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FIXED POINT THEORY OF APPROXIMABLE MULTIMAPS

SEHIE PARK

Abstract. We show that, for a subset X of a t.v.s. E, any compact closed approachable multimap $T : X \multimap X$ has a fixed point if the map has Klee approximable range. As applications of this theorem, we notice that the local convexity of E in many known results can be replaced by more general conditions, and obtain a number of generalizations of known fixed point or relevant theorems, all related to compact closed approachable or approximable multimaps.

1. Introduction

Early in 1991, Ben-El-Mechaiekh et al. [2, 8, 9] derived a fixed point theorem for a compact closed approachable multimap defined on a convex subset of a locally convex Hausdorff topological vector space E. In 1994, Ben-El-Mechaiekh and Idzik [10] obtained a Leray-Schauder alternative for approximable multimaps from a matching theorem of Fan. In 1996, the present author [14] gave a simple proof of their theorem by using an earlier fixed point theorem due to Ben-El-Mechaiekh et al. Moreover, we applied their theorem to obtain a Schaefer type theorem, the Birkhoff-Kellogg type theorems, a Penot type theorem for non-self multimaps, and quasi-variational inequalities with respect to compact closed approximable maps. Finally in [14], we indicated that our method works also for other classes of maps including compositions of acyclic maps.

In the present paper, by relaxing the local convexity of the t.v.s., we obtain a basic fixed point theorem for a continuous function having Klee approximable range and its various applications to compact closed approximable multimaps. Consequently, we show that the local convexity of E in many known results as in [14] can be replaced by more general conditions, and obtain a number of generalizations of known fixed point or relevant theorems, all related to compact closed approachable or approximable multimaps.

Section 2 deals with preliminaries on fixed point theory. In Section 3, we derive a basic fixed point theorem for a continuous function having Klee approximable range. Section 4 deals with preliminaries on approachable maps. In Section 5, we obtain a fixed point theorem for compact closed approachable multimaps having Klee approximable ranges. Section 6 deals with Leray-Schauder alternatives for such multimaps. From Sections 7 to 9, applications of the Leray-Schauder alternatives in Section 6 to a Schaefer type theorem, a Birkhoff-Kellogg type theorem on eigenvalues or invariant directions, and a fixed point theorem for non-self maps. Finally,

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in Section 10, our previous results are applied to quasi-variational or variational inequalities all related to compact closed approachable maps.

2. Preliminaries

All topological spaces are assumed to be Hausdorff unless explicitly stated otherwise. A t.v.s. means a topological vector space. Int, Bd, $\overline{}$, and co denote the interior, boundary, closure, and convex hull, respectively. Let $\mathcal{V}$ denote a fundamental system of neighborhoods of the origin 0 of a t.v.s. E.

For topological spaces X and Y, a *multimap* (or simply, a *map*) $T : X \multimap Y$ is a function from X into the power set of Y with nonempty *values* $T(x) \subset Y$ for $x \in X$ and *fibers* $T^-(y) := \{x \in X \mid y \in T(x)\}$ for $y \in Y$. For $A \subset X$, let $T(A) := \bigcup\{T(x) \mid x \in A\}$. T is said to be *closed* if it has the closed graph $\mathrm{Gr}(T) \subset X \times Y$, and *compact* if its range $T(X)$ is contained in a compact subset of Y. Recall that T is said to be *upper semicontinuous* (u.s.c.) if for any open subset $U \subset Y$, the set $T^+(U) :=\{x \in X \mid T(x) \subset U\}$ is open in X. Recall that a compact closed multimap is u.s.c. and compact-valued; and a u.s.c. multimap with closed values is closed whenever its range is regular.

A *polytope* P in a t.v.s. E is a homeomorphic image of a simplex.

