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COMMENTS ON RECENT STUDIES ON ABSTRACT CONVEX SPACES

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Abstract. Some modifications of the concept of generalized convex spaces are actually particular types of them. As an example, we introduce ϕ_A-spaces which include L-spaces due to Ben-El-Mechaiekh et al., FC-spaces due to Ding, and others. In this paper, we show that contents of sixteen recent papers on FC-spaces are consequences of the abstract convex space theory or the generalized convex space theory developed by the author.

1. Introduction

Since 1993, the author has studied generalized convex spaces (or G-convex spaces) and the better admissible class $\mathfrak{B}$ of multimaps as common generalizations of various general convexities without linear structures and of multimaps due to a large number of other authors, respectively. We have established within such a frame the foundations of the KKM theory initiated by Knaster, Kuratowski, and Mazurkiewicz (simply, KKM), as well as fixed point theorems and many other equilibrium results for multimaps; see [22-25,29,30,34-37]. This direction of study has been followed by a number of other authors.

In order to polish up the G-convex space theory, in our previous work [24], we suggested to destroy many of artificial terminology adopted by other authors in the KKM theory on such spaces. Moreover, in [29], we showed that a number of fixed point theorems or other results related to G-convex spaces appeared in many works are simple consequences of known results. Furthermore, in [26,27,31,32], we showed that G-convex space theory can be extended to an abstract convex space theory.

Recently there have appeared some modifications of the concept of generalized convex spaces. Typical examples of such modifications are L-spaces due to Ben-El-Mechaiekh et al. [1], FC-spaces due to Ding [2-12], and others; see [29,32]. Some authors incorrectly claimed that such modifications are generalizations of G-convex spaces without giving any justifications or any proper examples. Many of them even claimed that pairs of the forms $(X;\Gamma)$ or $(X;\{\varphi_A\})$ generalize the triple $(X,D;\Gamma)$.

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Moreover, a number of authors claimed to obtain new results on KKM theory within the frames of their own modifications; see [29,32].

Our main aim in this paper is to show that some modifications of G-convex spaces are unified to ϕ_A-spaces and that the contents of sixteen recent papers [2-14,17,39,40] on FC-spaces are simply consequences of the abstract convex space theory or the generalized convex space theory developed by the author.

A part of this paper was refined in our previous works [32,33].

2. Abstract convex spaces and the map classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KD}$

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. The following is well-known:

Definitions. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. In case $X \supset D$, the G-convex space is denoted by $(X \supset D; \Gamma)$. For details, see [22-25,29,30,34-37].

Recently, the above concept is extended as follows [26,31]:

Definitions. An *abstract convex space* $(E, D; \Gamma)$ consists of nonempty sets E, D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

[co is reserved for the convex hull in vector spaces.]

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D'\rangle})$ is called a Γ-*convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

An abstract convex space with a topology on E is called *an abstract convex topological space.*

Examples. 1. A convexity space $(E, \mathcal{C})$ in the classical sense; see [38], where the bibliography lists 283 papers.

2. A generalized convex space or a G-convex space $(X, D; \Gamma)$ due to Park.

3. A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde [21].

4. A G-convex space $(X, D; \Gamma)$ is called a *C-space* (or an *H-space*) if each Γ_A is ω-connected (that is, n-connected for all $n \geq 0$) and $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D \rangle$; see Horvath [19,20] for particular forms for $X = D$.

5. For $X = D$, a G-convex space $(X, D; \Gamma)$ reduces to an L-space (X, Γ) due to Ben-El-Mechaiekh et al. [1].

Definitions. [26,31] Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is said to have the *KKM property* and called *a $\mathfrak{K}$-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation KKM(E, Z) instead of $\mathfrak{KC}(E, Z)$.

3. Basic theorems in the KKM theory

In our KKM theory on abstract convex spaces given in [26,27,31], there exist some basic theorems from which we can deduce several equivalent formulations that can be used for applications. In this section, we introduce some of such basic theorems.

We begin with the following prototype of KKM type theorems:

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F \in \mathfrak{K}(E, Z)$. Let $G : D \multimap Z$ be a map such that*

(A.1) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$.*

Then for each $N \in \langle D \rangle$, $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$.

Remarks. 1. If Z is a topological space and G is open-valued [resp., closed-valued], then we can assume $F \in \mathfrak{KO}(E, Z)$ [resp., $F \in \mathfrak{KC}(E, Z)$].

2. If $E = Z$ and if the identity map $1_E = F \in \mathfrak{K}(E, E)$, then Condition (A.1) says that G is a KKM map.

3. For KKM type theorems on convex spaces, H-spaces, or G-convex spaces and their applications, there are a large number of works; see [22,23,31,34-37] and references therein.

The following coincidence theorem follows from Theorem A.

