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REMARKS ON FIXED POINTS, MAXIMAL ELEMENTS, AND EQUILIBRIA OF ECONOMIES IN ABSTRACT CONVEX SPACES

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. In this paper, KKM theorems or coincidence theorems on abstract convex spaces are applied to obtain the Fan-Browder type fixed point theorems, existence of maximal elements, existence of economic equilibria and some related results. Consequently, we obtain generalizations or improvements of a number of known equilibria results, especially, in a recent work of Ding and Wang [3] on the so-called FC-spaces.

1. Introduction

The KKM theory, first called by the author [8], is originated from the Knaster-Kuratowski-Mazurkiewicz theorem (simply, KKM principle), and nowadays regarded as the study of applications of various equivalent formulations of the KKM principle and their generalizations; see [10, 11] and references therein. At the beginning, the theory was devoted to convex subsets of topological vector spaces, and later, to convex spaces by Lassonde [7], to C-spaces (or H-spaces) by Horvath [5, 6], and to generalized convex (G-convex) spaces by the present author [11, 12, 18, 23, 24].

Recently there have appeared several imitations of G-convex spaces and a number of authors have tried to obtain generalizations of our G-convex space theory. Most of such imitations are unified to the class of ϕ_A-spaces; see [17, 19, 21]. Moreover, in our recent works [13-17, 19-22], we introduced a new class of abstract convex spaces more general than G-convex spaces and obtained basic results in the KKM theory and fixed point theory within the new classes.

Recall that the existence theorems on equilibria in economics due to Borglin and Keiding, Yannelis and Prabhakar, Toussiant, and Tulcea were generalized to various

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abstract convex spaces by Tarafdar, Tan, Ding, Yuan, and others in a large number of works; for the literature, see [3] and references therein. In [3], it is claimed that most of such works are improved and generalized in the frame of the so-called FC-spaces, which are particular forms of ϕ_A-spaces and hence of abstract convex spaces.

In the present paper, KKM theorems or coincidence theorems on abstract convex spaces in [13, 20] are applied to obtain the Fan-Browder type fixed point theorems, existence of maximal elements, and existence of economic equilibria. Consequently, we obtain generalizations or unifications of a number of known equilibria results, especially, in a recent work of Ding and Wang [3] on FC-spaces.

In Section 2, definitions and some basic facts on abstract convex spaces and the map classes $\mathfrak{KC}$, $\mathfrak{KO}$ are introduced. Section 3 deals with ϕ_A-spaces as a unified and generalized concept of various imitations of G-convex spaces. A KKM theorem for ϕ_A-spaces is also given. In Section 4, general forms of the Fan-Browder type coincidence and fixed point theorems are derived for abstract convex spaces. Section 5 deals with various existence theorems on maximal elements in abstract convex spaces. Finally, in Section 6, new existence theorems of maximal elements for $\mathcal{L}_F$-majorized correspondences and a new equilibrium existence theorem for one person game with $\mathcal{L}_F$-majorized correspondences are obtained in abstract convex spaces.

2. Abstract Convex Spaces

In this section, we recall definitions and some basic results on abstract convex spaces given in [13, 20].

A *multimap* (simply, a *map*) or a *correspondence* $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$. For any $B \subset Y$, the (*lower*) *inverse* of B under F is defined by

$$F^-(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

The following is the origin of the KKM theory; see [10, 11]:

The KKM Principle. *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map* (*that is,* co $A \subset G(A)$ *for each* $A \subset D$) *with closed* [*resp., open*] *values. Then* $\bigcap_{z\in D} G(z) \neq \emptyset$.

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definitions. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a map $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\text{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D' \rangle\}.$$

[co is reserved for the convex hull in vector spaces].

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to $D' \subset D$ if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\text{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D' \rangle})$ itself is an abstract convex space called a *subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\text{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

We already gave plenty of examples of abstract convex spaces in [13, 20, 21].

Definitions. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A map $F : E \multimap Z$ is said to have the *KKM property* and called *a* $\mathfrak{KC}$*-map* if, for any closed-valued KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{KC}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{KC}\text{-map}\}.$$

Similarly, we define a $\mathfrak{KO}$*-map* for open-valued maps G.

The following is given [20, Lemma 2]:

Lemma 2.1. *Let $(E, D; \Gamma)$ be an abstract convex space, $(X, D'; \Gamma|_{\langle D' \rangle})$ a subspace, and Z a topological space. If $F \in \mathfrak{KC}(E, Z)$, then $F|_X \in \mathfrak{KC}(X, Z)$. This also holds for $\mathfrak{KO}$ instead of $\mathfrak{KC}$.*

The following theorems are given in [13, 20]:

Theorem 2.2. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, and $F : E \multimap Z$ a map. Then $F \in \mathfrak{KC}(E, Z)$* [resp., $F \in \mathfrak{KO}(E, Z)$] *if and only if for any closed-valued* [resp., *open-valued*] *map $G : D \multimap Z$ satisfying*

(1) $F(\Gamma_N) \subset G(N)$ for any $N \in \langle D \rangle$,

we have $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.