A nonempty subset K of E is said to be *Klee approximable* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in E. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

Example 2.1. We give some examples of Klee approximable sets:

(1) A subset X of E is admissible (in the sense of Klee [13]) iff every compact subset K of X is Klee approximable into E.
(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.
(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.
(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.
(5) Any compact subset K of an admissible almost convex subset X of a t.v.s. is Klee approximable into X.
(6) Any compact subset of a Φ-space in a t.v.s. is Klee approximable.
(7) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X. See [19].

Note that (7)⇒(5)⇒(4)⇒(3).

Let X be a nonempty closed convex subset of a t.v.s. E. We say that X is *weakly admissible* (in the sense of Nhu and Arandelović) if for every $V \in \mathcal{V}$ there exist closed convex subsets $X_1, X_2, \ldots, X_n$ of X with $X = \mathrm{co}\,(\bigcup_{i=1}^n X_i)$ and continuous functions $f_i : X_i \to X \cap L$, $i = 1, 2, \ldots, n$, where L is a finite dimensional subspace of E, such that $\sum_{i=1}^n (f_i(x_i) - x_i) \in V$ for every $x_i \in X_i$ and $i = 1, 2, \ldots, n$.

A subset B of a t.v.s. E is said to be *convexly totally bounded* (simply, c.t.b.) if for every $V \in \mathcal{V}$, there exist a finite subset $\{x_i\}_{i=1}^n \subset B$ and a finite family of convex subsets $\{C_i\}_{i=1}^n$ of V such that $B \subset \bigcup_{i=1}^n (x_i + C_i)$.

A *Kakutani map* is a u.s.c. multimap with compact convex values.

The following is some of the most general known fixed point theorems for Kakutani maps:

Theorem 2.2. *Let X be a nonempty subset of a t.v.s. E. Then a compact Kakutani map $T : X \multimap X$ has a fixed point if one of the following conditions holds:*

1. (Idzik) *X is convex and $\overline{T(X)}$ is c.t.b.*
2. (Okon) *X is convex, compact, and weakly admissible.*
3. (Park) *$T(X)$ is Klee approximable into X.*

For the literature, see [18].

Let us say that a topological space X has the (*compact*) *fixed point property* (simply, f.p.p.) if any (compact) continuous selfmap $f : X \to X$ has a fixed point $x_0 \in X$.

3. A basic fixed point theorem

The following is the basic fixed point theorem in this paper:

Theorem 3.1. *Let X be a subset of a t.v.s. E and $f : X \to E$ continuous function.*

(a) *If $f(X)$ is Klee approximable into X, then f has the almost fixed point property (that is, for each $V \in \mathcal{V}$, f has a V-fixed point $x_V \in X$, that is, $f(x_V) - x_V \in V$).*

(b) *Further if f is compact and $\overline{f(X)} \subset X$, then f has a fixed point $x_0 \in X$ (that is, $x_0 = f(x_0)$).*

Proof. (a) Since $f(X)$ is Klee approximable into X, for any $V \in \mathcal{V}$, there exist a polytope P in X and a continuous function $h : f(X) \to P$ such that $y - h(y) \in V$ for all $y \in f(X)$. Then $f|_P : P \to f(X)$ and $h(f|_P) : P \to P$. Since any polytope has the f.p.p. by the Brouwer fixed point theorem, $h(f|_P)$ has a fixed point $x_V \in P$, that is, $x_V = hf(x_V)$. Let $y_V := f(x_V) \in f(P) \subset f(X) \subset \overline{f(X)}$. Then $x_V = h(y_V)$ and $y_V - x_V \in V$.