Theorem B. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, $S : D \multimap Z$, $T : E \multimap Z$ maps, and $F \in \mathfrak{K}(E, Z)$. Suppose that*

(B.1) *for each $z \in F(E)$, $\mathrm{co}_\Gamma S^-(z) \subset T^-(z)$; and*

(B.2) *$F(E) \subset S(N)$ for some $N \in \langle D \rangle$.*

Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Remark. If Z is a topological space and S has open [resp., closed] values, then we can assume $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KO}(E, Z)$].

Corollary B.1. *Let $(E, D; \Gamma)$ be an abstract convex space, $S : D \multimap E$, $T : E \multimap E$ maps, and $1_E \in \mathfrak{K}(E, E)$. Suppose that*

(1) *for each $y \in E$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*

(2) *$E \subset S(N)$ for some $N \in \langle D \rangle$.*

Then T has a fixed point $\bar{x} \in E$, that is, $\bar{x} \in T(\bar{x})$.

From Theorem B, we obtain the following Fan type matching theorem:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, $S : D \multimap Z$, and $F \in \mathfrak{K}(E, Z)$ satisfying $F(E) \subset S(N)$ for some $N \in \langle D \rangle$. Then there exists an $M \in \langle D \rangle$ such that $F(\Gamma_M) \cap \bigcap\{S(x) \mid x \in M\} \neq \emptyset$.*

Recall that, for a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$.

The following KKM theorem for G-convex spaces shows $1_X \in \mathfrak{KC}(X, X) \cap \mathfrak{KO}(X, X)$; see [23,24,37]:

Theorem D. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a multimap such that*

(1) *F has closed [resp., open] values; and*

(2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property (More precisely, for each $N \in \langle D \rangle$, we have $\phi_N(\Delta_n) \cap \bigcap_{z \in N} F(z) \neq \emptyset$).

Further, if

(3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have $\bigcap_{z \in D} \overline{F(z)} \neq \emptyset$.

Examples. 1. For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \mathrm{co}$, the convex hull, Theorem D reduces to the original KKM principle and its open version; see [22,23].

2. If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem D extends Fan's KKM lemma; see [16].

3. Note that any KKM theorem on spaces of the form $(X, \{\varphi_A\})$ can not generalize the original KKM principle or Fan's KKM lemma.

The following continuous selection theorem for multimaps with noncompact domain is given in [27, Lemma 3.5]:

Theorem E. *Let X be a normal space, $(Y, D; \Gamma)$ a G-convex space, and $S : X \multimap D$ a map such that $X = \bigcup\{\text{Int } S^-(y) \mid y \in A\}$ for some $A \in \langle D \rangle$. Then there exists a continuous function $s : X \to \Gamma_A$ such that $s(x) \in \Gamma(A \cap S(x))$ for all $x \in X$. In fact, if $|A| = n + 1$, then $s = \phi_A \circ p$, where $\phi_A : \Delta_n \to \Gamma_A$ and $p : X \to \Delta_n$ are continuous functions.*

There have appeared a lot of variants or particular forms of Theorem E; for examples, [15, Lemma 2.1] where it should be $D \subset Y$; [5, Theorems 2.1-2.3] where its author incorrectly claimed that a large number of known results follow from his; and [6, Theorem 2.1] where X should be Hausdorff and its author incorrectly stated that his result generalizes one of the present author's.

From now on, multimaps are sometimes called maps.

Definitions. Let $(X, D; \Gamma)$ be a G-convex space and Z a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Z as follows [22]:

$F \in \mathfrak{B}(X, Z) \iff F : X \multimap Z$ is a map such that for any $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$ and any continuous function $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n)$

For more details and examples of subclasses of $\mathfrak{B}$, see [22,28,30] and references therein. Authors of [2,17,40] adopted a certain restricted subclass $\mathcal{B}$ of $\mathfrak{B}$ and made a false claim that theirs includes $\mathfrak{B}$.

The following are given in [31]:

Theorem F. *Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space.*

(1) *If Y is Hausdorff, then every compact map $F \in \mathfrak{B}(X, Y)$ belongs to $\mathfrak{KC}(X, Y)$.*

(2) *If $F : X \multimap Y$ is a closed map such that $F\phi_N \in \mathfrak{KC}(\Delta_n, Y)$ for any $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, then $F \in \mathfrak{B}(X, Y)$.*

(3) *In the class of compact closed maps defined on a convex space (X, D) into a Hausdorff space Y, two subclasses $\mathfrak{KC}(X, Y)$ and $\mathfrak{B}(X, Y)$ are identical.*

4. ϕ_A-spaces

Recently, there have appeared a number of authors who introduced spaces of the form $(X, \{\varphi_A\})$; see [29,32,33] and references therein. Some authors tried to rewrite our works on G-convex spaces by simply replacing $\Gamma(A)$ by $\varphi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples.

Motivated by this fact, we are concerned with equivalent formulations of G-convex spaces as follows [32,33]:

Definition. A ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

Any G-convex space is a ϕ_A-space. The converse also holds:

Theorem G. *A* ϕ_A*-space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ *can be made into a* G*-convex space* $(X, D; \Gamma)$.