Theorem 2.3. *Let* $(E, D; \Gamma)$ *be an abstract convex space,* Z *a topological space,* $S : D \multimap Z$, $T : E \multimap Z$ *maps, and* $F \in \mathfrak{KC}(E, Z)$ [*resp.,* $F \in \mathfrak{KO}(E, Z)$]. *Suppose that*

(1) S *is open-valued* [*resp., closed-valued*];
(2) *for each* $z \in F(E)$, $\mathrm{co}_\Gamma S^-(z) \subset T^-(z)$ [*that is,* $T^-(z)$ *is* Γ*-convex relative to* $S^-(z)$]; *and*
(3) $F(E) \subset S(N)$ *for some* $N \in \langle D \rangle$.

Then there exists an $\bar{x} \in E$ *such that* $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

3. ϕ_A-Spaces

The following due to the present author is a typical example of abstract convex spaces:

Definition. *generalized convex space* or a *G-convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is a standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For details on G-convex spaces; see [11, 12, 18, 23, 24] and references therein.

When $E = D$, a G-convex space is called an L-space by Ben-El-Mechaiekh et al. [1].

Recently, there have appeared some authors who introduced spaces of the form $(X, \{\varphi_A\})$; see [1-3, 17, 19, 21]. Some of them tried to rewrite our works on G-convex spaces by simply replacing $\Gamma(A)$ by $\varphi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples.

Motivated by this fact, we are concerned with a reformulation of the class of G-convex spaces as follows [19, 21]:

Definition. A ϕ_A*-space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Note that any ϕ_A-space is an abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \mathrm{Im}\, \phi_A$ for $A \in \langle D \rangle$.

Any G-convex space is a ϕ_A-space. The converse also holds:

Proposition 3.1. *A ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ can be made into a G-convex space $(X, D; \Gamma)$.*

Proof. This can be done in two ways.

(1) For each $A \in \langle D\rangle$, by putting $\Gamma_A := X$, we obtain a trivial G-convex space $(X, D; \Gamma)$.

(2) Let $\{\Gamma^\alpha\}_\alpha$ be the family of maps $\Gamma^\alpha : \langle D\rangle \multimap X$ giving a G-convex space $(X, D; \Gamma^\alpha)$ such that $\phi_A(\Delta_n) \subset \Gamma^\alpha_A$ for each $A \in \langle D\rangle$ with $|A| = n + 1$. Note that, by (1), this family is not empty. Then, for each α and each $A \in \langle D\rangle$ with $|A| = n + 1$, we have

$$\phi_A(\Delta_n) \subset \Gamma^\alpha_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma^\alpha_J \quad \text{for} \quad J \subset A.$$

Let $\Gamma := \bigcap_\alpha \Gamma^\alpha$, that is, $\Gamma_A = \bigcap_\alpha \Gamma^\alpha_A$ for each $A \in \langle D\rangle$. Then

$$\phi_A(\Delta_n) \subset \Gamma_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma_J \quad \text{for} \quad J \subset A.$$

Therefore, $(X, D; \Gamma)$ is a G-convex space. ■

Consequently, ϕ_A-spaces are another names of G-convex spaces and they are essentially the same.

Definition. For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map*.

Proposition 3.2.

(1) *A KKM map $G : D \multimap X$ on a G-convex space $(X, D; \Gamma)$ is a KKM map on the corresponding ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$.*

(2) *A KKM map $T : D \multimap X$ on a ϕ_A-space $(X, D; \{\phi_A\})$ is a KKM map on a new G-convex space $(X, D; \Gamma)$.*

Proof.

(1) This is clear from the definition of a KKM map on a G-convex space.

(2) Define $\Gamma : \langle D\rangle \multimap X$ by $\Gamma_A := T(A)$ for each $A \in \langle D\rangle$. Then $(X, D; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n + 1$, we have a continuous function $\phi_A : \Delta_n \to T(A) =: \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle D\rangle$ and hence $T : D \multimap X$ is a KKM map on a G-convex space $(X, D; \Gamma)$. ■

The following is a KKM theorem for ϕ_A-spaces. The proof is just a simple modification of the corresponding one in [11-13]:

Proposition 3.3. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, let $G : D \multimap X$ be a KKM map with closed* [*resp., open*] *values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property.* (*More precisely, for each $N \in \langle D\rangle$ with $|N| = n + 1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.*)

Further, if

(3) $\bigcap_{z\in M} \overline{G(z)}$ *is compact for some $M \in \langle D\rangle$,*

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

Proof. Let $N = \{z_0, z_1, \ldots, z_n\}$. Since G is a KKM map, for each vertex e_i of Δ_n, we have $\phi_N(e_i) \in G(z_i)$ for $0 \leq i \leq n$. Then $e_i \mapsto \phi_N^{-1}G(z_i)$ is a closed [resp., open] valued map such that $\Delta_k = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1}G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the original KKM principle, $\Delta_n \supset \bigcap_{i=0}^{n} \phi_N^{-1}G(z_i) \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \left(\bigcap_{z\in N} G(z)\right) \neq \emptyset$.

The second conclusion is clear. ■

Remarks.

(1) We may assume that, for each $a \in D$ and $N \in \langle D\rangle$, $G(a)\cap\phi_N(\Delta_n)$ is closed [resp., open] in $\phi_N(\Delta_n)$. This is said by some authors that G has finitely closed [resp., open] values. However, by replacing the topology of X by its finitely generated extension, we can eliminate "finitely"; see [12].