(b) Since $\overline{f(X)}$ is compact, there exists a subnet of $\{y_V\}$ converging to an $x_0 \in \overline{f(X)}$. Since E is Hausdorff and $y_V - x_V \in V$ for each $V \in \mathcal{V}$, the subnet of $\{x_V\}$ corresponding to the subnet of $\{y_V\}$ also converges to x_0. Since $y_V = f(x_V)$ and f is continuous, we have $x_0 = f(x_0)$. This completes our proof. □

¿From Theorem 3.1, we have the following well-known consequences:

Corollary 3.2. *Let X be a subset of a t.v.s. E and $f : X \to X$ a compact continuous function. If $f(X)$ is Klee approximable into X, then f has a fixed point.*

Corollary 3.3. *Let X be an admissible convex subset of a t.v.s. E. Then any compact continuous function $f : X \to X$ has a fixed point.*

Corollary 3.4 (Hukuhara)**.** *Let X be a convex subset of a locally convex t.v.s. E. Then any compact continuous function $f : X \to X$ has a fixed point.*

4. Approachable maps

We adopt the following definitions from [9, 10].

Let X and Y be subsets of t.v.s. E and F, respectively, and $T : X \multimap Y$ a multimap. Given two open neighborhoods U and V of the origin 0 of E and F, respectively, a (U, V)-*approximative continuous selection* of T is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (T[(x + U) \cap X] + V) \cap Y \quad \text{for every } x \in X.$$

T is said to be *approachable* if it admits a (U, V)-approximative continuous selection for every U and V as above; and T *approximable* if its restriction $T|_K$ to any compact subset K of X is approachable.

Note that an approachable map is always approximable. Recall that Ben-El-Mechaiekh et al. [1-8] established a large number of properties and examples of approachable or approximable maps.

Example 4.1. We give some examples of approximable maps $T : X \multimap Y$ as follows [4, 16]:

(1) Any multimap having a continuous selection is approximable. There are a lot of continuous selection theorems due to Michael and others; for example, see [20].
(2) A locally selectionable map T with convex values is approximable whenever Y is a convex subset of a t.v.s.
(3) A u.s.c. map T with convex values is approachable whenever X is paracompact and Y is a convex subset of a locally convex t.v.s.
(4) A u.s.c. map T with compact contractible values is approachable whenever X is a finite polyhedron.
(5) A u.s.c. map T with compact values having trivial shape is approachable whenever X is a finite polyhedron.

Let $\mathbb{A}(X, Y)$ denote the class of all u.s.c. approachable maps $T : X \multimap Y$ with compact values, $\mathbb{A}^\kappa(X, Y)$ the class of all u.s.c. approximable maps $T : X \multimap Y$ with compact values, and $\mathbb{A}^\kappa_c(X, Y)$ the class of all finite compositions $T : X \multimap Y$ of u.s.c. approximable maps with compact values, where the intermediate spaces are subsets of t.v.s. Recall that the class $\mathbb{A}^\kappa_c$ is an example of the admissible class $\mathfrak{A}^\kappa_c$ and the better admissible class $\mathfrak{B}$ due to the author [15, 17, 19].

We say that a subset X of a t.v.s. has the (*compact*) *approachable fixed point property* (simply, $\mathbb{A}$-f.p.p.) if any (compact) map $T \in \mathbb{A}(X, X)$ has a fixed point; and the *approximable fixed point property* (simply, $\mathbb{A}^\kappa$-f.p.p.) if any map $T \in \mathbb{A}^\kappa(X, X)$ has a fixed point. Similarly, the $\mathbb{A}^\kappa_c$-f.p.p. can be defined.

Lemma 4.2 ([18])**.** *For a subset X of a t.v.s. E, the following are equivalent:*
(1) *X has the compact f.p.p.*
(2) *X has the compact $\mathbb{A}$-f.p.p.*

¿From Theorem 2.2 and Lemma 4.2, we have the following:

Theorem 4.3. *A nonempty subset X of a t.v.s. E has the compact $\mathbb{A}$-f.p.p. whenever one of the following holds:*

(1) *X is convex and admissible (in the sense of Klee).*
(2) *X is convex and any compact subset of X is c.t.b.*
(3) *X is convex, compact and weakly admissible.*
(4) *Every nonempty compact subset of X is Klee approximable into X.*

Remark. For a locally convex t.v.s. E, a particular form of Theorem 4.3 appeared in [2, Theorem 2.4], [8, Corollary 3.4], [9, Corollary 7.3].

