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Comments on the KKM Theory on ϕ_A-spaces

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Abstract

Basic results in the KKM theory on abstract convex spaces and the KKM maps are applied to ϕ_A-spaces which unify various imitations of G-convex spaces. We show that basic theorems on ϕ_A-spaces can be applied to correct and improve results on the so-called R-KKM maps on the so-called L-convex spaces in [3].

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1 Introduction

Since 1993, the author has initiated the study of the KKM theory on generalized convex spaces (or G-convex spaces) $(X, D; \Gamma)$ as a common generalization of various general convexities without linear structures due to other authors. We have established within such a frame the foundations of the KKM theory initiated by Knaster, Kuratowski, and Mazurkiewicz [14], as well as fixed point theorems and many other equilibrium results for multimaps. This direction of study has been followed by a number of other authors.

In the last decade, there have appeared authors in [3-9,13] and others who attempted to reformulate our works on G-convex spaces by replacing $\Gamma(A)$ by $\phi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples. Many of them even claimed that pairs of the forms $(X; \Gamma)$ called L-spaces or $(X; \{\varphi_N\})$ called

FC-spaces generalize the triple $(X, D; \Gamma)$. Those authors obtained KKM type theorems or equivalents which can not be applicable even to the original KKM principle [14] for $(\Delta_n, V; \mathrm{co})$ or to the celebrated Fan lemma [10] for $(E \supset D; \mathrm{co})$, where E is a topological vector space.

Moreover, recently in [3], its author defined L-convex spaces same as G-convex spaces and obtained 23 theorems which are improper modifications of known results. This might misguide naive readers to a wrong direction as the recent case of the so-called FC-spaces in more than 15 papers within two years.

In order to destroy such inadequate fake generalizations of our previous works due to other authors, we found that, in the framework of certain abstract convex spaces including G-convex spaces properly, the elements or basic results in the KKM theory can be established; see [21-27]. We also noted that the newly defined ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ in [25] are same to G-convex spaces.

Our principal aim in the present paper is to introduce basic results in the KKM theory on abstract convex spaces and the map classes $\mathfrak{KC}$ and $\mathfrak{KO}$, as in [22,26]. These are applied to simplify various modifications of the concept of G-convex spaces and the KKM type theorems on them. Especially we discuss the nature of results in [3].

In Section 2, we introduce our new abstract convex spaces, KKM maps, and the map classes $\mathfrak{KC}$ and $\mathfrak{KO}$ in [21-27]. Section 3 deals with a few basic theorems in our KKM theory for abstract convex spaces given in [21,26]. In Section 4, we introduce ϕ_A-spaces which unify various imitations of G-convex spaces. Section 5 deals with the nature of the so-called R-KKM theorems on the so-called L-convex spaces in [3]. Finally, in Section 6, we show that basic theorems in the KKM theory on ϕ_A-spaces can be applied to correct and improve results in [3].

2 Abstract convex spaces and the map classes $\mathfrak{KC}$, $\mathfrak{KO}$

In this section, we introduce new concepts in [21-23] as preliminaries.

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps. For a map $F : X \multimap Y$, the map $F^- : Y \multimap X$ is defined by $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to $D' \subset D$ if for any $A \in \langle D'\rangle$, we have $\Gamma_A \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D'\rangle})$ is called a *subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Examples. 1. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For details, see [18-20,28-30].

2. A G-convex space $(X, D; \Gamma)$ is called a *H-space* if each Γ_A is ω-connected (that is, n-connected for all $n \geq 0$) and $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D \rangle$. When $X = D$, an H-space reduces to a *C-space* due to Horvath [11,12].

3. A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde [15]. However he obtained several KKM type theorems w.r.t. $(X \supset D; \Gamma)$.

4. Let E be a topological vector space with a neighborhood system $\mathcal{V}$ of its origin. A subset X of E is said to be *almost convex* if for any $V \in \mathcal{V}$ and for any finite subset $A := \{x_1, x_2, \ldots, x_n\}$ of X, there exists a subset $B := \{y_1, y_2, \ldots, y_n\}$ of X such that $y_i - x_i \in V$ for each $i = 1, 2, \ldots, n$ and $\mathrm{co}\, B \subset X$. By choosing one of such B, let $\Gamma_A := B$ for each $A \in \langle X \rangle$. Then $(X; \Gamma)$ becomes an abstract convex space.