(2) For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \text{co}$, the convex hull, Theorem 3.3 reduces to the original KKM principle and its open version; see [10, 11].

(3) If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 3.3 extends Fan's KKM lemma; see [4,10].

(4) Note that any KKM theorem on spaces of the form $(X, \{\varphi_A\})$ can not generalize the original KKM principle or Fan's KKM lemma.

In 2005, Ding [2] introduced the following particular form of ϕ_A-spaces:

Definition. ([2]). $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset D of $(Y, \{\varphi_N\})$ is said to be a FC-subspace of Y if for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$ where $\Delta_k = \text{co}\{e_{i_j} : j = 0, \ldots, k\}$.

Note that for each N, there should be infinitely many φ_N's. In [3], its authors repeated the preceding definition where the restriction "where some elements in N may be same" in the original definition in [2] is removed.

Clearly an FC-space can be made into an L-space [1] by Proposition 3.1 and is a particular ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ for the case $X = D$. Further, Ding in [3] still insists that "It is clear that many topological spaces with abstract convexity structure are all FC-spaces. In particular, any convex subset of a topological vector space, any H-space introduced by Horvath [5], any G-convex spaces introduced by Park and Kim [24] and any L-convex spaces introduced by Ben-El-Mechaiekh et al. [1] are all FC-spaces. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces". This statement repeatedly appeared more than fifteen papers of Ding and his followers within the last two years. One wonders how could a pair $(Y, \{\varphi_N\})$ generalize a triple $(X, D; \Gamma)$ in [24].

In Lemma 2.4 [3], its authors showed that any generalized FC-KKM mapping is a generalized R-KKM mapping, which is already shown to be a simple KKM map for a G-convex space; see [17,21]. Lemmas 3.1 and 3.2 [3] are particular forms of Theorem 3.3. In [3], from their Lemma 3.2, it is routine in the KKM theory to deduce Fan-Browder type fixed point theorems (Section 3), Ky Fan type minimax inequalities and their geometric forms (Section 4), existence of maximal elements (Section 5), and equilibrium existence theorems (Section 6). The authors of [3] obtained these results for their FC-spaces, but it is evident that they can be stated more generally for G-convex spaces or ϕ_A-spaces, even for abstract convex spaces.

4. Coincidence and Fixed Points

From Theorem 2.3, we can generalize the Fan-Browder type coincidence theorem due to Park and Kim [23,24, Theorem 1]:

Theorem 4.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $S : D \multimap Z$, $T : E \multimap Z$ maps, and $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KO}(E, Z)$]. Suppose that*

(1) *S is open-valued [resp., closed-valued];*

(2) *for each $z \in F(E)$, $\mathrm{co}_\Gamma S^-(z) \subset T^-(z)$;*

(3) *there exists a nonempty subset K of Z such that $F(E) \cap K \subset S(N)$ for some $N \in \langle D\rangle$; and*

(4) *either*

(i) *$F(E) \setminus K \subset S(M)$ for some $M \in \langle D\rangle$; or*

(ii) *there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and $F(L_N) \setminus K \subset S(M)$ for some $M \in \langle D'\rangle$.*

Then there exists an $\overline{x} \in E$ such that $F(\overline{x}) \cap T(\overline{x}) \neq \emptyset$.

Proof.

Case (i). In view of Theorem 2.3, it suffices to show that condition (3) there holds. Since

$$F(E) = (F(E) \setminus K) \cup (F(E) \cap K) \subset S(M) \cup S(N) = S(N'),$$

where $N' := M \cup N$, the conclusion follows from Theorem 2.3.

Case (ii). Since L_N is a Γ-convex subset relative to D', $(L_N, D'; \Gamma \mid_{\langle D' \rangle})$ is a subspace. Instead of S and T, we can use $S|_{D'}$ and $T|_{L_N}$. Note that $F(L_N) \setminus K \subset S(M)$ for some $M \in \langle D' \rangle$. Now the conclusion follows from Case (i). ■

Corollary 4.2. *In Theorem 4.1, condition (ii) can be replaced by the following without affecting its conclusion:*
(ii)′ *$E \supset D$ and there exists a Γ-convex subset L_N of E containing N such that $F(L_N) \subset S(M)$ for some $M \in \langle L_N \cap D \rangle$.*

Proof. Choose $D' := L_N \cap D$ and apply (ii). ■

Corollary 4.2 generalizes [23, 24, Theorem 1] and leads too many modifications of the Fan-Browder coincidence theorem for G-convex spaces or ϕ_A-spaces in the literature. All of them might be particular cases of Theorem 4.1. Most of such modifications assume compactness of K and L_N and S has open values.