Proof. This can be done in two ways.

(1) For each $A \in \langle D\rangle$, by putting $\Gamma_A := X$, we obtain a trivial G-convex space $(X, D; \Gamma)$.

(2) Let $\{\Gamma^\alpha\}_\alpha$ be the family of maps $\Gamma^\alpha : \langle D\rangle \multimap X$ giving a G-convex space $(X, D; \Gamma^\alpha)$ such that $\phi_A(\Delta_n) \subset \Gamma^\alpha_A$ for each $A \in \langle D\rangle$ with $|A| = n+1$. Note that, by (1), this family is not empty. Then, for each α and each $A \in \langle D\rangle$ with $|A| = n + 1$, we have

$$\phi_A(\Delta_n) \subset \Gamma^\alpha_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma^\alpha_J \quad \text{for} \quad J \subset A.$$

Let $\Gamma := \bigcap_\alpha \Gamma^\alpha$, that is, $\Gamma_A = \bigcap_\alpha \Gamma^\alpha_A$. Then

$$\phi_A(\Delta_n) \subset \Gamma_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma_J \quad \text{for} \quad J \subset A.$$

Therefore, $(X, D; \Gamma)$ is a G-convex space.

Consequently, G-convex spaces and ϕ_A-spaces are essentially the same.

Definition. For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map*.

Theorem H. (1) *A KKM map* $G : D \multimap X$ *on a* G*-convex space* $(X, D; \Gamma)$ *is a KKM map on the corresponding* ϕ_A*-space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$.

(2) *A KKM map* $T : D \multimap X$ *on a* ϕ_A*-space* $(X, D; \{\phi_A\})$ *is a KKM map on a new* G*-convex space* $(X, D; \Gamma)$.

Proof. (1) This is clear from the definition of a KKM map on a G-convex space.

(2) Define $\Gamma : \langle D\rangle \multimap X$ by $\Gamma_A := T(A)$ for each $A \in \langle D\rangle$. Then $(X, D; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n+1$, we have a continuous function $\phi_A : \Delta_n \to T(A) =: \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle D\rangle$ and hence $T : D \multimap X$ is a KKM map on a G-convex space $(X, D; \Gamma)$.

The following is a restatement of the KKM Theorem D for ϕ_A-spaces. The proof is just a simple modification of that in [23,24,37]:

Theorem I. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, let $G : D \multimap X$ be a KKM map with closed [resp., open] values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D\rangle$ with $|N| = n+1$, we have $\phi_N(\Delta_n)\cap\bigcap_{z\in N} G(z) \neq \emptyset$.)*

Further, if

(3) $\bigcap_{z\in M} \overline{G(z)}$ is compact for some $M \in \langle D\rangle$,

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

Proof. Let $N = \{z_0, z_1, \ldots, z_n\}$. Since G is a KKM map, for each vertex e_i of Δ_n, we have $\phi_N(e_i) \in G(z_i)$ for $0 \leq i \leq n$. Then $e_i \mapsto \phi_N^{-1}G(z_i)$ is a closed [resp., open] valued map such that $\Delta_k = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1}G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the original KKM principle, $\Delta_n \supset \bigcap_{i=0}^{n} \phi_N^{-1}G(z_i) \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \left(\bigcap_{z\in N} G(z)\right) \neq \emptyset$.

The second conclusion is clear.

Remark. We may assume that, for each $a \in D$ and $N \in \langle D\rangle$, $G(a) \cap \phi_N(\Delta_n)$ is closed [resp., open] in $\phi_N(\Delta_n)$. This is said by some authors that G has finitely closed [resp., open] values. However, by replacing the topology of X by its finitely generated extension, we can eliminate "finitely"; see [24].

From now on, numbers attached to Theorems and Definitions are the ones in their original sources.

5. Ding's FC-spaces

In 2005, Ding [2] introduced the following notion of "a finitely continuous" topological space (in short, FC-space):

Definitions 1.1 [2]. $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset D of $(Y, \{\varphi_N\})$ is said to be a FC-subspace of Y if for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$ where $\Delta_k = \mathrm{co}\{e_{i_j} : j = 0, \ldots, k\}$.

Then Ding [2] wrote that it is clear that the class of G-convex spaces $(Y; \Gamma)$ is a true subclass of FC-spaces without giving any justification.

Note that for each N, there should be infinitely many φ_M's with $M = N$ since some elements in M may be the same.

In 2006, Ding [3] added the following to Definition 1.1 [2]:

Definitions 2.1 [3]. If A and B are two subsets of Y, B is said to be a FC-subspace of Y relative to A if for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ and for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset A \cap N$, $\varphi_N(\Delta_k) \subset B$ where $\Delta_k = \mathrm{co}\{e_{i_j} : j = 0, \ldots, k\}$. If $A = B$, then B is called a FC-subspace of Y.