5. Let $\mathcal{C} := \mathcal{C}[0, 1]$ be the class of all real continuous functions on $[0, 1]$ and $\mathcal{P} := \mathcal{P}[0, 1]$ the subclass of all polynomials $p(x)$ on $x \in [0, 1]$ with real coefficients. Let $\varepsilon > 0$. For each $f \in \mathcal{C}$, choose a fixed $p_f \in \mathcal{P}$ which is ε-near to f, that is, $\max_{x \in [0,1]} |f(x) - p_f(x)| < \varepsilon$. Let $\Gamma : \langle \mathcal{C} \rangle \multimap \mathcal{P}$ be defined by $\Gamma_A := \mathrm{co}\, \{p_{f_i}\}_{i=0}^n \in \mathcal{P}$ for each $A = \{f_i\}_{i=0}^n \in \langle \mathcal{C} \rangle$. Moreover, let $\phi_A : \Delta_n \to \Gamma_A$ be a linear map such that $e_i \mapsto p_{f_i}$. Then $(X, D; \Gamma) := (\mathcal{P}, \mathcal{C}; \Gamma)$ is a G-convex space satisfying $X \subsetneq D$.

6. Similarly, by choosing a proper subset D of $\mathcal{C}$, we can obtain G-convex spaces $(X, D; \Gamma)$ satisfying $X \not\subset D$ or $X \not\supset D$. This is why we assumed X and D are not comparable in general.

7. Since there are various forms of the Stone-Weierstrass approximation theorem, we can construct a large number of examples similar to the ones in 5 or 6.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp., a $\mathfrak{KO}$*-map*] [21-26] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{KC}(X, Z) := \{F : X \multimap Z \mid F \text{ is a } \mathfrak{KC}\text{-map}\}.$$

Similarly, $\mathfrak{KO}(X, Z)$ is defined. Some authors use the notation KKM(X, Z) instead of $\mathfrak{KC}(X, Z)$.

Examples. 1. If $1_E \in \mathfrak{KC}(E, E)$, then $f \in \mathfrak{KC}(E, Z)$ for any continuous function $f : E \to Z$. This also holds for $\mathfrak{KO}$.

2. For a G-convex space $(X, D; \Gamma)$ and a topological space Z, we defined the classes $\mathfrak{KC}$ and $\mathfrak{KO}$ of multimaps $F : X \multimap Z$ [20]. It is well-known that for a G-convex space $(X, D; \Gamma)$, we have the identity map $1_X \in \mathfrak{KC}(X, X) \cap \mathfrak{KO}(X, X)$; see [18,19,24]. Moreover, for any topological space Y, if $F : X \to Y$ is a continuous single-valued map or if $F : X \multimap Y$ has a continuous selection, then $F \in \mathfrak{KC}(X, Y) \cap \mathfrak{KO}(X, Y)$.

3. There are many examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps; see [22-24].

3 Basic theorems in the KKM theory

In our KKM theory on abstract convex spaces given in [21,26], there exist some basic theorems from which we can deduce several equivalent formulations that can be used for applications. In this section, we introduce some of such basic theorems.

The following is a prototype of KKM type theorems:

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, Y a topological space, and $F \in \mathfrak{KO}(E, Y)$ [resp., $F \in \mathfrak{KC}(E, Y)$]. Let $G : D \multimap Y$ be a map such that*

(A.1) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$; and*
(A.2) *G is open-valued [resp., closed-valued].*

Then for each $N \in \langle D \rangle$, $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$.

Remarks. 1. If $E = Y$ and $F = 1_E$, then Condition (A.1) says that G is a KKM map.

2. If $E = Y = \Delta_n$ is an n-simplex, D is the set of its vertices, and $\Gamma = \mathrm{co}$ is the convex hull operation, then the celebrated KKM theorem [14] says that $1_E \in \mathfrak{KC}(E, E)$.