For $E = Z$ and $F = 1_E$, the identity map on E, Theorem 4.1 reduces to the following Fan-Browder type fixed point theorem:

Theorem 4.3. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying $1_E \in \mathfrak{KC}(E, E)$ [resp., $1_E \in \mathfrak{KO}(E, E)$], and $S : D \multimap E$, $T : E \multimap E$ maps. Suppose that*

(1) *S is open-valued [resp., closed-valued];*
(2) *for each $x \in E$, $\mathrm{co}_\Gamma S^-(x) \subset T^-(x)$;*
(3) *there exists a nonempty subset K of E such that $K \subset S(N)$ for some $N \in \langle D \rangle$; and*
(4) *either*

(i) *$E \setminus K \subset S(M)$ for some $M \in \langle D \rangle$; or*
(ii) *there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and $L_N \setminus K \subset S(M)$ for some $M \in \langle D' \rangle$.*

Then T has a fixed point $\overline{x} \in E$, that is, $\overline{x} \in T(\overline{x})$.

Remarks.

(1) Note that Theorem 4.3 works for G-convex spaces or ϕ_A-spaces and has numerous particular forms. Moreover, there are abstract convex spaces $(E, D; \Gamma)$ satisfying $1_E \in \mathfrak{KC}(E, E)$ [resp., $1_E \in \mathfrak{KO}(E, E)$] which are not G-convex spaces; see [22]. Therefore Theorem 4.3 properly generalizes the particular case for G-convex spaces.

(2) Usually Condition (ii) is called a coercivity or compactness condition. On the surface, Case (ii) seems to generalize Case (i), but both cases are equivalent as can be seen in the proof of Theorem 4.1.

The following is a new variant of Theorem 4.3(ii):

Theorem 4.4. *Let $(X \supset D; \Gamma)$ be an abstract convex space satisfying $1_X \in \mathfrak{KC}(X, X)$, $F : D \multimap X$, $G : X \multimap X$, and K a nonempty compact subset of X such that*

(1) *for each $x \in X$, $F^-(x) \subset G^-(x)$ and $G^-(x)$ is Γ-convex;*
(2) *for each $z \in D$, $F(z)$ is open in X;*
(3) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ containing N such that*

$$\bigcap_{z \in D'} \mathrm{cl}_{L_N}(L_N \setminus G(z)) \subset K;$$

(4) *for each $x \in K$, $F^-(x) \neq \emptyset$.*

Then there exists $\hat{y} \in X$ such that $\hat{y} \in G(\hat{y})$.

Proof 1. By (4), $K \subset \bigcup_{z \in D} F(z)$. Note that $F(z) \subset G(z)$ for each $z \in D$ by (1). Since each $F(z)$ is open by (2) and K is compact, we have an $N \in \langle D \rangle$ such that $K \subset \bigcup_{z \in N} F(z) \subset \bigcup_{z \in N} \mathrm{int}_X G(z)$ by (1). Note that

$$\mathrm{cl}_{L_N}(L_N \setminus (G(z)) = \mathrm{cl}_{L_N}(L_N \setminus (G(z) \cap L_N)) = L_N \setminus \mathrm{int}_{L_N}(G(z) \cap L_N).$$

Hence, by (3), we have

$$L_N \setminus K \subset \bigcup_{z \in D'} [\mathrm{cl}_{L_N}(L_N \setminus G(z))]^c = \bigcup_{z \in D'} [L_N \setminus \mathrm{int}_{L_N}(G(z) \cap L_N)]^c$$

$$\subset \bigcup_{z \in D'} \mathrm{int}_{L_N}(G(z) \cap L_N) \subset \bigcup_{z \in D'} \mathrm{int}_X G(z),$$

where ^{c} denotes the complement with respect to L_N. Therefore

$$L_N \subset (L_N \setminus K) \cup K \subset \bigcup_{z \in D'} \mathrm{int}_X G(z) \cup \bigcup_{z \in N} \mathrm{int}_X G(z).$$

Since L_N is compact and $N \subset D'$, there exists an $M \in \langle D' \rangle$ such that $L_N \setminus K \subset \bigcup_{z \in M} \mathrm{int}_X G(z)$.

Now let $S(z) := \mathrm{int}_X G(z) \subset G(z)$ for $z \in D$ and $T(x) := G(x)$ for $x \in X$. Then $\mathrm{co}_\Gamma S^-(x) \subset \mathrm{co}_\Gamma G^-(x) = G^-(x) = T^-(x)$. Hence all of the requirements of Theorem 4.3(ii) are satisfied. Therefore there exists $\hat{y} \in X$ such that $\hat{y} \in T(\hat{y}) = G(\hat{y})$. This completes our proof. ■

Proof 2. Suppose that the conclusion does not hold; that is, $x \notin G(x)$ for each $x \in X$. Define $T, S : D \multimap X$ by

$$T(z) := \mathrm{cl}_X(K \setminus G(z)) \quad \text{and} \quad S(z) := K \setminus F(z)$$

for $z \in D$.

Let $N \in \langle D \rangle$. By (3), there exists a compact Γ-convex subset L_N of X relative to some D' containing N. Define $T_0,\ S_0 : D' \multimap L_N$ by

$$T_0(z) := \mathrm{cl}_{L_N}(L_N \setminus G(z)) \quad \text{and} \quad S_0(z) := L_N \setminus F(z)$$

for $z \in D'$. Note that each $T_0(z)$ is compact, each $S_0(z)$ is relatively closed by (2), and $T_0(z) \subset S_0(z)$ by (1).

(i) T_0 is a KKM map.

