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GENERALIZATIONS OF THE HIMMELBERG FIXED POINT THEOREM

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ABSTRACT. This is to review various generalizations of the Himmelberg fixed point theorem within the category of topological vector spaces. We consider the Lassonde type, the Idzik type, and the KKM type generalizations for Kakutani maps, and other types of generalizations for acyclic maps. Finally, generalizations for various 'better' admissible maps on admissible almost convex domains to Klee approximable ranges are discussed.

1. INTRODUCTION

In 1972, Himmelberg [6] obtained two generalizations of a well-known fixed point theorem of Fan [4] in 1952 and applied them to generalize the minimax theorem due to von Neumann by following Kakutani's method in [10]. One of the generalizations is usually called the Himmelberg fixed point theorem which unifies and generalizes historically well-known theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, and others. For the literature, see [19].

The Himmelberg fixed point theorem says that *a compact Kakutani map from a convex subset of a locally convex Hausdorff topological vector space into itself has a fixed point.* Since its appearance, numerous applications and a number of generalizations have followed. Most of such generalizations concerned with almost convex subsets, non-locally convex topological vector spaces, or multimaps more general than Kakutani maps. Moreover, there have also appeared generalizations for abstract convex spaces without any linear structure, e.g., see [21].

This paper is to review various generalizations of the Himmelberg fixed point theorem within topological vector spaces. We consider the Lassonde type, the Idzik type, and the KKM type generalizations for Kakutani maps, and other type of generalizations for acyclic maps. Finally, generalizations for various 'better' admissible maps on admissible almost convex domains or maps having Klee approximable ranges are discussed.

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In the present paper, we will not cover the so-called condensing maps and any maps on abstract convex spaces.

2. Preliminaries

All topological spaces are assumed to be Hausdorff unless explicitly stated otherwise. A t.v.s. means a topological vector space and $\mathcal{V}$ denotes a fundamental system of neighborhoods of the origin 0 of a t.v.s. E. The convex hull and closure operations are denoted by co and $\overline{}$, resp.

A *multimap* or *map* $F : X \multimap Y$ is a function from a set X into the set 2^Y of *nonempty* subsets of a set Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) = \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{F(x) \mid x \in A\}$. For any $B \subset Y$, the (*lower*) *inverse* of B under F is defined by

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

For topological spaces X and Y, a map $F : X \multimap Y$ is said to be *closed* if its graph

$$\mathrm{Gr}(F) = \{(x, y) \mid y \in F(x),\ x \in X\}$$

is closed in $X \times Y$, and *compact* if $F(X)$ is contained in a compact subset of Y.

$F : X \multimap Y$ is said to be *upper semicontinuous* (*u.s.c.*) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X; *lower semicontinuous* (*l.s.c.*) if, for each open set $B \subset Y$, $F^-(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c.

If F is u.s.c. with closed values and Y is regular, then F is closed. The converse is true whenever Y is compact.

A *Kakutani map* is a u.s.c. multimap with compact convex values.

A topological space is said to be *acyclic* if all of its reduced Čech homology groups over the rational field vanish. A u.s.c. map is said to be *acyclic* if it has compact acyclic values.

A *polytope* P in a t.v.s. E is a homeomorphic image of a simplex.

Let X and Y be topological spaces. An *admissible class* $\mathfrak{A}^\kappa_c(X, Y)$ of maps $T : X \multimap Y$ is the one such that, for each compact subset K of X, there exists a map $S \in \mathfrak{A}_c(K, Y)$ satisfying $S(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite compositions of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(1) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(2) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(3) for each polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$, the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$, the Powers maps $\mathbb{V}_c$, the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of uniform spaces), admissible maps of Górniewicz, the Simons maps $\mathbb{K}_c$, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, and others. Further, the Fan-Browder maps (codomains are convex sets), locally selectionable maps having convex values, $\mathbb{K}_c^+$ due to Lassonde, $\mathbb{V}_c^+$ due to Park et al., and approximable maps $\mathbb{A}_c^\kappa$ due to Ben-El-Mechaiekh and Idzik are examples of $\mathfrak{A}_c^\kappa$. For the literature, see [15]-[18].

