) *for each $y \in E$, $S^*(y)$ is Γ-convex; and*
(3) *$\overline{R(x_0)}$ is compact for some $x_0 \in D$.*
Then $\bigcap_{x \in D} \overline{R(x)} \neq \emptyset$.

Proof. By Lemma 4.1, $\Gamma_J \subset S(J)$ for each $J \in \langle D \rangle$ and hence S and R are KKM maps. Therefore, by Theorem 3.2, the family $\{R(x) \mid x \in D\}$ has the finite intersection property. Since $\overline{R(x_0)}$ is compact for some $x_0 \in D$, the conclusion follows. □

Remark. Theorem 4.3 generalizes Horvath [10, Section 4, Theorem 1].

By taking $A = R^*$ and $B = S^*$ we obtain at once the following fixed point theorem:

Theorem 4.4. *Let $(E \supset D; \Gamma)$ be a KKM space and $A, B : E \multimap E$ two maps such that*

(1) *for each $y \in E$, $A^{-}(y)$ is open [resp., closed], $A(y) \neq \emptyset$ and $A(y) \subset B(y)$;*
(2) *for each $x \in E$, $B(x)$ is Γ-convex; and*
(3) $\overline{E \setminus A^{-}(y_0)}$ *is compact for some $y_0 \in D$.*

Then there is a $\overline{x} \in E$ such that $\overline{x} \in B(\overline{x})$.

Remark. Theorem 4.4 generalizes Horvath [10, Section 4, Theorem 2].

5. Fan–Browder type fixed point theorems

The following coincidence theorem was given in [24,28]:

Theorem 5.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, $S : D \multimap Z$, $T : E \multimap Z$ maps, and $F \in \mathfrak{K}(E, Z)$. Suppose that*

(1) *for each $z \in F(E)$, $\operatorname{co}_\Gamma S^{-}(z) \subset T^{-}(z)$ [that is, $T^{-}(z)$ is Γ-convex relative to $S^{-}(z)$]; and*
(2) *$F(E) \subset S(N)$ for some $N \in \langle D \rangle$.*

Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Remark. If Z is any topological space and S has open [resp., closed] values, then we can assume that $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KO}(E, Z)$] in Theorem 5.1.

From Theorem 5.1 with $F = 1_E$, as in [24], we have the following prototype of the Fan–Browder fixed point theorem:

Theorem 5.2. *Let $(E, D; \Gamma)$ be a KKM space, and $G : E \multimap D$, $F : E \multimap E$ maps satisfying*

(1) *for each $x \in E$, $\operatorname{co}_\Gamma G(x) \subset F(x)$ [that is, $F(x)$ is Γ-convex relative to $G(x)$];*
(2) *$E = G^{-}(N)$ for some $N \in \langle D \rangle$; and*
(3) *G^{-} has open [resp., closed] values.*

Then F has a fixed point $\bar{x} \in E$, that is, $\bar{x} \in F(\bar{x})$.

Any binary relation R in a set X can be regarded as a map $T : X \multimap X$ and conversely by the following obvious way:

$$y \in T(x) \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a map T if $T(x_0) = \emptyset$.

The Fan–Browder fixed point theorem is used by Borglin and Keiding [2] and Yannelis and Prabhakar [39] to the existence of maximal elements in mathematical economics.

From Theorem 5.2, we have the following maximal element theorem:

Theorem 5.3. *Let $(E, D; \Gamma)$ be a KKM space and $G : E \multimap D$, $F : E \multimap E$ maps satisfying*

(1) *for each $x \in E$, $\operatorname{co}_\Gamma G(x) \subset F(x)$;*
(2) *$F^{-}(E) \subset G^{-}(N)$ for some $N \in \langle D \rangle$;*
(3) *G^{-} has open [resp., closed] values; and*
(4) *$x \notin F(x)$ for all $x \in E$.*

Then F has a maximal element $\bar{x} \in E$, that is, $F(\bar{x}) = \emptyset$.

Proof. Note that (1) and (3) are the same to conditions (1) and (3) of Theorem 5.2, resp. Suppose that $F(x) \neq \emptyset$ for each $x \in E$. Then $E = F^{-}(E) = \bigcup\{F^{-}(y) \mid y \in E\}$. By (2), condition (2) of Theorem 5.2 holds. Therefore, by Theorem 5.2, F has a fixed point. This violates (4). □

For a G-convex space, Theorem 5.3 generalizes the results in [18–21]. Note that the conclusions of Theorems 5.2 and 5.3 follow from the KKM principle.