Then Ding [3] wrote: "It is easy to see that the class of FC-spaces includes the classes of convex sets in topological vector spaces, C-spaces (or H-spaces) [19], G-convex spaces [35], L-convex spaces [1], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces." Here again he failed to give any justification or any proper example of his space which is not G-convex. One wonders how could a pair $(Y, \{\varphi_N\})$ generalize a triple $(X, D; \Gamma)$ in [35].

The above definition and Ding's claim appeared also in [4-15,17, 39,40], and possibly more. Sixteen such papers on FC-spaces have appeared within two years! In these papers, known results in KKM theory on G-convex spaces are restated or modified. In order to prevent such unnecessary efforts, something has to be done.

The following is clear:

Proposition 1. *An FC-space $(Y, \{\varphi_N\})$ can be made into a particular abstract convex space $(Y; \Gamma)$ with $\Gamma_N := \varphi_N(\Delta_n)$ whenever $N \in \langle Y \rangle$ with $|N| = n + 1$.*

Moreover, on the contrary to Ding's claim, from Theorem G, we have the following:

Proposition 2. *An FC-space $(Y, \{\varphi_N\})$ can be made into an L-space $(Y; \Gamma)$, a particular type of G-convex spaces $(Y, D; \Gamma)$.*

Recall that our G-convex space $(X, D; \Gamma)$ or our abstract convex space $(E, D; \Gamma)$ are originated from the space $(\Delta_n, V; \mathrm{co})$ due to KKM and $(E \supset D; \mathrm{co})$ due to Fan [16], where D is a nonempty subset of a topological vector space E. Note that the particular case $X = D$ or $E = D$ can not cover the original KKM principle or the celebrated Ky Fan lemma from the beginning. This is the most serious defect of L-spaces or FC-spaces.

Now Ding's FC-subspace relative to A in Definition 2.1 [3] can be extended as follows:

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and $X \subset E$, $D' \subset D$. Then X is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. Recall that, in such case, we used to say that a subset X of E is Γ-*convex* if, for any $N \in \langle X \cap D \rangle$, we have $\Gamma_N \subset X$. This is just saying that X is Γ-convex relative to $D' := X \cap D$.

Therefore, instead of using the concept of an FC-subspace of $(Y, \{\varphi_N\})$ relative to A, we may use the Γ-convex subset of the abstract or G-convex space $(Y; \Gamma)$ relative to A. Any interested reader can check this matter in all of [2-14,17,39,40].

For a topological space $(X, \mathcal{T})$, the compactly generated extension (or the k-extension) $\mathcal{T}_k$ of the original topology $\mathcal{T}$ is a new topology of X finer than $\mathcal{T}$ such that $\mathcal{T}_k$ is the collection of all compactly open [resp., compactly closed] subsets of $(X, \mathcal{T})$. Note that the artificial terminology of compact interior, compact closure, etc., are not practical and can be eliminated by switching the original topology of the underlying space to its compactly generated extension; see [24].

Such inadequate artificial terminology were used in [2-4,6,14,17,40], but disappeared or withdrawn in [5,7-13,39].

6. Comments on papers on FC-spaces

In this section, we show that main results in sixteen papers [2-14,17,39,40] on FC-spaces are simply either consequences of our previous works or have more simplified general formulations. We will not check all the results in those papers, but give comments on some sample results. Now each of (I)-(XVI) corresponds [2-14,17,39,40] in this order.

(I) In [2], its author introduced a restricted subclass $\mathcal{B}$ of our better admissible class $\mathfrak{B}$ for his space. His basic existence theorem on maximal elements runs as follows:

Theorem 2.1 [2]. *Let X be a topological space, $(Y, \{\varphi_N\})$ be a FC-space, $F \in \mathcal{B}(Y, X)$ be a u.s.c. with compact values, and $A : X \to 2^Y$ such that,*

(i) *for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N$,*

$$F(\varphi_N(\Delta_k)) \cap (\bigcap_{j=0}^{k} \operatorname{cint} A^{-1}(y_{i_j})) = \emptyset,$$

(ii) *$A^{-1} : Y \to 2^X$ is transfer compactly open-valued,*

(iii) *there exists a nonempty set $Y_0 \subset Y$ and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a compact FC-subspace L_N of Y containing $Y_0 \cup N$ such that $K = \bigcap_{y \in Y_0} (\operatorname{cint} A^{-1}(y))^c$ is empty or compact in X.*

Then there exists a point $\hat{x} \in X$ such that $A(\hat{x}) = \emptyset$.

We show that this is a simple consequence of our Theorem C. Note that X should be Hausdorff, otherwise Ding's proof does not work. We also ignore 'compactly' and adopt int instead of cint.

Proof using Theorem C. Suppose that $A(x) \neq \emptyset$ for each $x \in X$. Then condition (ii) implies $X = \bigcup_{y \in Y} A^{-1}(y) = \bigcup_{y \in Y} \operatorname{int} A^{-1}(y)$. Since K is empty or compact, $K \subset \bigcup_{y \in N} \operatorname{int} A^{-1}(y)$ for some $N \in \langle Y \rangle$. Therefore,

$$X = K^c \cup K = \bigcup_{y \in Y_0 \cup N} \operatorname{int} A^{-1}(y).$$

Choose L_N as in hypothesis. Now we apply Theorem C.