Indeed, it suffices to show the map $T^* : D' \multimap L_N$ defined by

$$T^*(z) := L_N \setminus G(z) \quad \text{for} \quad z \in D'$$

is a KKM map. If this were not, there exist $A \in \langle D' \rangle$ and $x \in \mathrm{co}_\Gamma(A)$ such that

$$x \notin T^*(A) = L_N \setminus \bigcap_{z \in A} G(z) \subset X \setminus \bigcap_{z \in A} G(z).$$

Hence $x \in \bigcap_{z \in A} G(z)$ and $A \subset G^-(x)$. Therefore $x \in \mathrm{co}_\Gamma(A) \subset G^-(x)$ by (1). This contradicts our assumption.

(ii) T is a KKM map.

Since $1_X \in \mathfrak{KC}(X, X)$, this also holds for $(L_N, D'; \Gamma|_{\langle D' \rangle})$ by Lemma 2.1. Since T_0 is a closed-compact-valued KKM map, $\bigcap_{z \in D'} T_0(z) \neq \emptyset$. Then there exists

$$\hat{y} \in \bigcap_{z \in D'} T_0(z) = \bigcap_{z \in D'} \mathrm{cl}_{L_N}(L_N \setminus G(z)) \subset L_N \cap K$$

by (3). Therefore

$$\hat{y} \in \bigcap_{z \in D'} \mathrm{cl}_X(K \setminus G(z)) \subset \bigcap_{z \in D'} T(z) \subset \bigcap_{z \in N} T(z).$$

Hence, T is a closed-compact-valued KKM map. Since $1_X \in \mathfrak{KC}(X, X)$, we have $\bigcap_{z \in D} T(z) \neq \emptyset$. Since $T(z) \subset S(z)$ for each $z \in D$, we have

$$K \setminus \bigcup_{z \in D} F(z) = \bigcap_{z \in D} (K \setminus F(z)) = \bigcap_{z \in D} S(z) \neq \emptyset.$$

This contradicts (4). This completes our proof. ■

Remarks.

(1) Note that Theorem 4.4 works for G-convex spaces or ϕ_A-spaces.

(2) In Proof 2, we followed the proof of [3, Theorem 3.1] and its notation. This might be convenient to compare those proofs.

The following main result for FC-spaces in [3] is a particular form:

Corollary 4.5. ([3, Theorem 3.1]). *Let (X, φ_N) be an FC-space, $F, G : X \to 2^X$ and K be a nonempty compact subset of X such that*

(1) *for each $x \in X$, $F(x) \subset G(x)$,*

(2) *for each $y \in X$, $F^{-1}(y)$ is compactly open in X,*

(3) *for each $N \in \langle X \rangle$, there exists a compact FC-subspace L_N of X containing N such that*

$$L_N \bigcap \bigcap_{x \in L_N} \mathrm{cl}_{L_N} \left((X \setminus (FC(G))^{-1}(x)) \bigcap L_N \right) \subset K,$$

(4) *for each $x \in K$, $F(x) \neq \emptyset$.*

Then there exists $\hat{y} \in X$ such that $\hat{y} \in FC(G(\hat{y}))$.

Proof. In Theorem 4.4, let $X = D$, $\Gamma_N := \mathrm{Im}\, \varphi_N$, and $(F, G) := (F^-, (FC(G))^{-1})$ where $FC(G) = \mathrm{co}_\Gamma G$. Then (X, Γ) becomes an abstract convex space satisfying $1_X \in \mathfrak{KC}(X, X)$ by Theorem 3.3. Moreover condition (3) can be simplified to that of Theorem 4.4. Now the conclusion follows. ■

Remarks.

(1) In condition (2), "compactly" is obsolete; see [12].

(2) Note that the coercivity or compactness condition (3) of Corollary 4.5 is artificial, not practical, not elegant, and it should be replaced by the one in Theorem 4.4. The condition (3) or its equivalents are used to all of the key results in [3].

(3) As our Theorem 4.4 improves [3, Theorem 3.1], other theorems in Sections 3 and 4 of [3] can be easily improved by following Theorem 4.4.

(4) In [3], its authors stated that Corollary 4.5 generalizes corresponding results due to Ding, Ding and Tarafdar, Ding and Tan, and Taraftar. Similar remarks are also stated for other key results in [3].

5. Maximal Elements

Any binary relation R in a set X can be regarded as a map $T : X \multimap X$ and conversely by the following obvious way:

$$y \in T(x) \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a map T if $T(x_0) = \emptyset$.

From Theorem 4.1, by interchanging S and S^-, we have the following existence theorem for maximal elements:

Theorem 5.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $S : Z \multimap D$, $T : E \multimap Z$ maps, and $F \in \mathfrak{KC}(E, Z)$. Suppose that*

(1) *for each $y \in D$, $S^-(y)$ is open in Z;*

(2) *for each $z \in F(E)$, $\mathrm{co}_\Gamma S(z) \subset T^-(z)$;*

(3) *there exists a nonempty compact subset K of Z; and*

(4) *either*

(i) *$F(E) \setminus K \subset S^-(M)$ for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and $F(L_N) \setminus K \subset S^-(M)$ for some $M \in \langle D' \rangle$.*

If $F(x) \cap T(x) = \emptyset$ for all $x \in E$, then there exists a $z \in \overline{F(E)} \cap K$ such that $S(z) = \emptyset$.