6. Ky Fan type minimax inequalities

From Theorem 5.1, we have the following analytic alternative, which is a basis of various equilibrium problems:

Theorem 6.1. *Let* $(E, D; \Gamma)$ *be an abstract convex space,* Z *a set,* $F \in \mathfrak{K}(E, Z)$, $\alpha, \beta \in \mathbb{R}$, *and* $f : D \times Z \to \overline{\mathbb{R}}$, $g : E \times Z \to \overline{\mathbb{R}}$ *extended real-valued functions. Suppose that*

(1) *for each* $z \in F(E)$,

$$\mathrm{co}_\Gamma\{y \in D \mid f(y, z) > \alpha\} \subset \{x \in E \mid g(x, z) > \beta\}.$$

Then either

(a) *for each* $N \in \langle D\rangle$, *there exists a* $z_N \in F(E)$ *such that* $f(y, z_N) \le \alpha$ *for all* $y \in N$; *or*

(b) *there exists an* $(\hat{x}, \hat{z}) \in F$ *such that* $g(\hat{x}, \hat{z}) > \beta$.

Proof. Consider the maps $S : D \multimap Z$ and $T : E \multimap Z$ given by

$$S(y) := \{z \in Z \mid f(y, z) > \alpha\} \quad \text{for } y \in D$$

and

$$T(x) := \{z \in Z \mid g(x, z) > \beta\} \quad \text{for } x \in E.$$

Then (1) implies (1) in Theorem 5.1. Suppose that (a) does not hold. Then, for some $N \in \langle D\rangle$ and for each $z \in F(E)$ there exists an $y \in N$ such that $f(y, z) > \alpha$; that is, $F(E) \subset S(N)$. Hence (2) in Theorem 5.1 holds. Therefore, by Theorem 5.1, F and T have a coincidence point $\hat{x} \in X$ with $F(\hat{x}) \cap T(\hat{x}) \neq \emptyset$; that is, (b) holds. □

Remark. If Z is a topological space and $\{z \in Z \mid f(y, z) > \alpha\}$ is open [resp., closed] for each $y \in D$, then we can assume that $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KO}(E, Z)$] as in Theorem 5.1.

From Theorem 6.1, we clearly have the following generalized form of the Ky Fan minimax inequality [5]:

Theorem 6.2. *Under the hypothesis of Theorem 6.1, if* $\alpha = \beta = \sup\{g(x, z) \mid (x, z) \in F\}$, *then for each* $N \in \langle D\rangle$,

(c) *there exists a* $z_N \in F(E)$ *such that*

$$f(y, z_N) \le \sup_{(x,z)\in F} g(x, z) \quad \textit{for all } y \in N; \textit{ and}$$

(d) *we have the following minimax inequality*

$$\inf_{z\in F(E)} \sup_{y\in N} f(y, z) \le \sup_{(x,z)\in F} g(x, z).$$

Recall that an extended real-valued function $g : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X \mid g(x) > r\}$ [resp., $\{x \in X \mid g(x) < r\}$] is open for each $r \in \overline{\mathbb{R}}$.

From Theorem 4.3, we obtain the following form of the minimax inequality of Ky Fan as in Horvath [10, Section 5, Proposition 1].

Theorem 6.3. *Let* $(E \supset D; \Gamma)$ *be a KKM space and* $f, g : E \times E \to \mathbb{R}$ *two functions such that*

(1) $f \le g$;

(2) *for each* $y \in E$, $x \mapsto g(x, y)$ *is l.s.c.;*

(3) *for each* $x \in E$, $\{y \in E \mid f(x, y) > 0\}$ *is* Γ*-convex; and*

(4) $\{x \in E \mid g(x, y_0) \le 0\}$ *is compact for some* $y_0 \in D$.

Then the following alternative holds:

(a) *there is a* $x_0 \in E$ *such that* $\sup\{g(x_0, y) \mid y \in E\} \le 0$; *or*

(b) *there is a* $y_0 \in E$ *such that* $f(y_0, y_0) > 0$.

Proof. Use Theorem 4.3 with

$$R(y) := \{x \in E \mid g(x, y) \leq 0\}$$

and

$$S(y) := \{x \in E \mid f(x, y) \leq 0\}.$$

Then the conclusion follows. □

For an abstract convex space $(X; \Gamma)$, a real function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbb{R}}$.