Let $(E; \Gamma) := (L_N; \Gamma)$ be an abstract convex space such that $\Gamma_M := \varphi_M(\Delta_n)$ for $M \in \langle E \rangle$. As in our Proposition 2 in Section 5, by replacing Γ by a suitable one, we may assume $(E; \Gamma)$ is a G-convex space. Since $F|_{L_N} \in \mathfrak{B}(L_N, X)$ and X is Hausdorff, by Theorem F(1), $F|_{L_N} \in \mathfrak{KC}(L_N, X)$. Since L_N is compact and F is u.s.c. with compact values, $F(L_N)$ is compact. Since

$$F(L_N) \subset X = \bigcup_{y \in Y_0 \cup N} \operatorname{int} A^{-1}(y),$$

$$F(L_N) \subset \bigcup_{y \in N'} S(y) \subset X \text{ for some } N' \in \langle L_N \rangle,$$

where $S(y) := \operatorname{int} A^{-1}(y)$. Therefore all of the requirements of Theorem C are satisfied, and hence, there exists an $M \in \langle D \rangle$ such that $F(\Gamma_M) \cap \bigcap \{S(x) \mid x \in M\} \neq \emptyset$. This contradicts (i).

Remark. Condition (i) can be simplified. Ding's original proof is one–and–half page long. Similarly, other results in [2] can be simply derived from known results in the G-convex space theory; see [23,31,35] and references therein

(II) In [3], the following KKM type theorem is given:

Theorem 2.1 [3]. *Let X be a nonempty set, (Y, φ_N) be a FC-space, and Z be a topological space. Let $s : X \to Y$ be a surjective mapping, $F : X \to 2^Z$ be a set-valued mapping and $T \in s\text{-}KKM(X, Y, Z)$ such that*

(1) $\overline{T(s(X))}$ is compact in Z,

(2) F is a generalized s-KKM mapping with respect to T with compactly closed values.

Then $\overline{T(s(X))} \cap (\bigcap_{x \in X} F(x)) \neq \emptyset$.

Proof using Theorem A. Note that $s(X) = Y$ and $T \in \mathfrak{KC}(Y, Z)$. We apply Theorem A with $E := Y$, $D := X$, $\Gamma_N := \text{Im } \varphi_N$, $F := T$, and $G := F$. Then (2) implies (A.1). Applying Theorem A, $\{\overline{T(Y)} \cap F(x)\}_{x \in X}$ has the finite intersection property. Since $\overline{T(Y)} \cap F(x)$ is closed, in view of (1), we immediately have the conclusion.

Remark. In [3], its author applied Theorem 2.1 [3] to various section theorems and coincidence theorems by a routine method in the KKM theory. Recall that this theory is already done for generalized convex spaces or abstract convex spaces in more general setting; see [23,31,35] and references therein.

(III) In [4], the following main theorem is applied to some existence theorems of solutions for three classes of generalized vector equilibrium problems under noncompact setting of FC-spaces.

Theorem 3.2 [4]. *Let (Y, φ_N) be a FC-space and X be a topological space. Let $T \in \text{KKM}(Y, X)$ and $F, G, M : Y \to 2^X$ be set-valued mappings such that*

(1) F has transfer compactly closed values,

(2) for each $y \in Y$, $G(y) \subset F(y)$ and $T(y) \subset M(y)$,

(3) G is a generalized KKM mapping with respect to M,

(4) for each compact subset D of Y, $\overline{T(D)}$ is compact in X,

(5) there exists a compact subset K of X such that, for each $N \in \langle Y \rangle$, there exists a compact FC-subspace L_N of Y containing N such that

$$\emptyset \neq T(L_N) \cap (\bigcap_{y \in L_N} \text{ccl } F(y)) \subset K.$$

Then $\bigcap_{y \in Y} F(y) \neq \emptyset$.

Proof using Theorem A. Set $\Gamma_N := \varphi_N(\Delta_n)$ for each $N \in \langle Y \rangle$ as in Proposition 1. Condition (3) implies

$$M(\Gamma_N) \subset G(N) \ \text{ for each } \ N \in \langle Y \rangle.$$

By (2), this implies

$$T(\Gamma_N) \subset F(N) \ \text{ for each } \ N \in \langle Y \rangle.$$

Since $T \in \mathfrak{KC}(Y, X)$, by Theorem A, we have

$$\overline{T(Y)} \cap \bigcap_{y \in N} F(y) \neq \emptyset \ \text{ for each } \ N \in \langle Y \rangle.$$

Let $K' := \overline{T(Y)} \cap \bigcap_{y \in N} \overline{F(y)} \cap K \supset \overline{T(L_N)} \cap \bigcap_{y \in L_N} \overline{F(y)} \cap K \neq \emptyset$. Then $\{\overline{F(y)} \cap K'\}_{y \in Y}$ is a family of closed sets in the compact set K and has the finite intersection property. Hence it has the whole intersection property, that is, $\bigcap_{y \in Y} \overline{F(y)} \cap \overline{T(Y)} \cap K \neq \emptyset$. By (1) the conclusion follows.