5. Fixed points of compact approachable maps

¿From Theorem 3.1 or Corollary 3.2, we deduce the following:

Theorem 5.1. *Let X be a subset of a t.v.s. E and $T : X \multimap X$ a compact closed approachable map. If $T(X)$ is Klee approximable into X, then T has a fixed point $\hat{x} \in X$, that is, $\hat{x} \in T(\hat{x})$.*

Proof. For any $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ such that $V + V \subset U$. Since T is approachable, we have a continuous function $s : X \to X$ satisfying

$$s(x) \in T[(x + V) \cap X] + V \quad \text{for every } x \in X.$$

We may assume that V is symmetric and that $s(x) \in \overline{T(X)}$ for all $x \in X$. [Otherwise, by the regularity of E, we may have $[s(x) - V] \cap T[(x + V) \cap X] = \emptyset$ for some $V \in \mathcal{V}$.] Since $s(X) \subset \overline{T(X)}$, s is compact and $s(X)$ is Klee approximable into X. Therefore s has a fixed point $x_0 \in X$ by Corollary 3.2. Then

$$x_0 = s(x_0) \in T[(x_0 + V) \cap X] + V,$$

and hence there exist $y_U \in T[(x_0 + V) \cap X]$ and $x_U \in (x_0 + V) \cap X$ such that $y_U \in T(x_U)$ and $x_0 \in y_U + V$. Then we have $x_U - y_U \in (x_0 + V) - (x_0 - V) = V + V \subset U$. Since $\{y_U \mid U \in \mathcal{V}\}$ is a net in the compact set $\overline{T(X)}$, it has a subnet converging to some $\hat{x} \in \overline{T(X)}$. Since E is Hausdorff and $x_U - y_U \in U$, $\{x_U \mid U \in \mathcal{V}\}$ has a corresponding subnet converging to $\hat{x}$. Since the graph of T is closed and $(x_U, y_U) \in \mathrm{Gr}(T)$, we have $(\hat{x}, \hat{x}) \in \mathrm{Gr}(T)$. This completes our proof. □

Remark. Since every continuous function is approachable, Corollary 3.2 follows from Theorem 5.1. Hence, Theorem 5.1 and Corollary 3.2 are equivalent.

Corollary 5.2. *Let X be an admissible almost convex subset of a t.v.s. E and $T : X \multimap X$ a compact closed approachable map. Then T has a fixed point.*

Remark. If X is a convex subset of a locally convex t.v.s. E, Corollary 5.2 reduces to [2, Theorem 2.4], [8, Corollary 3.4], [9, Corollary 7.3].

¿From the definition of approximable maps, we have the following:

Corollary 5.3. *Let X be a compact subset of a t.v.s. E and $T : X \multimap X$ a closed approximable map. If $T(X)$ is Klee approximable into X, then T has a fixed point.*

¿From Theorem 5.1, we can deduce the following well-known result:

Corollary 5.4 (Himmelberg)**.** *Let X be a convex subset of a locally convex t.v.s. E and $T : X \multimap X$ a compact Kakutani map. Then T has a fixed point.*

Proof. Let $L :=\text{co}\,\overline{T(X)} \subset X$. Then it is well-known that L is paracompact, and hence, $T|_L : L \multimap L$ is approachable. Therefore, by Theorem 5.1, $T|_L$ has a fixed point. □

6. Leray-Schauder alternatives

¿From Theorem 5.1, we obtain the following nonlinear alternative:

Theorem 6.1. *Let X be a closed subset of a t.v.s. E such that $0 \in \text{Int}\ X$ and $T : X \multimap E$ a compact closed approachable map. If $T(X)$ is Klee approximable, then either*

(1) *T has a fixed point; or*
(2) *$\lambda x \in T(x)$ for some $\lambda > 1$ and $x \in \text{Bd}\ X$.*

Proof. Let $R \subset X$ be defined by

$$R := \{x \in X \mid x \in tT(x) \text{ for some } t \in [0,1]\},$$

which is nonempty since $0 \in R$. Moreover, it is closed since T is closed. Therefore, R is compact since T is compact.