An abstract convex space $(X; \Gamma)$ is said to be *compact* if X is a compact topological space. From now on, for simplicity, we are mainly concerned with compact KKM spaces $(X; \Gamma)$. For example, any compact G-convex space, any compact H-space, or any compact convex space of the form $(X; \Gamma)$ is such a space.

The following is a particular form of the minimax Theorem 6.2:

Theorem 6.4. *Let $(X; \Gamma)$ be a compact KKM space and $f, g : X \times X \to \mathbb{R} \cup \{+\infty\}$ be functions such that*

(1) *$f(x, y) \leq g(x, y)$ for each $(x, y) \in X \times X$,*
(2) *for each $x \in X$, $g(x, \cdot)$ is quasiconcave on X; and*
(3) *for each $y \in X$, $f(\cdot, y)$ is l.s.c. on X.*

Then we have

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} g(x, x).$$

Proof. Observe that $\sup_{x \in X} f(x, y)$ is by (3) a l.s.c. function of y on the compact space X, and therefore its minimum exists. If $\sup_{x \in X} g(x, x) = +\infty$, then the inequality in the conclusion holds automatically. If $\gamma = \sup_{x \in X} g(x, x) < +\infty$, then by Theorem 6.2 with $F = 1_X$, we have the conclusion. □

Remarks. 1. For $f = g$, Theorem 6.4 reduces to Fan's minimax inequality [5] on convex subsets of a topological vector space. Fan obtained his inequality from his own generalization of the KKM principle, and applied it to deduce fixed point theorems, theorems on sets with convex sections, a fundamental existence theorem in potential theory, and so on.

2. Later, the inequality has been an important tool in nonlinear analysis, game theory, and economic theory; see [17,18].

In particular, we have the following:

Corollary 6.5. *Under the hypothesis of Theorem 6.4, if $g(x, x) \leq 0$ for all $x \in X$, then there exists a $y_0 \in X$ such that $f(x, y_0) \leq 0$ for all $x \in X$. Thus in particular*

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq 0.$$

7. Variational inequalities and best approximations

Theorem 6.4 can be applied to the existence of solutions of certain variational inequalities:

Theorem 7.1. *Let $(X; \Gamma)$ be a compact KKM space and $p, q : X \times X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ functions satisfying*

(1) *$p(x, y) \leq q(x, y)$ for each $(x, y) \in X \times X$, and $q(x, x) \leq 0$ for all $x \in X$;*
(2) *for each $x \in X$, $q(x, \cdot) + h(\cdot)$ is quasiconcave on X; and*
(3) *for each $y \in X$, $p(\cdot, y) - h(\cdot)$ is l.s.c. on X.*

Then there exists a $y_0 \in X$ such that

$$p(x, y_0) + h(y_0) \leq h(x) \quad \text{for all } x \in X.$$

Proof. Let

$$f(x, y) := p(x, y) + h(y) - h(x), \qquad g(x, y) := q(x, y) + h(y) - h(x)$$

for $(x, y) \in X \times Y$. Then f and g satisfy the requirements of Theorem 6.4. Furthermore, $g(x, x) = q(x, x) \leq 0$ for all $x \in X$. Therefore, by Corollary 6.5, the conclusion follows. □

Remarks. 1. Putting $h = 0$, Theorem 7.1 reduces to Corollary 6.5.

2. Theorem 7.1 is a basis of existence theorems of many results concerning variational inequalities; see [18] and the references therein.

Theorem 7.2. *Let $(X; \Gamma)$ be a compact KKM space and $p, q : X \times X \to \mathbb{R}$ functions such that*

(1) *$p \leq q$ on the diagonal $\Delta := \{(x, x) \mid x \in X\}$ and $q \leq p$ on $(X \times X) \setminus \Delta$;*

(2) *for each $x \in X$, $y \mapsto q(y, y) - q(x, y)$ is quasiconcave on X; and*

(3) *for each $y \in X$, $x \mapsto p(x, y)$ is u.s.c. on X.*

Then there exists a $y_0 \in X$ such that

$$p(y_0, y_0) \leq p(x, y_0) \quad \text{for all } x \in X.$$

Proof. Define $f, g : X \times X \to R$ by

$$f(x, y) := p(y, y) - p(x, y), \qquad g(x, y) := q(y, y) - q(x, y).$$

Then f and g satisfy the hypothesis of Theorem 6.4. Since $g(x, x) = 0$ for all $x \in X$, Corollary 6.5 implies that $f(x, y_0) \leq 0$ for all $x \in X$. This implies the conclusion. □

Remark. For a convex space X and $p = q$, Theorem 7.2 reduces to a result of Fan [5], which was shown to be very useful in nonlinear functional analysis. In fact, the Tychonoff (and hence, the Brouwer) fixed point theorem, Browder's variational inequality, and many other applications follow from his result.