Note that condition (4) seems to be redundant.

In [4], its author claimed that his theorems improve and generalize many known results in literature. This is a nonsense since his results can not be applied to G-convex spaces.

(IV) In [5], its author claimed that "Some new continuous selection theorems are first proved in noncompact topological spaces. As applications, some new collectively fixed point theorems and coincidence theorems for two families of set-valued mappings defined on product space of noncompact topological spaces are obtained under very weak assumptions. These results generalize many known results in recent literature."

Contrary to his claim, all results in [5] concerned with the so-called FC-spaces and are particular forms of known results for G-convex spaces. For example, one of his new continuous selection theorems [5, Theorem 2.1] is a particular form of Theorem E.

(V) In [6], using the inadequate terminology like ccl and cint, its author begins with almost same or particular results in [5] under "very weak assumptions. As applications, some nonempty intersection theorems, inclusion theorems and existence theorems of solutions for systems of inequalities in the product space of noncompact FC-spaces." He also claimed that his theorems improve and generalize many known results from the recent literature, based on his misconception that G-convex spaces are FC-spaces.

(VI) In [7], its author stated as follows:

"Let X be a topological space and (Y, ϕ_N) be a FC-space. The class $\mathcal{B}(Y, X)$ of better admissible mappings was introduced by Ding [2] as follows: $F \in \mathcal{B}(Y, X) \Leftrightarrow F : Y \to 2^X$ is a upper semi-continuous set-valued mapping with compact values such that for any $N = \{y_0, ..., y_n\} \in \langle Y \rangle$ and any continuous mapping $\psi : F(\varphi_N(\Delta_n)) \to \Delta_n$, the composition mapping $\psi \circ F|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point.

Clearly the our class $\mathcal{B}(Y, X)$ of better admissible mappings includes many classes of set-valued mappings as true subclasses (see [2])."

Note that Ding's better admissible class $\mathcal{B}$ is an imitation and a proper subclass of our $\mathfrak{B}$. For example, a Fan-Browder map or a Φ-map belongs to $\mathfrak{B}$, but not Ding's.

In [7], a nonempty intersection theorem [3, Theorem 3.1] is used to obtain some existence theorems on weak Pareto equilibria for generalized multi-objective games. Its author claimed that [4, Theorem 3.1] improves and generalizes Theorems 2 and 4 of Park and Kim [34] in several aspects and his argument method is completely different from that in [34]. This statement is false since Ding's is for FC-spaces, but [34] is for G-convex spaces. Note that all results in [7] can be extended to G-convex spaces.

(VII) In [8], its author introduced some new families of G_{KKM}-mappings and G_{KKM}-majorized mappings from a topological space X into finite continuous topological spaces (Y_i, φ_{N_i}) (in short, FC-spaces) involving a set-valued mapping $T \in \mathrm{KKM}(Y, X)$ with KKM property, where $Y = \prod_{i \in I} Y_i$.

In order to show that all results in [8] can be generalized to G-convex spaces, we give the following definition of families of G_{KKM}-maps:

Definition. Let I be any index set and X be a topological space. For each $i \in I$, let $(Y_i, D_i; \Gamma^i)$ be a G-convex space, $Y := \prod_{i \in I} Y_i$, $D := \prod_{i \in I} D_i$ and $T \in \mathfrak{KC}(Y, X)$. For each $i \in I$, a multimap $G_i : X \to D_i$ is called a G_{KKM}-*map* if

(a) for each $N := \{a_0, \dots, a_n\} \in \langle D \rangle$ and each $J := \{a_{i_0}, \dots, a_{i_k}\} \subset N$, $T(\Gamma_J) \cap (\bigcap_{j=0}^k G_i^-(\pi_i(a_{i_j}))) = \emptyset$, where π_i is the projection of D onto D_i;

(b) $G_i^- : D_i \multimap X$ has open values.

With this definition, we can state [8, Theorem 3.1] for G-convex spaces as follows:

Theorem J. *Let $(Y, D; \Gamma)$ be a G-convex space and X a topological space. Let $T \in \mathfrak{KC}(Y, X)$ be a compact map and $G : X \multimap D$ a G_{KKM}-map. Then there exists a point $\hat{x} \in \overline{T(Y)}$ such that $G(\hat{x}) = \emptyset$.*