Proof. Suppose that for each $z \in \overline{F(E)} \cap K$, there exists a $y \in S(z) \subset D$. Then $\overline{F(E)} \cap K \subset S^-(D)$. Since $\overline{F(E)} \cap K$ is compact, by (1), there exists an $N \in \langle D \rangle$ such that $\overline{F(E)} \cap K \subset S^-(N)$. Therefore, by Theorem 4.1, there exists an $\overline{x} \in E$ such that $F(\overline{x}) \cap T(\overline{x}) \neq \emptyset$, a contradiction. ■

Remark. If $X = D$ is a convex space, Theorem 5.1 reduces to Park [9, Theorem 3.1], which includes earlier works of Yannelis and Prabhakar, Mehta, Kim, and Mehta and Sessa.

For $E = Z$ and $F = 1_E$, Theorem 5.1 reduces to the following:

Theorem 5.2. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying $1_E \in \mathfrak{KC}(E, E)$, and $S : E \multimap D$, $T : E \multimap E$ maps. Suppose that*

(1) *S^- is open-valued;*

(2) *for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$ and $x \notin T(x)$;*

(3) *there exists a nonempty compact subset K of E; and*

(4) *either*

(i) *$E \setminus K \subset S^-(M)$ for some $M \in \langle D \rangle$; or*

(ii) *there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and $L_N \setminus K \subset S^-(M)$ for some $M \in \langle D' \rangle$.*

Then S has a maximal point $\overline{x} \in K$, that is, $S(\overline{x}) = \emptyset$.

From Theorem 4.4, we have the following:

Theorem 5.3. *Let $(X \supset D; \Gamma)$ be an abstract convex space satisfying $1_X \in \mathfrak{KC}(X, X)$, $F : X \multimap D$, $G : X \multimap X$ and K a nonempty compact subset of X such that*

(1) *for each $x \in X$, $F(x) \subset G(x)$ and $G(x)$ is Γ-convex;*

(2) *for each $z \in D$, $F^-(z)$ is open in X;*

(3) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ containing N such that*

$$\bigcap_{z \in D'} \mathrm{cl}_{L_N} \left(L_N \setminus G^-(z) \right) \subset K;$$

(4) *for each $x \in X$, $x \notin G(x)$.*

Then there exists $\hat{y} \in K$ such that $F(\hat{y}) = \emptyset$.

Corollary 5.4. ([3, Theorem 5.1]). *Let (X, φ_N) be an FC-space, $F, G : X \to 2^X$ and K be a nonempty compact subset of X such that*

(1) *for each $x \in X$, $F(x) \subset G(x)$,*

(2) *for each $y \in X$, $F^-(y)$ is compactly open in X,*

(3) *for each* $N \in \langle X \rangle$, *there exists a compact FC-subspace* L_N *of* X *containing* N *such that*

$$L_N \bigcap \bigcap_{x \in L_N} \mathrm{cl}_{L_N} \Big((X \setminus (FC(G))^{-1}(x)) \bigcap L_N \Big) \subset K,$$

(4) *for each* $x \in K$, $x \notin FC(G(x))$.

Then F *has a maximal element* $\hat{y} \in K$, *that is,* $F(\hat{y}) = \emptyset$.

6. Equilibria of Abstract Economies in Abstract Convex Spaces

We introduce the following generalization of [3, Definition 2.4]:

Definitions. Let X be a topological space, $(E \supset Y; \Gamma)$ an abstract convex space, $\theta : X \to E$ a function, and $\phi : X \multimap Y$ a map. Then

(1) ϕ is said to be of class $\mathcal{L}_{\theta,F}$ if

- (a) for each $x \in X$, $\mathrm{co}_\Gamma\, \phi(x) \subset Y$ and $\theta(x) \notin \mathrm{co}_\Gamma\, \phi(x)$ for each $x \in X$,
- (b) there exists a map $\psi : X \multimap Y$ such that $\psi(x) \subset \phi(x)$ for each $x \in X$ and $\psi^{-}(y)$ is open in X for each $y \in Y$, and
- (c) $\{x \in X \mid \phi(x) \neq \emptyset\} = \{x \in X \mid \psi(x) \neq \emptyset\}$;

(2) (ϕ_x, ψ_x, N_x) is called a $\mathcal{L}_{\theta,F}$-majorant of ϕ at x if $\phi_x, \psi_x : X \multimap Y$ and N_x is an open neighborhood of x in X such that

- (a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \mathrm{co}_\Gamma\, \phi_x(z)$,
- (b) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ and $\mathrm{co}_\Gamma\, \phi_x(z) \subset Y$,
- (c) for each $y \in Y$, $\psi_x^{-}(y)$ is open in X;

(3) ϕ is said to be $\mathcal{L}_{\theta,F}$-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an $\mathcal{L}_{\theta,F}$-majorant (ϕ_x, ψ_x, N_x) of ϕ at x such that for any nonempty finite subset A of the set $\{x \in X \mid \phi(x) \neq \emptyset\}$,

$$\left\{ z \in \bigcap_{x \in A} N_x \mid \bigcap_{x \in A} \mathrm{co}_\Gamma \phi_x(z) \neq \emptyset \right\} = \left\{ z \in \bigcap_{x \in A} N_x \mid \bigcap_{x \in A} \mathrm{co}_\Gamma \psi_x(z) \neq \emptyset \right\}.$$

It is clear that every map of class $\mathcal{L}_{\theta,F}$ is $\mathcal{L}_{\theta,F}$-majorized. The above definitions generalize the corresponding ones for FC-spaces in [3], which were stated there to generalize the ones due to Ding and Tan; Ding, Kim and Tan; Tan and Yuan; Ding and Tarafdar; and Tulcea.