A simple consequence of Theorem 7.2 is the following well-known existence result on best approximations originated from Ky Fan [4]:

Corollary 7.3. *Let X be a compact convex subset of a topological vector space E and $f : X \to E$ a continuous function. Then for any continuous seminorm p on E, there exists a point $y_0 \in X$ such that*

$$p(y_0 - f(y_0)) \leq p(x - f(y_0)) \quad \text{for all } x \in X.$$

Proof. For each $y \in X$, $x \mapsto p(y - f(y)) - p(x - f(y))$ is convex on X, and for each $x \in X$, $y \mapsto p(x - f(y))$ is continuous. Therefore, by Theorem 7.2, we have a $y_0 \in X$ satisfying the conclusion. □

Remark. Further if E is a normed vector space and p is a norm on E, then Corollary 7.3 reduces to the well-known existence result on best approximation due to Ky Fan [4, Theorem 2], which immediately implies the Schauder fixed point theorem; that is, the normed space version of the Brouwer theorem. Therefore, Corollary 7.3 generalizes and implies the Brouwer theorem. Note also that Corollary 7.3 partially generalizes Reich [32, Lemma 1.4], [33, Corollary 2.2].

8. von Neumann type minimax theorems

We begin this section with that the product of any family of abstract convex spaces is also an abstract convex space. In fact, we clearly have the following:

Lemma 8.1. *Let $\{(E_i, D_i; \Gamma_i)\}_{i \in I}$ be any family of abstract convex spaces. Let $E := \prod_{i \in I} E_i$ and $D := \prod_{i \in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D \rangle$, define $\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$. Then $(E, D; \Gamma)$ is an abstract convex space.*

Note also that for the case $E_i = D_i$ for each i, the product of abstract convex subsets is also abstract convex in the product abstract convex space.

In this section, we show that a typical classical application of the KKM principle can be extended to abstract convex spaces.

As a direct application of Theorem 5.2, we have the following generalization of the von Neumann–Sion minimax theorem [36,37]:

Theorem 8.2. *Let $(X; \Gamma)$ and $(Y; \Gamma')$ be compact abstract convex spaces and $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions such that*

(1) *$f(x, y) \le g(x, y)$ for each $(x, y) \in X \times Y$;*

(2) *for each $x \in X$, $f(x, \cdot)$ is l.s.c. and $g(x, \cdot)$ is quasiconvex on Y; and*

(3) *for each $y \in Y$, $f(\cdot, y)$ is quasiconcave and $g(\cdot, y)$ is u.s.c. on X.*

If $(X \times Y; \Gamma_{X\times Y})$ is a KKM space, where $\Gamma_{X\times Y}$ is the product convexity defined as above, then we have

$$\min_{y\in Y} \sup_{x\in X} f(x, y) \le \max_{x\in X} \inf_{y\in Y} g(x, y).$$

Proof. Note that $y \mapsto \sup_{x\in X} f(x, y)$ is l.s.c. on Y and $x \mapsto \inf_{y\in Y} g(x, y)$ is u.s.c. on X. Therefore, both sides of the inequality exist. Suppose that there exists a real c such that

$$\max_x \inf_y g(x, y) < c < \min_y \sup_x f(x, y).$$

Note that $(X \times Y; \Gamma_{X\times Y})$ is a compact KKM space. Define a map $T : X \times Y \multimap X \times Y$ by

$$T(x, y) = \{\bar{x} \in X \mid f(\bar{x}, y) > c\} \times \{\bar{y} \in Y \mid g(x, \bar{y}) < c\}$$

for $(x, y) \in X \times Y$. Then $T(x, y)$ is nonempty and Γ-convex for each $(x, y) \in X \times Y$ and $T^-(x, y)$ is open. By using Theorem 5.2, we have an $(x_0, y_0) \in X \times Y$ such that $(x_0, y_0) \in T(x_0, y_0)$. Therefore, $c < f(x_0, y_0) \le g(x_0, y_0) < c$, a contradiction. □

Remarks. 1. If $f = g$ and if X is a convex space, Theorem 8.2 reduces to Sion's generalization [36] of the von Neumann minimax theorem [37]:

$$\min_x \max_y f(x, y) = \max_y \min_x f(x, y).$$

2. If $(X; \Gamma)$ and $(Y; \Gamma')$ are G-convex spaces, so is $X \times Y$. Therefore, Theorem 8.2 works for G-convex spaces.