Proof. Since G is a G_{KKM}-map with a singleton I, for each $N := \{a_0, \dots, a_n\} \in \langle D \rangle$, by (a), we have

$$T(\Gamma_J) \subset X \setminus \bigcap_{j=0}^{k} G^-(a_i) = \bigcup_{j=0}^{k} (X \setminus G^-(a_j)) = G^*(J)$$

where $G^* : D \multimap X$ is defined by $G^*(a) := X \setminus G^-(a)$ for $a \in D$. Then G^* is a KKM map with respect to T and closed-valued by (b). Let $F : D \multimap \overline{T(Y)}$ be a map defined by $F(a) := \overline{T(Y)} \cap G^*(a)$ for $a \in D$. Then

$$T(\Gamma_N) \subset \overline{T(Y)} \cap G^*(N) = F(N)$$

for each $N \in \langle D \rangle$. Then F is a compact closed-valued KKM map with respect to T. Since $T \in \mathfrak{KC}(Y, X)$, $\{F(a)\}_{a \in D}$ has the finite intersection property, there exists an element $\hat{x} \in F(a) = \overline{T(Y)} \cap G^*(a)$ for all $a \in D$, that is, $\hat{x} \in \overline{T(Y)}$ and $\hat{x} \in X \setminus G^-(a)$ for all $a \in D$. This shows $G(\hat{x}) = \emptyset$.

Remark. Variations of Theorem J can be obtained in the following cases:
(1) $G^*(a)$ is compact for some $a \in D$.
(2) $T(Y)$ is compact. In this case $\hat{x} \in T(Y)$.

From a particular form of Theorem J, in [8], some equilibrium existence theorems of generalized games are obtained in non-compact product FC-spaces. These can be easily extended to G-convex spaces by applying our Theorem J.

(VIII) In [9], which is a continuation of [8], four classes of systems of generalized vector quasi-equilibrium problems are introduced and studied in FC-spaces. By applying these notions and the results in [8], some existence theorems of solutions for these systems of generalized vector quasi-equilibrium problems are obtained in non-compact product FC-spaces.

These results can be easily extended to G-convex spaces.

(IX) In [10], Ding remedied his previous definition as follows:

Definition 2.2 [10]. $(X, \{\varphi_N\})$ is said to be a finitely continuous topological space (in short, FC-space) if X is a topological space and for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to X$. If D is a subset of X, D is said to be a FC-subspace of X if for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and for each $\{x_{i_0}, \ldots, x_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$ where $\Delta_k = \text{co}\{e_{i_j} : j = 0, \ldots, k\}$.

He introduced multimap classes V, $\mathbf{L}$, $\mathbf{U}_c^k$ without mentioning their sources. (Actually they are $\mathbb{V}$, $\mathbb{L}$, $\mathfrak{A}_c^\kappa$ due to the present author.) Moreover, Ding claimed in 2005 that, if $(X, \{\varphi_N\})$ is an FC-space and Y is a topological space, then $V(X, Y) \subset \mathbf{U}_c^k(X, Y) \subset KKM(X, Y)$. The correct form of this fact is already given in 1997 [35] for a G-convex space $(X, D; \Gamma)$ and a Hausdorff space Y.

It is evident that all results in [10] can be stated more generally for G-convex spaces or ϕ_A-spaces.

(X) In 2002, the present author [25] extended the celebrated Himmelberg fixed point theorem [18] to locally G-convex spaces. (For further generalized forms, see [30,31].)

This result is restated for FC-spaces in [11, Theorem 2.1]. Its author falsely stated that this generalizes Theorem 2 of Park [25] from locally G-convex space to locally FC-uniform space without any convexity structure. This is another evidence of our claims in Section 5.

Ding applied [11, Theorem 2.1] to generalized games and systems of generalized vector quasi-equilibrium problems in locally FC-uniform spaces. These results can be extended to locally G-convex uniform spaces by applying results in [25].

(XI) In [12], its author repeated his original definition of FC-spaces and insisted that his class of such spaces includes G-convex spaces [35], L-convex spaces [1], and many topological spaces with abstract convexity structure as true subclasses. This false statement is the basis of nearly (perhaps more than) twenty papers of its author and his blind followers.

In order to justify his claim, he gave two examples of two FC-spaces (E, φ_N) which are not L-spaces (E, Γ) with $\Gamma(N) = \varphi_N(\Delta_n)$ for each $N \in \langle E \rangle$. This is wrong. By defining $\Gamma'(N) = X_i$ if $N \subset X_i, i = 1, 2$, and $\Gamma'(N) = E$ otherwise, (E, Γ) becomes an L-space.

(XII) In [13], its authors repeated his previous definition where the restriction "where some elements in N may be same" in the original definition in [2] was removed. Clearly an FC-space is a particular ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ for the case $X = D$. Further, the authors in [13] still insists that "any G-convex spaces introduced by Park and Kim [35] are all FC-spaces. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces".

In Lemma 2.4 [13], its authors showed that any generalized FC-KKM mapping is a generalized R-KKM mapping, which is already shown to be a simple KKM map for a G-convex space; see [32,33]. Lemmas 3.1 and 3.2 [13] are particular forms of Theorem I.