For a subset X of a t.v.s. E and a topological space Y, we defined the *"better" admissible class* $\mathfrak{B}$ of maps as follows [17], [18]:

$T \in \mathfrak{B}(X,Y) \Longleftrightarrow T : X \multimap Y$ is a map such that for any polytope P in X and any continuous function $f : T(P) \to P$, the composition $f(T|_P) : P \multimap P$ has a fixed point.

Note that $\mathfrak{A}_c^\kappa(X,X) \subset \mathfrak{B}(X,X)$ and some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ are known. Moreover, compact closed maps in the class KKM due to Chang and Yen [2] and in the class s-KKM due to Chang et al. [1] also belong to the class $\mathfrak{B}$; see [24].

3. The Himmelberg fixed point theorem

Himmelberg [6] defined that a subset A of a t.v.s. E is said to be *almost convex* if for any neighborhood $V \in \mathcal{V}$ of the origin 0 in E and for any finite set $\{w_1, \ldots, w_n\}$ of points of A, there exist $z_1, \ldots, z_n \in A$ such that $z_i - w_i \in V$ for all i, and $\mathrm{co}\{z_1, \ldots, z_n\} \subset A$.

In 1972, Himmelberg [6] derived the following from the Kakutani fixed point theorem [10]:

Theorem 3.1. *Let K be a nonvoid compact subset of a separated locally convex space L and $G : K \to K$ be a u.s.c. multifunction such that $G(x)$ is closed for all x in K and convex for all x in some dense almost convex subset A of K. Then G has a fixed point.*

Theorem 3.2. *Let T be a nonvoid convex subset of a separated locally convex space L. Let $F : T \to T$ be a u.s.c. multifunction such that $F(x)$ is closed and convex for all $x \in T$, and $F(T)$ is contained in some compact subset C of T. Then F has a fixed point.*

In the above we stated their original forms. Usually Theorem 3.2 is called the Himmelberg fixed point theorem.

4. The Lassonde type generalizations

In 1983, Lassonde [13] obtained the following generalization of Theorem 3.2:

Theorem 4.1. *Let X and C be nonempty convex subsets of a locally convex t.v.s. E. Let $T : X \multimap X + C$ be a compact Kakutani map. Suppose that one of the following holds:*

(1) *X is closed and C is compact.*
(2) *X is compact and C is closed.*
(3) $C = \{0\}$.

Then there is an $x_0 \in X$ such that $T(x_0) \cap (x_0 + C) \neq \emptyset$.

Note that Case (3) is the Himmelberg theorem.

Theorem 4.1 is generalized by the present author to acyclic maps $\mathbb{V}$ (1992, [14]), compositions of acyclic maps $\mathbb{V}_c$ (1994), admissible maps $\mathfrak{A}_c$ (1994), and better admissible maps $\mathfrak{B}$ (1997, [17]) by step by step.

5. The Idzik type generalizations

A subset B of a t.v.s. E is said to be *convexly totally bounded* (c.t.b.) (Idzik [7]) if for every $V \in \mathcal{V}$ there exist a finite subset $\{x_i \mid i \in I\}$ of B and a finite family $\{C_i \mid i \in I\}$ of convex subsets of V such that $B \subset \bigcup\{x_i + C_i \mid i \in I\}$. Note that $\{x_i \mid i \in I\}$ can be chosen in the whole space E; see Idzik and Park [8].

Idzik [7] showed that any locally convex compact subset and any compact subset of the Zima type are c.t.b. Moreover, every subset of a locally convex t.v.s. E is locally convex and of the Zima type. For other examples of c.t.b. sets, see [3].

In 1988, the following was given by Idzik [7, Theorem 4.3]:

Theorem 5.1. *Let D, C be nonempty almost convex sets in a t.v.s. E and C a dense subset of D. Let $\varphi : D \to 2^D$ be a u.s.c. map with nonempty closed values such that $\varphi(x)$ is convex for all $x \in C$. If $\overline{\varphi(D)}$ is a compact c.t.b. subset of D, then there exists $x \in D$ such that $x \in \varphi(x)$.*

The proof of Theorem 5.1 is not elementary. Note that Theorem 5.1 generalizes Theorem 3.1 and, for $D = C$, Theorem 3.2.