9. Nash type equilibrium theorems

In this section, from a Fan–Browder type fixed point result for abstract convex spaces, we deduce the Ky Fan intersection theorem and the Nash equilibrium theorem for abstract convex spaces.

Given a cartesian product $X = \prod_{i=1}^n X_i$ of sets, let $X^i = \prod_{j\ne i} X_j$ and $\pi_i : X \to X_i$, $\pi^i : X \to X^i$ be the projections; we write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x, y \in X$, we let

$$(y_i, x^i) := (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n).$$

From Theorem 5.2, we have the following Ky Fan type intersection theorem [3] generalizing the well-known von Neumann intersection lemma [38]:

Theorem 9.1. *Let $X = \prod_{i=1}^n X_i$ and $(X; \Gamma)$ be a compact KKM space, and $A_1, A_2, \ldots, A_n$ n subsets of X such that*

(1) *for each $x \in X$ and $i = 1, \ldots, n$, the set $A_i(x) = \{y \in X \mid (y_i, x^i) \in A_i\}$ is Γ-convex and nonempty; and*

(2) *for each $y \in X$ and $i = 1, \ldots, n$, the set $A_i(y) = \{x \in X \mid (y_i, x^i) \in A_i\}$ is open.*

Then $\bigcap_{i=1}^n A_i \ne \emptyset$.

Proof. Define a map $T : X \multimap X$ by $T(x) := \bigcap_{i=1}^n A_i(x)$ for $x \in X$. Then each $T(x)$ is Γ-convex being an intersection of Γ-convex sets by (1). For each $x \in X$ and each i, there exists a $y^{(i)} \in A_i(x)$ by (1), or $(y_i^{(i)}, x^i) \in A_i$. Hence, we have $(y_1^{(1)}, \ldots, y_n^{(n)}) \in \bigcap_{i=1}^n A_i(x)$. This shows that $T(x) \ne \emptyset$. Moreover, $T^-(y) = \bigcap_{i=1}^n A_i(y)$ is open for each $y \in X$ by (2). Now, the conclusion follows from Theorem 5.2. □

Remark. If each X_i is a compact abstract convex space, so is X with the product topology; see Lemma 8.1. Note that Theorem 8.2 can be also deduced from Theorem 9.1.

From Theorem 9.1, we deduce the following Nash equilibrium theorem [14] for abstract convex spaces:

Theorem 9.2. *Let $X = \prod_{i=1}^{n} X_i$ and $(X; \Gamma)$ be a compact KKM space, and $f_1, \dots, f_n : X \to \mathbb{R}$ continuous functions such that*

(1) *for each $x \in X$, each $i = 1, \dots, n$, and each $r \in \mathbb{R}$, the set $\{(y_i, x^i) \in X \mid f_i(y_i, x^i) > r\}$ is Γ-convex.*

Then there exists a point $x \in X$ such that

$$f_i(x) = \max_{y_i \in X_i} f_i(y_i, x^i) \quad \text{for } i = 1, \dots, n.$$

Proof. Let $\varepsilon > 0$ and, for each i, let

$$A_i^\varepsilon = \{x \in X \mid f_i(x) > \max_{y_i \in X_i} f_i(y_i, x^i) - \varepsilon\}.$$

Then the sets $A_1^\varepsilon, \dots, A_n^\varepsilon$ satisfy conditions (1) and (2) of Theorem 9.1, and hence $\bigcap_{i=1}^{n} A_i^\varepsilon \neq \emptyset$. Then $H_\varepsilon = \bigcap_{i=1}^{n} \overline{A_i^\varepsilon}$ is a nonempty compact set. Since $H_{\varepsilon_1} \subset H_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$, we have $\bigcap_{\varepsilon > 0} H_\varepsilon \neq \emptyset$. Then $x \in \bigcap_{\varepsilon > 0} H_\varepsilon$ satisfies the conclusion. □

10. Himmelberg type fixed point theorems

We need the following:

Definition. A *KKM uniform space* $(E, D; \Gamma; \mathcal{U})$ is a KKM space with a basis $\mathcal{U}$ of a Hausdorff uniform structure of E.