Suppose that

$$T(y) \cap \{\lambda y \mid \lambda > 1\} = \emptyset \quad \text{for all } y \in \text{Bd}\, X. \tag{LS}$$

Then $R \cap \text{Bd}\, X = \emptyset$. Since X is completely regular, there exists a continuous function $r : X \to [0,1]$ such that $r(x) = 1$ for $x \in R$ and $r(x) = 0$ for $x \in \text{Bd}\, X$.

Let $S : E \multimap E$ be defined by $S(x) := r(x)T(x)$ if $x \in X$ and $S(x) := \{0\}$ if $x \notin X$. Since T is compact and closed, so is S.

Moreover, S is approachable. In fact, if $s : X \to E$ is a (U,V)-approximative continuous selection of T, then $rs : X \to E$ defined by $(rs)(x) := r(x)s(x)$ for $x \in X$ is a (U,V)-approximative continuous selection of $S|_X$. Define $f : E \to E$ by $f(x) := (rs)(x)$ for $x \in X$ and $f(x) := 0$ for $x \notin X$. Then f is a (U,V)-approximative continuous selection of $S : E \multimap E$.

Further, $S(E) \subset [0,1]T(X)$ is Klee approximable into E. In fact, since $T(X)$ is Klee approximable, for each $V \in \mathcal{V}$, there exists a continuous $h : T(X) \to X$ such that $y - h(y) \in V$ for all $y \in T(X)$ and $h(T(X)) \subset P \subset E$ for a polytope P. For each $\alpha y \in S(E)$ with $\alpha \in [0,1]$ and $y \in T(X)$, define $g : S(E) \to E$ by $g(\alpha y) := \alpha h(y)$. Then $\alpha y - g(\alpha y) = \alpha(y - h(y)) \in \alpha V \subset V$ (since V can be shrinkable [13]) for all $\alpha y \in S(E)$. Note that $g(S(E)) \subset [0,1]h(T(X)) \subset [0,1]P \subset E$ and $[0,1]P$ is a polytope in E.

Therefore, S has a fixed point by Theorem 5.1 with $X = E$. Now $x \in S(x)$ implies $x \in R$ and $r(x) = 1$. Therefore, $x \in X$ and $x \in T(x)$. This completes our proof. □

Remark. 1. We followed the method of Schöneberg [25]. Note that if X itself is compact, Theorem 6.1 works for approximable maps.

2. For a long time, the alternatives like Theorem 6.1 and the condition (LS) have been called the Leray-Schauder alternatives and the Leray-Schauder boundary condition, resp. These are known to be not directly related to the works of Leray and Schauder; see [17].

Similarly, we have the following:

Theorem 6.2. *Let X be a closed subset of a t.v.s. E such that $0 \in \operatorname{Int} X$ and $\Phi : X \multimap E$ a compact closed approachable map. If E has the compact f.p.p., then either*

(1) *Φ has a fixed point; or*
(2) *$\lambda x \in \Phi(x)$ for some $\lambda > 1$ and $x \in \operatorname{Bd} X$.*

Recall that an admissible t.v.s. has the compact f.p.p. Many examples of admissible t.v.s. are known; a locally convex one is one of them.

Corollary 6.3. *Let X be a closed subset of a t.v.s. E such that $0 \in \operatorname{Int} X$ and $\Phi : X \multimap E$ a compact closed approximable map. If E has the compact $\mathbb{A}^\kappa$-f.p.p., then the conclusion of Theorem 6.2 holds.*

Proof. In fact, for any compact subset K of X, if $s : K \to E$ is a (U, V)-approximative continuous selection of $\Phi|_K$, then $rs : K \to E$ in the proof of Theorem 6.1 is a (U, V)-approximative continuous selection of $\Phi|_K$. □

Remark. Corollary 6.3 includes the main result in [10], [1, Theorem 5] and many others. See [15].