6. The KKM type generalizations

As the author once stated, the classical KKM principle implies many fixed point theorems [23]. This section deals with generalizations of Theorems 3.1 and 3.2 which can be deduced from the KKM principle and its open version.

In 2000, for Kakutani maps, we had the following in Park and Tan [27]:

Theorem 6.1. *Let X be a subset of a locally convex t.v.s. E and Y an almost convex dense subset of X. Let $T : X \multimap X$ be a compact u.s.c. map with nonempty closed values such that $T(y)$ is convex for all $y \in Y$. Then T has a fixed point.*

In particular, for $Y = X$, we obtain the following:

Theorem 6.2. *Let X be an almost convex subset of a locally convex t.v.s. Then any compact Kakutani map $T : X \multimap X$ has a fixed point.*

Note that Theorems 6.1 and 6.2 generalize Theorems 3.1 and 3.2, resp., and Theorem 6.2 is a consequence of Theorem 5.1. Note also that the proofs of Theorems 6.1 and 6.2 are transparent and based on the KKM principle.

In 2000 [20], we have the following almost fixed point theorem:

Theorem 6.3. *Let X be a subset of a t.v.s. and Y an almost convex dense subset of X. Let $T : X \multimap E$ be an l.s.c. [resp., a u.s.c.] map such that $T(y)$ is convex for all $y \in Y$. If there is a totally bounded subset K of $\overline{X}$ such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$, then for any convex neighborhood U of the origin 0 of E, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Note that a t.v.s. is not necessarily Hausdorff in Theorem 6.3. The point x_U in the conclusion of Theorem 6.3 is called a *U-almost fixed point* of the multimap T. It is routine to deduce Theorems 3.1 and 3.2 from Theorem 6.3.

In 2004 [11], from the KKM principle and its open version, we deduced a very general almost fixed point theorem on almost convex subsets having a certain form of the Zima type as follows:

Theorem 6.4. *Let X be a subset of a t.v.s. E and Y an almost convex subset of X. Let $T : X \multimap E$ be an l.s.c. [resp., a u.s.c.] map such that $T(y)$ is convex for all $y \in Y$. Suppose that*

(Z_1) *for each $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ such that*

$$\mathrm{co}(V \cap (T(Y) - Y)) \subset U.$$

If there is a totally bounded subset K of $\overline{X}$ such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$ and $Y \cap K$ is dense in K, then for any $U \in \mathcal{V}$, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Note that if $X = Y$ then the condition (Z_1) reduces to the following:

(Z_2) *for each $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ such that*

$$\mathrm{co}(V \cap (T(X) - X)) \subset U.$$

According to Hadžić [5], a subset K of E is said to be *of the Zima type* if for each $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ such that $\mathrm{co}(V \cap (K - K)) \subset U$.

Note that Theorem 6.4 generalizes Theorem 6.3 and has a number of particular forms; see [11] and [23]. For example, the following is a particular form:

Corollary 6.5. *Let X be a convex subset of a locally convex t.v.s. E and $T : X \multimap X$ a u.s.c. [resp., an l.s.c.] map with convex values such that $T(X)$ is totally bounded. Then T has the almost fixed point property; that is, for each $V \in \mathcal{V}$, there exists an $x_V \in X$ such that $T(x_V) \cap (x_V + V) \neq \emptyset$.*

Moreover, Theorem 6.4 is useful to deduce various forms of fixed point theorems including that of the Zima type. In fact, we have

Theorem 6.6. *Under the hypothesis of Theorem 6.4, if K is compact, then there exists a point $x_0 \in X$ such that $(x_0, x_0) \in \overline{\mathrm{Gr}(T)}$.*

Now we apply Theorem 6.6 to compact maps as in [26]:

Corollary 6.7. *Let X be a convex subset of a locally convex t.v.s. E and $T : X \multimap X$ a compact u.s.c. [resp., l.s.c.] map with convex values. Then there exists a point $x_0 \in X$ such that $(x_0, x_0) \in \overline{\mathrm{Gr}(T)}$.*

For a u.s.c. map in Theorem 6.6, we have the following:

Theorem 6.8. *Let X be a subset of a t.v.s. E and Y an almost convex subset of X. Let $T : X \multimap X$ be a compact u.s.c. map with closed values such that $T(y)$ is convex for all $y \in Y$, and $Y \cap T(X)$ is dense in $T(X)$. If the condition (Z_1) holds, then T has a fixed point $x_0 \in X$, that is, $x_0 \in T(x_0)$.*

If E is locally convex and Y is dense in X, then Theorem 6.8 reduces to Theorems 6.1 and 6.2. In particular, for $Y = X$, Theorem 6.8 reduces to

Theorem 6.9. *Let X be an almost convex subset of a t.v.s. E. Then any compact Kakutani map $T : X \multimap X$ has a fixed point in X whenever the condition (Z_2) holds.*

From Theorem 6.9, we have Theorem 6.2.

Moreover, it is well-known that the Brouwer fixed point theorem is equivalent to the KKM principle and, since each of theorems and corollaries in this section implies the Brouwer theorem and is deduced from the KKM principle, they are all equivalent to the Brouwer theorem.

7. Kakutani maps on admissible sets

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset K of X and every $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E. Examples of admissible sets can be seen in [5].

In 2007 [25], we obtained another generalized versions of results of Himmelberg [6]. One of them for maps having convex values is as follows:

Theorem 7.1. *Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Let $G : Y \multimap Y$ be a compact closed map such that $G(x)$ is convex for all $x \in X$. Then G has a fixed point.*

For $X = Y$, Theorem 7.1 reduces to

Corollary 7.2. *Let X be an almost convex admissible subset of a t.v.s. E. Let $G : X \multimap X$ be a compact Kakutani map. Then G has a fixed point.*

This implies Theorems 6.2 and 3.2; see also [21, Corollary 1]. Note that our proof in [25] is quite different from that in [6] or [21].

8. Acyclic maps $\mathbb{V}$

In 1992, Theorem 4.1 was extended to acyclic maps in [14]. The following case (3) of which is known to be very useful in many applications:

Theorem 8.1. *Let X be a convex subset of a locally convex t.v.s. Then every compact acyclic map $T : X \multimap X$ has a fixed point.*

Theorem 8.1 was extended to the class $\mathfrak{A}_c$ (1993), $\mathbb{V}_c^\sigma$ (1994), $\mathfrak{A}_c^\sigma$ (1994), and $\mathfrak{B}$ (1997, [17]). In [25], this is extended to a non-locally convex t.v.s. In fact, for maps having acyclic values or values of trivial shape (that is, contractible in each neighborhood), one of our results in [25] can be stated as follows:

Theorem 8.2. *Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Let $G : Y \multimap Y$ be a compact closed map such that $G(x)$ is acyclic (resp., has trivial shape) for all $x \in X$. Then G has a fixed point.*

Corollary 8.3. *Let X be an almost convex dense subset of a closed subset Y of a locally convex t.v.s. E. Let $G : Y \multimap Y$ be a compact u.s.c. map with closed values such that $G(x)$ is acyclic (resp., has trivial shape) for all $x \in X$. Then G has a fixed point.*

Corollary 8.4. *Let X be an almost convex admissible subset of a t.v.s. E. Then any compact closed map $G : X \multimap X$ such that $G(x)$ is acyclic (resp., has trivial shape) for all $x \in X$ has a fixed point.*

If X is a convex subset of a locally convex t.v.s. E and G has convex values, then Corollary 8.4 reduces to Theorem 3.2.

9. $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps

Let X and D be subsets of a t.v.s. such that $\operatorname{co} D \subset X$ and Y a topological space. A multimap $F : X \multimap Y$ is said to have *the KKM property* if, for any map $G : D \multimap Y$ with closed [resp., open] values satisfying

$$F(\operatorname{co} A) \subset G(A) \qquad \text{for all } \ A \in \langle D \rangle,$$

the family $\{G(z)\}_{z \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(X, Y) := \{F : X \multimap Y \mid F \text{ has the KKM property}\}.$$

Some authors use the notation $KKM(X, Y)$. Note that $1_X \in \mathfrak{K}(X, X)$ by the KKM principle. Moreover, if $F : X \to Y$ is a continuous single-valued map or if $F : X \multimap Y$ has a continuous selection, then it is easy to check that $F \in \mathfrak{K}(X, Y)$. Note that there are many known selection theorems due to Michael and others; see [26].