In this section, we introduce particular subclasses or subsets of KKM uniform spaces.

Definition. A KKM uniform space $(E \supset D; \Gamma; \mathcal{U})$ is called an *$L\Gamma$-space* if D is dense in E and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] := \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset E$ is Γ-convex.

In particular, for G-convex spaces or C-spaces $(E \supset D; \Gamma; \mathcal{U})$, we can define LG-spaces or LC-spaces ($l.c.$-spaces), resp.

Remarks. 1. The LG-spaces are introduced in [21].

2. A singleton is not necessarily Γ-convex in an $L\Gamma$-space.

Examples 10.1. For a C-space $(X; \Gamma)$, an $L\Gamma$-space reduces to an LC-space [10,11]. Any nonempty convex subset X of a locally convex t.v.s. E is an obvious example of an LC-space $(X; \Gamma)$ with $\Gamma_A = \operatorname{co} A$ for $A \in \langle X \rangle$. For other examples, see [10,11].

Examples 10.2. A G-convex space $(X \supset D; \Gamma)$ is called a *metric LG-space* if X is equipped with a metric d such that (1) D is dense in X, (2) for any $\varepsilon > 0$, the set $\{x \in X \mid d(x, C) < \varepsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, and (3) open balls are Γ-convex. This concept generalizes that of metric LC-spaces due to Horvath [10].

Examples 10.3 (*Horvath [11]*). Any hyperconvex metric space (H, d) is a complete metric LC-space $(H; \Gamma)$.

The following is the main result of this section:

Theorem 10.1. *Let $(X \supset D; \Gamma; \mathcal{U})$ be an $L\Gamma$-space and $T : X \multimap X$ a compact u.s.c. map with closed Γ-convex values. Then T has a fixed point $x_0 \in X$.*

Proof. For any closed $V \in \mathcal{U}$, there exists an open member W of $\mathcal{U}$ such that $W \subset V$. Note that for each $x \in X$, $W[x]$ is an open neighborhood of x. Since $K = \overline{T(X)}$ is compact and D is dense in X, there exists an $M = \{y_1, \dots, y_n\} \in \langle D \rangle$ such that $K \subset \bigcup_{y \in M} W[y]$.

For each $y_i \in M$, let $G(y_i) := \{x \in X \mid T(x) \cap V[y_i] = \emptyset\}$. Since T is u.s.c., each $G(y_i)$ is open. Moreover, since $T(X) \subset K \subset \bigcup_{i=1}^{n} V[y_i]$, we have

$$\bigcap_{i=1}^{n} G(y_i) \subset \left\{ x \in X \mid T(x) \cap \bigcup_{i=1}^{n} V[y_i] = \emptyset \right\} = \emptyset.$$

We will apply Theorem 3.2 to the KKM space $(X, M; \Gamma)$. Since $G : M \multimap X$ cannot be a KKM map; that is, there exist an $N \in \langle M \rangle$ and an $x_V \in \Gamma_N$ such that $x_V \notin G(N) = \bigcup_{y \in N} G(y)$. Hence $T(x_V) \cap V[y] \neq \emptyset$ for all $y \in N$, and

$$N \subset L := \{y \in X \mid T(x_V) \cap V[y] \neq \emptyset\}.$$

Since $T(x_V)$ is Γ-convex and $(X, D; \Gamma)$ is an $L\Gamma$-space, L is Γ-convex. Therefore, $x_V \in \Gamma_N \subset L$ and hence $T(x_V) \cap V[x_V] \neq \emptyset$.

So, for each basis element V, there exist $x_V, y_V \in X$ such that $y_V \in T(x_V)$ and $y_V \in V[x_V]$. Since $T(X)$ is relatively compact, we may assume that y_V converges to some $x_0 \in K$. Since X is Hausdorff, x_V also converges to x_0. Since T is u.s.c. with closed values, the graph of T is closed in $X \times \overline{T(X)}$, and hence we have $x_0 \in Tx_0$. This completes our proof. □

Examples 10.4. 1. (Himmelberg [6]) X is a convex subset of a Hausdorff locally convex t.v.s.

2. (Park [21]) $(X, D; \Gamma)$ is an LG-space. Many particular forms were stated there.

3. The extended long line L^* is a compact $L\Gamma$-space. Now it has *the fixed point property*. This is a proper example of Theorem 10.1 which is not an LG-space.

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