It is not known yet whether Theorem 5.1 holds for an approximable map instead of an approachable map. This is affirmative for particular types of convex sets.

The following is a particular form of our earlier work [15]:

Theorem 6.4. *Let X be an admissible convex subset of a t.v.s. E and $T \in \mathbb{A}^\kappa_c(X, X)$. If T is compact, then T has a fixed point.*

¿From Theorem 6.4, we can deduce the following Leray-Schauder type alternative as in the proof of Theorem 6.1:

Theorem 6.5. *Let X be a closed subset of a t.v.s. E such that $0 \in \operatorname{Int} X$, Y an admissible convex subset of E containing X, and $T : X \multimap Y$ a compact closed approximable map. Then either*

(1) *T has a fixed point; or*
(2) *$\lambda x \in T(x)$ for some $\lambda > 1$ and $x \in \operatorname{Bd}_Y X$.*

Remark. Theorem 6.5 includes [1, Theorem 5] and others for locally convex t.v.s.; see [14].

For $Y = E$, Theorem 6.5 reduces to the following:

Corollary 6.6. *Let X be a closed subset of an admissible t.v.s. E such that $0 \in \operatorname{Int} X$ and $T : X \multimap E$ a compact closed approximable map. Then either*

(1) *T has a fixed point; or*
(2) *$\lambda x \in T(x)$ for some $\lambda > 1$ and $x \in \operatorname{Bd} X$.*

Remark. 1. Recall that Corollary 6.6 is due to Ben-El-Mechaiekh and Idzik [10] for a locally convex t.v.s. E.

2. In our previous work [14], for a locally convex t.v.s. E, we gave a Schöneberg type proof of Corollary 6.6 and apply it to deduce several results all related to compact closed approximable maps. In view of Corollary 6.6, all those theorems in [14] hold for admissible t.v.s.

7. A Schaefer type theorem

¿From the Leray-Schauder alternative in Theorem 6.2, as in [14], we can deduce a Schaefer type theorem, a Birkhoff-Kellogg type theorem on eigenvalues or invariant directions, a fixed point theorem for non-self maps, and a quasi-variational inequality — all related to compact closed approachable maps.

¿From Theorem 6.2, we have the following:

Theorem 7.1. *Let E be a t.v.s. having the compact f.p.p. and $\Phi : E \multimap E$ a closed approachable map which sends bounded sets into compact sets. Then either*

(1) *Φ has a fixed point; or*

(2) *the set $A = \{x \in E \mid x \in t\Phi(x)$ for some $t \in (0, 1)\}$ is not bounded.*

Proof. Suppose that A is bounded. Let X be a bounded neighborhood of 0 such that $A \subset \text{Int } X$. Then no $y \in \text{Bd } X$ satisfies $\lambda y \in \Phi(y)$ for any $\lambda > 1$. Therefore, by Theorem 6.2, Φ has a fixed point in $\overline{X}$. □

Remark. 1. Theorem 7.1 was first obtained by Schaefer [23, 24] for a completely continuous map $f : E \to E$ on a complete locally convex t.v.s. E.