In [13], from the KKM Lemma 3.2, it is routine in the KKM theory to deduce Fan-Browder type fixed point theorems (Section 3), Ky Fan type minimax inequalities and geometric forms (Section 4), existence of maximal elements (Section 5), and equilibrium existence theorems (Section 6). The authors of [13] obtained these results for their FC-spaces, but it is evident that they can be stated more generally for G-convex spaces or ϕ_A-spaces.

The following is the main theorem of [13] with one-page proof:

Theorem 3.1 [13]. *Let (X, φ_N) be an FC-space, $F, G : X \to 2^X$ and K be a nonempty subset of X such that*

(1) *for each $x \in X$, $F(x) \subset G(x)$,*

(2) *for each $y \in X$, $F^{-1}(y)$ is compactly open in X,*

(3) *for each $N \in \langle X \rangle$, there exists a compact FC-subspace L_N of X containing N such that*

$$L_N \bigcap \bigcap_{x \in L_N} \mathrm{cl}_{L_N} \left((X \setminus (FC(G))^{-1}(x)) \bigcap L_N \right) \subset K,$$

(4) *for each $x \in K$, $F(x) \neq \emptyset$.*

Then there exists $\hat{y} \in X$ such that $\hat{y} \in FC(G(\hat{y}))$.

Note that, in the coercivity or compactness condition (3), the formula can be replaced by

$$\bigcap_{x \in L_N} \mathrm{cl}_{L_N} \left((L_N \setminus (FC(G))^{-1}(x) \right) \subset K.$$

Hence (3) seems to be artificial, not practical, not elegant, and it should be destroyed. The condition is used to all of the key results in [13].

(XIII) In [14], within the frame of FC-spaces, some KKM type theorems and existence results of generalized vector equilibrium problems are obtained.

The following is a sample result:

Theorem 3.1 [14]. *Let (Y, φ_N) be a FC-space and X be a topological space. Let $T \in KKM(X, Y)$ be a compact mapping and $F : Y \to 2^X$ be a set-valued mapping such that F is a generalized KKM mapping with respect to T with compactly closed values. Then* $\overline{T(Y)} \cap (\bigcap_{y\in Y} F(y)) \neq \emptyset$.

Note that this is a simple consequence of Theorem A and a particular form of [3, Theorem 2.1] for $X = Y$ and $s = 1_Y$; see (II).

(XIV) In [17], its authors followed [2] and obtained the following coincidence theorem:

Theorem 3.1 [17]. *Let X be a topological space and (Y, φ_N) be a FC-space. Let $F, G : X \to 2^Y$ and $T \in \mathcal{B}(Y, X)$ be set-valued mappings such that*
(i) *for each $x \in X$ and $N \in \langle F(x)\rangle$, $\varphi_N(\Delta_k) \subseteq G(x)$;*
(ii) *for each nonempty compact subset K of X, $K = \bigcup_{y\in Y}(\text{cint } F^{-1}(y) \cap K)$;*
(iii) *there exists a compact FC-subspace L_N of Y containing N such that*

$$T(L_N) \setminus K \subseteq \bigcup\{\text{cint } F^{-1}(y) : y \in L_N\}.$$

Then there exist $\hat{y} \in Y$ and $\hat{x} \in X$ such that $\hat{x} \in T(\hat{y})$ and $\hat{y} \in G(\hat{x})$.

This simply follows from Theorem B.

(XV) In [39], its authors stated that, "Recently, Ding [2] introduced FC-space which extended G-convex space further and proved the corresponding KKM theorem. From this, many new KKM type theorems and applications were founded in FC-spaces" in a number of Ding's papers.

In [39], "by introducing *W-G-F-KKM* mapping with respect to T which extends the corresponding notion of Ding in [14], we will prove a new matching theorem and some intersection theorems in noncompact FC-spaces and study some properties of KKM mappings. As applications, some new minimax inequalities are established. The results presented in this paper extend some corresponding known results in the literature."

Since FC-spaces are particular to G-convex spaces, the authors of [39] should correct their claims.

(XVI) In [40], its authors also followed methods in [2] and claimed as follows in their Abstract: "By applying the existence theorem of maximal element [2, Theorem 2.1], some new collectively fixed-point theorems for a family of set-valued mappings defined on the product space of noncompact FC-space are proved and some new theorems about minimax inequality involving two functions are given to show the relations of fixed-point theorem and minimax inequality in FC-spaces. These results improve and generalize many important results in the recent literature."

The authors of [40] used artificial terminology like ccl and cint, defined the better admissible class $\mathcal{B}^*$ (which is Ding's $\mathcal{B}$), and made a false statement that $\mathcal{B}^*$ properly contains the classes $\mathcal{B}$ and U_c^K (which are $\mathfrak{B}$ and $\mathfrak{A}_c^\kappa$ due to the present author). Note that all results on FC-spaces in [40] are particular or modified forms of known ones on G-convex spaces.

Added in proof. After this paper was accepted, we found three more papers on FC-spaces [10,12,13] and added some comments on them in this paper.

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