In the present paper, we will restrict ourselves to the case $X = Y$ is an abstract convex space and $\theta = 1_X$ and write $\mathcal{L}_F$ instead of $\mathcal{L}_{\theta,F}$.

Theorem 6.1. *Let $(X;\Gamma)$ be an abstract convex space satisfying $1_X \in \mathfrak{KC}(X,X)$, $G : X \multimap X$ be of class $\mathcal{L}_F$ and K a nonempty compact subset of X. Suppose that*

(1) *for each $x \in X$, $G(x)$ is Γ-convex;*

(2) *for each $N \in \langle X \rangle$, there exists a compact Γ-convex subset L_N of X containing N such that*

$$\bigcap_{x\in L_N} \mathrm{cl}_{L_N}\big(L_N \setminus G^-(x)\big) \subset K.$$

Then there exists $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$.

Proof. Since G is of class $\mathcal{L}_F$, we have

(i) for each $x \in X$, $x \notin \mathrm{co}_\Gamma G(x) = G(x)$;

(ii) there exists a map $F : X \multimap X$ such that (a) $F(x) \subset G(x)$ for each $x \in X$; (b) $F^-(y)$ is open in X for each $y \in X$; and (c) $\{x \in X \mid F(x) \neq \emptyset\} = \{x \in X \mid G(x) \neq \emptyset\}$.

Suppose $G(x) \neq \emptyset$ for all $x \in K$. Then, by (iii), $F(x) \neq \emptyset$ for all $x \in K$. We apply Theorem 4.4 with $X = D$ and replacing F and G by F^- and G^-, respectively. Then G has a fixed point $\hat{y} \in X$ and this contradicts (i). Hence the conclusion follows. ■

Remark. Theorem 6.1 works for G-convex spaces or ϕ_A-spaces and generalizes [3, Theorem 5.2], which was stated there to generalize corresponding ones due to Ding, Ding and Tarafdar, Ding and Tan, Tan and Yuan, and others.

Closely examining the proof of [3, Lemma 5.1] with replacing $FC(\cdot)$ by $\mathrm{co}_\Gamma(\cdot)$, it leads to the following generalization:

Theorem 6.2. *Let X be a regular topological space and $(E \supset Y;\Gamma)$ an abstract convex space. Let $\theta : X \to E$ and $P : X \multimap Y$ be $\mathcal{L}_{\theta,F}$-majorized. If each open subset of X containing $B := \{x \in X \mid P(x) \neq \emptyset\}$ is paracompact, then there exists a map $\phi : X \multimap Y$ of class $\mathcal{L}_{\theta,F}$ such that $P(x) \subset \phi(x)$ for all $x \in X$.*

Remark. For FC-spaces, Lemma 6.2 reduces to [3, Lemma 5.1], which was said in [3] to generalize results of Ding; Ding and Tan; Ding, Kim and Tan; Tan and Yuan, and Tulcea.

From Theorem 4.4 and Lemma 6.2, we deduce the following generalization of Theorem 6.2 on the existence of maximal elements of $\mathcal{L}_F$-majorized maps:

Theorem 6.3. *Let $(X;\Gamma)$ be an abstract convex paracompact space satisfying $1_X \in \mathfrak{KC}(X,X)$, $P : X \multimap X$ an $\mathcal{L}_F$-majorized map, and K a nonempty compact subset of X. Suppose that*

(1) *for each $x \in X$, $P(x)$ is Γ-convex;*
(2) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X containing N such that*

$$\bigcap_{x \in L_N} \mathrm{cl}_{L_N} \left(L_N \setminus P^{-}(x) \right) \subset K.$$

Then there exists $\hat{y} \in K$ such that $P(\hat{y}) = \emptyset$.

Proof. We prove first that $P(y) \neq \emptyset$ for each $y \in X \setminus K$. For this y, by (2), we have

$$y \notin \bigcap_{x \in L_{\{y\}}} \left(L_{\{y\}} \setminus P^{-}(x) \right).$$

Hence, there exists an $x \in L_{\{y\}}$ such that $y \in P^{-}(x)$. So $x \in P(y)$ and $P(y) \neq \emptyset$.