3. If D is a nonempty subset of a topological vector space $E = Y$ (not necessarily Hausdorff), Fan's KKM lemma [10] says that $1_E \in \mathfrak{KC}(E, E)$.

4. For another forms of the KKM theorem for convex spaces, C-spaces, or G-convex spaces and their applications, there are a large number of works; see [18-20,26-30] and references therein.

From Theorem A, we have another finite intersection property as follows:

Theorem B. *Let $(E, D; \Gamma)$ be an abstract convex space, Y a topological space, and $F \in \mathfrak{KO}(E, Y)$ [resp., $F \in \mathfrak{KC}(E, Y)$]. Let $G : D \multimap Y$ and $H : E \multimap Y$ be maps satisfying*

(B.1) *G is open-valued [resp., closed-valued];*
(B.2) *for each $x \in E$, $F(x) \subset H(x)$; and*
(B.3) *for each $y \in F(E)$, $M \in \langle D \setminus G^-(z) \rangle$ implies $\Gamma_M \subset E \setminus H^-(z)$.*

Then $F(E) \cap \bigcap\{G(z) \mid z \in N\} \neq \emptyset$ for all $N \in \langle D \rangle$.

The following coincidence theorem follows from Theorem B.

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Y a topological space, $S : D \multimap Y$, $T : E \multimap Y$ maps, and $F \in \mathfrak{KC}(E, Y)$ [resp., $F \in \mathfrak{KO}(E, Y)$]. Suppose that*

(C.1) *S is open-valued [resp., closed-valued];*
(C.2) *for each $y \in F(E)$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*
(C.3) *$F(E) \subset S(N)$ for some $N \in \langle D \rangle$.*

Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

From Theorem C, we obtain the following Fan type matching theorem:

Theorem D. *Let $(E, D; \Gamma)$ be an abstract convex space, Y a topological space, $S : D \multimap Y$, and $F \in \mathfrak{KC}(E, Y)$ [resp., $F \in \mathfrak{KO}(E, Y)$] satisfying* (C.1) *and* (C.3). *Then there exists an $M \in \langle D \rangle$ such that $F(\Gamma_M) \cap \bigcap\{S(x) \mid x \in M\} \neq \emptyset$.*

Theorem D can be stated in its contrapositive form and in terms of the complement $G(z)$ of $S(z)$ in Y. Then we obtain Theorem A. Therefore, Theorems A–D are equivalent.

4 ϕ_A-spaces

Recently, we are concerned with reformulations of G-convex spaces as follows [25]:

Definition. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Note that, by putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$, a ϕ_A-space becomes an abstract convex space.

Examples. 1. [1] An L-space is a G-convex space $(X; \Gamma)$.

2. [13] A topological space Y is said to have property (H) if, for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$.

3. [6,7] $(Y, \{\varphi_N\})$ is called an FC-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$.

4. We can consider a ψ_A-space $(X, D; \{\psi_A\}_{A \in \langle D \rangle})$, similar to a ϕ_A-space, where $\psi_A : [0, 1]^n \to X$ is continuous for each $A \in \langle D \rangle$ with $|A| = n + 1$. Such types of spaces are given by Michael [17], Llinares [16], and Cain and Gonzáles [2]. For each $n \geq 0$, considering continuous functions $g_n : \Delta_n \to [0, 1]^n$ given by

$$g_n : u = \sum_{i=0}^{n} \lambda_i(u) e_i \mapsto (\lambda_0(u), \ldots, \lambda_{n-1}(u))$$

for $u \in \Delta_n$ and by putting $\phi_A := \psi_A g_n$, a ψ_A-space becomes a ϕ_A-space.

5. Any G-convex space is a ϕ_A-space. The converse also holds:

Proposition 1. *A ϕ_A-space $(X, D; \{\phi_A\})$ can be made into a G-convex space $(X, D; \Gamma)$.*

Proof. This can be done at least in three ways.

(1) For each $A \in \langle D \rangle$, by putting $\Gamma_A := X$, we obtain a trivial G-convex space $(X, D; \Gamma)$.