From now on, $\mathfrak{KC}$ denote the class $\mathfrak{K}$ for closed-valued maps G, and $\mathfrak{KO}$ for open-valued maps G. We follow [22]:

For convex subsets of a t.v.s., from the KKM principle, we had the following almost fixed point theorem for the class $\mathfrak{KC}$:

Theorem 9.1. *Let X be a convex subset of a t.v.s. E and $F \in \mathfrak{KC}(X,X)$ such that $F(X)$ is totally bounded. Then for any convex neighborhood V of 0 in E, there exists an $x_* \in X$ such that $F(x_*) \cap (x_* + V) \neq \emptyset$.*

Similarly, we obtained the following for the class $\mathfrak{KO}$:

Theorem 9.2. *Let X be a totally bounded convex subset of a t.v.s. E and $F \in \mathfrak{KO}(X,X)$. Then for each closed convex neighborhood V of 0 in E, there exists an $x_* \in X$ such that $F(x_*) \cap (x_* + V) \neq \emptyset$.*

Note that E is not necessarily Hausdorff in Theorems 9.1 and 9.2. From Theorem 9.1, we immediately have the following:

Corollary 9.3 (Chang–Yen [2])**.** *Let X be a convex subset of a locally convex t.v.s. E. Then any compact closed map $F \in \mathfrak{KC}(X,X)$ has a fixed point.*

10. Better admissible maps $\mathfrak{B}$

In 1998, we obtained the following [18]:

Theorem 10.1. *Let E be a t.v.s. and X an admissible* (*in the sense of Klee*) *convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

In [18], it was shown that Theorem 10.1 subsumes more than sixty known or possible particular cases and generalizes them in terms of the involving spaces and multimaps as well. Later, further examples of maps in the class $\mathfrak{B}$ were known.

It is not known whether the admissibility of X can be eliminated in Theorem 10.1.

Theorem 10.1 can be generalized by switching the admissibility of domain of the map to the Klee approximability of its ranges as follows.

Let X be a subset of a t.v.s. E. A compact subset K of X is said to be *Klee approximable into X* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in X.

Examples. We give some examples of Klee approximable sets:

(1) If a subset X of E is admissible (in the sense of Klee), then every compact subset K of X is Klee approximable into E.

(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.

(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.

(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.

(5) Any compact subset K of an admissible convex subset X of a t.v.s. is Klee approximable into X.

(6) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.

Note that (6)⇒(5)⇒(4)⇒(3).

In 2004 [24], Theorem 10.1 is generalized as follows:

Theorem 10.2. *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}(X,X)$ a compact closed multimap. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

The following are recently obtained in 2007 [25], where $\mathfrak{B}^p$ should be replaced by $\mathfrak{B}$:

Corollary 10.3. *Let X be an almost convex admissible subset of a t.v.s. E and $F \in \mathfrak{B}(X,X)$ a compact closed map. Then F has a fixed point.*

Corollary 10.4. *Let X be an almost convex subset of a locally convex t.v.s. E and $F \in \mathfrak{B}(X,X)$ a compact closed map. Then F has a fixed point.*

One of the most simple known example is that every compact continuous selfmap on an almost convex subset in a Euclidean space has a fixed point. This generalizes the Brouwer fixed point theorem.

Moreover, since the class $\mathfrak{B}(X,X)$ contains a large number of special types of maps, we can apply Theorem 10.2 to them. Since any Kakutani map belongs to $\mathfrak{B}$, Theorem 10.2 and Corollaries 10.3 and 10.4 generalize Theorem 3.2. Moreover, Corollary 10.3 also generalizes Corollary 9.3.

More recently, some new types of generalizations of the Himmelberg theorem appeared in [9] and [12].

Final Remark. The author personally met Professor Himmelberg in Summer of 1972 at Lawrence, Kansas, when he was visiting the States. That time, he did not know about the theorem. Since then he met the theorem from time to time in so many articles. This remark reflects his personal recollections.

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