Suppose the conclusion does not hold, that is, $P(y) \neq \emptyset$ for all $y \in X$. Hence the set $\{x \in X \mid P(x) \neq \emptyset\} = X$ is paracompact. By Lemma 6.2, there exists a Γ-convex-valued map $\phi : X \multimap X$ of class $\mathcal{L}_F$ such that $P(x) \subset \phi(x)$ for each $x \in X$. Then for each $N \in \langle X \rangle$, we have

$$\bigcap_{x \in L_N} \mathrm{cl}_{L_N} \left(L_N \setminus \phi^{-}(x) \right) \subset \bigcap_{x \in L_N} \mathrm{cl}_{L_N} \left(L_N \setminus P^{-}(x) \right) \subset K.$$

Hence, by Theorem 6.1, there exists a $\hat{y} \in K$ such that $\phi(\hat{y}) = \emptyset$ so that $P(\hat{y}) = \emptyset$, which is a contradiction. Therefore the conclusion follows. ■

Remark. For FC-spaces, Theorem 6.3 reduces to [3, Theorem 5.3], which was said in [3] to generalize results of Ding and Tan; Tan and Yuan; Ding, Ding and Tan; and Borglin and Keiding.

As an application of Theorem 6.1, we deduce the following equilibrium existence theorem for a one-person game:

Theorem 6.4. *Let $(X;\Gamma)$ be an abstract convex space satisfying $1_X \in \mathfrak{KC}(X,X)$, $A, B, P : X \multimap X$, and K a nonempty compact subset of X. Suppose that*

(1) *for each* $x \in X$, $\mathrm{co}_\Gamma A(x) \subset \overline{B}(x)$;

(2) *for each* $y \in X$, $A^-(y)$ *is open in* X;

(3) $A \cap P$ *is of class* $\mathcal{L}_F$ *and has* Γ*-convex values;*

(4) *for each* $N \in \langle X \rangle$, *there exists a compact* Γ*-convex subset* L_N *of* X *containing* N *such that*

$$\bigcap_{x \in L_N} \mathrm{cl}_{L_N} \big(L_N \setminus (A \cap P)^-(x) \big) \subset K;$$

(5) *for each* $x \in K$, $A(x) \neq \emptyset$.

Then there exists $\hat{x} \in K$ *such that* $\hat{x} \in \overline{B}(\hat{x})$ *and* $A(\hat{y}) \cap P(\hat{y}) = \emptyset$.

Proof. Let $Y := \{x \in X \mid x \notin \overline{B}(x)\}$. Then Y is open. Define $\phi : X \multimap X$ by

$$\phi(x) := \begin{cases} A(x) \cap P(x), & \text{for } x \notin Y; \\ A(x), & \text{for } x \in Y. \end{cases}$$

Since $A \cap P$ is of class $\mathcal{L}_F$, for each $x \in X$, we have $x \notin \mathrm{co}_\Gamma(A(x) \cap P(x)) = A(x) \cap P(x)$ and a map $\beta : X \multimap X$ such that

(a) for each $x \in X$, $\beta(x) \subset A(x) \cap P(x)$;

(b) for each $y \in X$, $\beta^-(y)$ is open in X; and

(c) $\{x \in X \mid \beta(x) \neq \emptyset\} = \{x \in X \mid A(x) \cap P(x) \neq \emptyset\}$.

We define a map $\psi : X \multimap X$ by

$$\psi(x) := \begin{cases} \beta(x), & \text{for } x \notin Y; \\ A(x), & \text{for } x \in Y. \end{cases}$$

It is clear that for each $x \in X$, $\psi(x) \subset \phi(x)$ and $\{x \in X \mid \psi(x) \neq \emptyset\} = \{x \in X \mid \phi(x) \neq \emptyset\}$ by (c). Note that, for each $y \in X$, $\psi^-(y) = (Y \cup \beta^-(y)) \cap A^-(y)$ is open in X by (2) and (b). Let $x \in X$. If $x \in Y$, then $x \notin \overline{B}(x)$ and $x \notin \mathrm{co}_\Gamma \phi(x)$; if $x \notin Y$, then $x \notin \mathrm{co}_\Gamma(A(x) \cap P(x))= \mathrm{co}_\Gamma \phi(x)$. Consequently, ϕ is of class $\mathcal{L}_F$. From (4) and the definition of ϕ, for each $N \in \langle X \rangle$, we have

$$\bigcap_{x \in L_N} \mathrm{cl}_{L_N} \big(L_N \setminus \phi^-(x) \big) \subset K.$$

Now by Theorem 6.1, we have a point $\hat{x} \in K$ such that $\phi(\hat{x}) = \emptyset$. Note that $A(x) \neq \emptyset$ for each $x \in K$ and that (4) implies $A(x) \neq \emptyset$ for each $x \in X \setminus K$. Therefore $A(x) \neq \emptyset$ for each $x \in X$ so that $\hat{x} \in \overline{B}(\hat{x})$ and $A(\hat{y}) \cap P(\hat{y}) = \emptyset$. ■

Remark. For FC-spaces, Theorem 6.4 reduces to [3, Theorem 6.1], which was claimed in [3] to improve and generalize results of Ding and Tarafdar; Ding and Tan; Tan and Yuan; Ding; and Ding and Tan.

Final Remark. Until now we showed that Lemmas 3.1, 3.2, and 5.1, Theorems 3.1, 5.1, 5.2, 5.3, and 6.1 for FC-spces in [3] are generalized to our abstract convex spaces in better forms. Other results in [3] can also be improved by following our method in this paper.

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