(2) Let $\{\Gamma^\alpha\}_\alpha$ be the family of maps $\Gamma^\alpha : \langle D \rangle \multimap X$ giving a G-convex space $(X, D; \Gamma^\alpha)$. Note that, by (1), this family is not empty. Then, for each α and each $A \in \langle D \rangle$ with $|A| = n + 1$, we have

$$\phi_A(\Delta_n) \subset \Gamma_A^\alpha \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma_J^\alpha \quad \text{for} \quad J \subset A.$$

Let $\Gamma := \bigcap_\alpha \Gamma^\alpha$, that is, $\Gamma_A = \bigcap_\alpha \Gamma_A^\alpha$ for each $A \in \langle D \rangle$. Then

$$\phi_A(\Delta_n) \subset \Gamma_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma_J \quad \text{for} \quad J \subset A.$$

Therefore, $(X, D; \Gamma)$ is a G-convex space.

(3) Let $N \in \langle D \rangle$ with $|N| = n + 1$. For each $M \in \langle D \rangle$ with $N \subset M$, $M = \{a_0, \ldots, a_m\}$ and $N = \{a_{i_0}, \ldots, a_{i_n}\}$, there exists a subset $\phi_M(\Delta_n^M)$ of X such that $\Delta_n^M := \mathrm{co}\{e_{i_j} \mid j = 0, \ldots, n\} \subset \Delta_m$. Now let

$$\Gamma_N = \Gamma(N) := \bigcup_{M \supset N} \phi_M(\Delta_n^M).$$

Then $\Gamma : \langle D \rangle \multimap X$ is well-defined and $(X, D; \Gamma)$ becomes a G-convex space: For each $A \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous map $\phi_A : \Delta_n \multimap \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. □

Therefore, G-convex spaces and ϕ_A-spaces are essentially the same and so are FC-spaces and L-spaces.

Proposition 2. *For a ϕ_A-space $(X, D; \{\phi_A\})$, any map $T : D \multimap X$ satisfying*

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is a KKM map on a G-convex space $(X, D; \Gamma)$.

Proof. Define $\Gamma : \langle D \rangle \multimap X$ by $\Gamma_A := T(A)$ for each $A \in \langle D \rangle$. Then $(X, D; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n + 1$, we have a continuous function $\phi_A : \Delta_n \to T(A) =: \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle D \rangle$ and hence $T : D \multimap X$ is a KKM map on a G-convex space $(X, D; \Gamma)$. □

Recall that Ding [7] wrote: "It is easy to see that the class of FC-spaces includes the classes of convex sets in topological vector spaces, C-spaces (or H-spaces) [11], G-convex spaces [28], L-convex spaces [1], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces." Here he failed to give any justification or any proper example of his space which is not G-convex. One wonders how could a pair $(Y, \{\varphi_N\})$ generalize a triple $(X, D; \Gamma)$ in [28,29].

Ding's claim on his FC-spaces has appeared repeatedly in [6,8,9] and many more of his papers. More than one dozen of such papers on FC-spaces have appeared within two years! In these papers, known results in KKM theory on G-convex spaces are restated or modified for the so-called FC-spaces. In order to prevent such unnecessary efforts, something has to be done. This is why we introduced the concept of ϕ_A-spaces.

5 Chen's R-KKM theorems on L-convex spaces

Recently, there have appeared authors of [3-9,13] and others who attempted to reformulate resultss in the G-convex space theory by replacing $\Gamma(A)$ by $\phi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples. Such attempts are all failed since G-convex spaces and ϕ_A-spaces are essentially the same.

In 2003, the following appeared:

Definition. [4, Definition 2.1] Let X be a nonempty set and Y be a topological space. $T : X \to 2^Y$ is said to be generalized relatively KKM (R-KKM) mapping if for any $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, there exists a continuous mapping $\phi_N : \Delta_n \to Y$ such that, for each $e_{i_0}, e_{i_1}, \ldots, e_{i_k}$,

$$\phi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

where Δ_k is a standard k-simplex of Δ_n with vertices $e_{i_0}, e_{i_1}, \ldots, e_{i_k}$.