2. In our [14], Theorem 7.1 for a locally convex t.v.s. E is obtained whenever Φ is approximable.

8. A Birkhoff-Kellogg type theorem

As an application of Theorem 6.2, we have the following generalization of the Birkhoff-Kellogg theorem [12]:

Theorem 8.1. *Let X be a closed subset of a t.v.s. E having the compact f.p.p. such that $0 \in \text{Int } X$ and $\Phi : X \multimap E$ a compact closed approachable map such that $\lambda\Phi(X) \cap X = \emptyset$ for some λ. Then $\Phi|_{\text{Bd } X}$ has an eigenvalue; that is, $\mu x \in \Phi(x)$ for some $\mu \neq 0$ and $x \in \text{Bd } X$.*

Proof. Note that $\lambda \neq 0$ and $\lambda\Phi : X \multimap E$ is a compact closed approachable map. Moreover, $\lambda\Phi$ has no fixed point. Therefore, by Theorem 6.2, there exist $x \in \text{Bd } X$ and $\mu > 1$ such that $\mu x \in \lambda\Phi(x)$, whence we have $(\lambda^{-1}\mu)x \in \Phi(x)$, where $\lambda^{-1}\mu \neq 0$. This completes our proof. □

Remark. If $\lambda > 0$ in Theorem 8.1, then $\Phi|_{\text{Bd } X}$ has an invariant direction (a positive eigenvalue); that is, $\mu x \in \Phi(x)$ for some $\mu > 0$ and $x \in \text{Bd } X$.

¿From Theorem 8.1, we obtain

Theorem 8.2. *Let S be the unit sphere of a normed vector space E of infinite dimension, and $\Phi : S \multimap E$ a compact closed approachable map such that $0 \notin \overline{\Phi(S)}$. Then Φ has an invariant direction.*

Proof. Since E is infinite dimensional, by the Dugundji extension theorem, there exists a retraction $r : E \to S$ such that $r(x) = x/\|x\|$ if $\|x\| \geq 1$ and $\|r(x)\| = 1$ if $\|x\| \leq 1$. Let $\Psi = \Phi r : E \multimap E$. Then Ψ is a compact closed approachable map. Let B be the closed unit ball. Then $\lambda\Psi(B) \cap B = \emptyset$ for some $\lambda > 0$ since $\Psi(B) \subset \overline{\Phi(S)}$ and $0 \notin \overline{\Phi(S)}$. Therefore, by Theorem 8.1 with $X = B$, $\Psi|_S$ has an eigenvalue. Since $\lambda > 0$, this eigenvalue is positive. This completes our proof. □

Theorem 8.2 reduces immediately to the following fixed point theorem:

Theorem 8.3. *Let S be the unit sphere of a normed vector space E. Then E is of infinite dimension if and only if any compact closed approachable map $\Phi : S \multimap S$ has a fixed point.*

9. Non-self maps

Combining Lemma 4.2 and Theorem 6.2, we obtain the following fixed point theorem for approachable maps:

Theorem 9.1. *Let X be a closed convex subset of a t.v.s. E having the compact f.p.p. and $\Phi : X \multimap E$ a compact closed approachable map. If $\Phi(\mathrm{Bd}\ X) \subset X$, then Φ has a fixed point.*

Proof. If Int $X = \emptyset$, then $X = \mathrm{Bd}\ X$ and $\Phi : X \multimap X$ has a fixed point by Lemma 4.2. If Int $X \neq \emptyset$, then we may assume $0 \in \mathrm{Int}\ X$. Now for each $x \in \mathrm{Bd}\ X$, $\Phi(x) \subset X$ implies $\Phi(x) \cap \{\lambda x \mid \lambda > 1\} = \emptyset$ since X is shrinkable; that is, (LS) holds. Therefore, by Theorem 6.2, Φ has a fixed point. □

Proof. For a compact closed map $\Phi : X \multimap E$ with convex values, Theorem 9.1 reduces to Penot [22, Proposition 1.4], which contains the particular case for a single-valued continuous map due to Brezis; see [22]. □

Corollary 9.2. *Let X be a compact convex subset of a t.v.s. E having the compact f.p.p. and $\Phi : X \multimap E$ a Kakutani map. If $\Phi(\mathrm{Bd}\ X) \subset X$, then Φ has a fixed point.*

Remark. For a locally convex t.v.s. E, Corollary 9.2 is due to Brezis; see [22]. This contains the classical results due to Knaster-Kuratowski-Mazurkiewicz and Rothé; see [17]. Brezis' theorem is generalized to a compact Kakutani map by Penot [22].