Its authors claimed that their definition unifies and extends a lot of similar definitions due to other authors.

In our previous work [25], we mentioned that the key result in [4] is a simple consequence of known results. Moreover, in [4], its authors claimed also that, applying their key result, they obtained new theorems which unify and extend many known results in recent literature. However, theirs are all disguised forms of known results and their applicability is doubtful.

In 2005 [13], its author also defined a generalized R-KKM mapping on a space having property (H) and obtained modifications of some known results in the G-convex space theory in which we supplied a large number of examples of such spaces. It is noteworthy that the authors of [8,9,13] adopted R-KKM maps and claimed to obtain generalizations of known results without giving any justifications or any proper examples.

Note that, in [4], by choosing $\Gamma_N := \phi_N(\Delta_n)$ for each $N \in \langle X \rangle$, $(Y, X; \Gamma)$ becomes a ϕ_A-space and the generalized relatively KKM (R-KKM) mapping becomes simply a KKM map.

Contrary to this fact, Ding in [8] claimed as follows: "The above class of generalized *R-KKM* mappings include those classes of *KKM* mappings, *H-KKM* mappings, *G-KKM* mappings, generalized *G-KKM* mappings, generalized *S-KKM* mappings, *GLKKM* mappings and *GMKKM* mappings defined in topological vector spaces, H-spaces, G-convex spaces, *G-H*-spaces, L-convex spaces and hyperconvex metric spaces, respectively, as true subclasses."

Therefore, all of the KKM type theorems on such variants are simple consequences of our G-convex space theory. Consequently, all results in [8] are artificial disguised forms of known ones having no proper examples.

Moreover, in a recent paper [3], its author introduced a generalized *R-KKM* mapping and a new class R_{ψ_N}-$KKM(X,Y)$, and claimed some fixed point theorems, matching theorems, coincidence theorems, and minimax inequalities on the so-called L-convex spaces.

Since 1998, the present author has adopted the concept of a generalized convex (simply, G-convex) space $(X, D; \Gamma)$, where D is a nonempty set. When D is a subset of X, this space is called an L-convex space in [3], where it is incorrectly stated that L-convex spaces are due to Ben-El-Mechaiekh. Recall that particular forms of G-convex spaces $(X;\Gamma) = (X, X;\Gamma)$ was called L-spaces by Ben-El-Mechaiekh et al. [1].

In [3], the concept of G-convex spaces is used only in the definition of a generalized *R-KKM* mapping and all of other usages of L-convex spaces in [3] are simply for L-spaces in the sense of Ben-El-Mechaiekh et al. [1]. Note that generalized *R-KKM* mappings in [3] are different from the one of other authors mentioned above.

This usage with the incorrect application of the KKM principle make failure to justify [3, Theorems 1-3, Lemma 16, Corollary 17, Theorem 18, and Corollary 19], whose correct forms are already well-known in the G-convex space theory.

Moreover, the following definition is given:

Definition. [3] Let X be an L-convex space, Y a topological space such that for each $N \in \langle X \rangle$ with $|N| = n+1$, there exists a continuous mapping $\psi_N : \Delta_N \to X$. If $T, F : X \to 2^Y$ are two set-valued function satisfying that $T(\psi_N(\Delta_N)) \subset F(N)$ for each $N \in \langle X \rangle$ with $|N| = n+1$, then F is said to be a generalized R_{ψ_N}-KKM mapping with respect to T and ψ_N. Moreover, if the set-valued function $T : X \to 2^Y$ satisfies the requirement that for any generalized R_{ψ_N}-KKM mapping with respect to T and ψ_N the family $\{\overline{Fx} \mid x \in X\}$ has the finite intersection property, then T is said to have the R_{ψ_N}-KKM property. The class R_{ψ_N}-$KKM(X,Y)$ is defined to be the set $\{T : X \to 2^Y \mid T$ has the R_{ψ_N}-KKM property$\}$. (* This ψ_N may be different from the ϕ_N of the definition for the L-convex space.)