10. Quasi-variational or variational inequalities

¿From Theorem 9.1, we have the following quasi-variational inequality:

Theorem 10.1. *Let X be a closed convex subset of a t.v.s. E having the compact f.p.p., Y a compact subset of E, and $f : X \times Y \to \mathbb{R}$ an u.s.c. function. Let $T : X \multimap Y$ be a closed map such that $T(\mathrm{Bd}\ X) \subset X \cap Y$. Suppose that*

(i) *the function M defined on X by*

$$M(x) = \sup_{y \in T(x)} f(x, y) \quad \text{for} \quad x \in X$$

is l.s.c.; and

(ii) *the map $\Phi : X \multimap Y$ defined on X by*

$$\Phi(x) = \{y \in T(x) \mid f(x, y) = M(x)\} \quad \text{for} \quad x \in X$$

is approximable.

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in T(\hat{x}) \quad \text{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

Proof. Note that the marginal function M in (i) is actually continuous since f is u.s.c. and T is a compact-valued u.s.c. map by the well-known result of Berge [11]. Now, each $\Phi(x)$ is nonempty. Moreover, Φ is a closed map. In fact, let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(\Phi)$, the graph of Φ, and $(x_\alpha, y_\alpha) \to (x, y)$ in $X \times Y$. Then

$$f(x,y) \geq \overline{\lim_\alpha} f(x_\alpha, y_\alpha) = \overline{\lim_\alpha} M(x_\alpha) \geq \underline{\lim_\alpha} M(x_\alpha) \geq M(x)$$

and, since $\mathrm{Gr}(T)$ is closed in $X \times Y$, $y_\alpha \in T(x_\alpha)$ implies $y \in T(x)$. Hence $(x, y) \in \mathrm{Gr}(\Phi)$. Therefore, $\Phi : X \multimap E$ is a compact closed approximable map satisfying $\Phi(\mathrm{Bd}\ X) \subset T(\mathrm{Bd}\ X) \subset X$. Hence, by Theorem 9.1, Φ has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T\hat{x}$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$. This completes our proof. □

The following is an immediate consequence of Theorem 10.1.

Theorem 10.2. *Let X be a compact convex subset of a locally convex t.v.s., $f : X \times X \to \mathbb{R}$ a continuous function such that, all of the sets*

$$\{y \in X \mid f(x,y) = \inf_{y \in X} f(x,y)\}$$

for $x \in X$ are (1) *convex,* (2) *contractible,* (3) *decomposable, or* (4) *∞-proximally connected. Then there exists an $\hat{x} \in X$ such that*

$$f(\hat{x}, \hat{x}) \leq f(\hat{x}, y) \quad \textit{for all} \quad y \in X.$$

Proof. In any case (1)-(4), the map $\Phi : X \multimap X$ defined by

$$\Phi(x) := \{y \in X \mid f(x,y) = \inf_{y \in X} f(x,y)\} \quad \text{for} \quad x \in X$$

is a compact closed approximable map; see [10]. Now, the conclusion follows from Theorem 10.1 by putting $X = Y$ and $T(x) = X$ for all $x \in X$. □

Remark. As in [21], Theorem 10.2 can be used to obtain variational or variational-like inequalities due to Hartman-Stampacchia, Browder, Lions-Stampacchia, Mosco, Juberg-Karamardian, Park, Karamardian, Parida-Sahoo-Kumar, Behera-Panda, and Siddiqi-Khaliq-Ansari. For the literature, see [21].

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Sehie Park
The National Academy of Sciences, Republic of Korea, Seoul 137–044; and
Department of Mathematical Sciences, Seoul National University, Seoul 151–747, Korea
E-mail address: shpark@math.snu.ac.kr