In the above definition of a class R_{ψ_N}-$KKM(X,Y)$, the role of G-convex spaces (its author's L-convex spaces) is not clear. This remark also works all of Theorems 4–20 in [3]. Moreover, Theorems 21-23 in [3] are concerned with G-convex spaces and follow easily from the known results in the G-convex space theory.

The above definition can be improved as follows:

Definition. For a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

with $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$, $|A| = n+1$, and a topological space Z, a closed-valued [resp., open-valued] multimap $G : D \multimap Z$ is called a *KKM map* with respect to a nonempty-valued multimap $F : E \multimap Z$ if

$$F(\phi_A(\Delta_n)) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle.$$

A *KKM map* $G : D \multimap X$ is a KKM map with respect to the identity map 1_X.

By putting $\Gamma_A := \phi_A(\Delta_n)$, the above definition reduces to the one for abstract convex spaces.

In [3], its author's aim seems to generalize the well-known results in the KKM theory in his own setting, but he failed to give any proper examples or any suggested applications.

6 Basic theorems in the KKM theory on ϕ_A-spaces

In this section, we show correct forms of some theorems in [3] with simple proofs. We already mentioned about [3, Theorems 1-3] in the preceding section.

In Theorem C with $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$, $|A| = n+1$, we have the following:

Theorem 1. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ and a topological space Y, let $S : D \multimap Y$ be an open-valued [resp., a closed-valued] map, $T : E \multimap Y$ a map, and $F \in \mathfrak{KC}(X, Y)$[resp., $F \in \mathfrak{KO}(X, Y)$]. Suppose that*

(1) *for each $y \in F(X)$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$[that is, $N \in \langle S^-(y)\rangle$, $|N| = n+1$, implies $\phi_N(\Delta_n) \subset T^-(y)$]; and*

(2) *$F(E) \subset S(N)$ for some $N \in \langle D \rangle$.*

Then there exists an $\bar{x} \in X$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Note that from Theorem 1, we can deduce improved versions of [3, Theorems 4-8] with more simple proofs.

In Theorem D with $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$, $|A| = n+1$, we have the following:

Theorem 2. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ and a topological space Y, let $S : D \multimap Y$ be an open-valued [resp., a closed-valued] map and $T : X \multimap Y$ a map. Suppose that*

(1) *$F \in \mathfrak{KC}(X, Y)$[resp., $F \in \mathfrak{KO}(X, Y)$]; and*

(2) *$F(E) \subset S(N)$ for some $N \in \langle D \rangle$.*

Then there exists an $M \in \langle D \rangle$ with $|M| = m+1$ such that $F(\phi_M(\Delta_m)) \cap \bigcap\{S(x) \mid x \in M\} \neq \emptyset$.

Note that Theorem 2 improves [3, Theorems 11, 12, 14 and 15] with more simple proofs.

In Theorem A with $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$, $|A| = n+1$, we have the following:

Theorem 3. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ and a topological space Y, let $S : D \multimap Y$ be an open-valued [resp., a closed-valued] map. Suppose that*

(1) *$F \in \mathfrak{KO}(X, Y)$[resp., $F \in \mathfrak{KC}(X, Y)$]; and*

(2) *for any $N \in \langle D \rangle$ with $|N| = n+1$, $F(\phi_N(\Delta_n)) \subset G(N)$.*

Then for each $N \in \langle D \rangle$, $F(X) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$.

From Theorem 3, if $\overline{F(X)}$ is compact, then $\overline{F(X)} \cap \bigcap\{\overline{G(y)} \mid y \in N\} \neq \emptyset$. This implies [3, Theorems 13 and 14] as very special cases.

Finally, note that, in [3, Theorems 21-23], its author assumed a very particular form of a nonempty L-convex space X. Instead of such X, he could assume a ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ such that for each $J \subset A \in \langle D \rangle$ with $|A| = n+1, |J| = r+1$, we have $\phi_A(\Delta_J) = \phi_J(\Delta_r)$. In fact, all of [3, Theorems 21-23] are variants of known results. All of other results in [3] can be similarly generalized and improved.

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