J Appl Math Comput (2008) 26: 555–564
DOI 10.1007/s12190-007-0022-3

JAMC

Compact Browder maps and equilibria of abstract economies

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Received: 11 December 2006 / Published online: 5 January 2008

Abstract We obtain a partial resolution of a conjecture raised by Ben-El-Mechaiekh; that is, for a convex subset X of a Hausdorff t.v.s., any compact Browder map $T : X \multimap X$ (a multimap with nonempty convex values and open fibers) has a fixed point. From this new result, we deduce a collectively fixed point theorem with applications to existences of equilibrium points and maximal elements of an abstract economy. Consequently, some known results are extended.

Keywords Schauder conjecture · Klee approximable subset · Φ-map · (Collectively) Fixed point theorem · Abstract economy · Equilibrium point · Maximal element · Qualitative game

Mathematics Subject Classification (2000) Primary 47H10 · 49A29 · 90D06

1 Introduction

In 1990, Ben-El-Mechaiekh [3, 4] raised a conjecture that, for a convex subset X of a Hausdorff topological vector space, any compact Browder multimap $T : X \multimap X$ (with nonempty convex values and open fibers) would have a fixed point. The present author [20], in 1999, found that the affirmativity of the Schauder conjecture implies that of the Ben-El-Mechaiekh conjecture and obtained a number of partial resolutions of the latter.

In this paper, from one of our recent fixed point theorems, we deduce a new partial resolution of the Ben-El-Mechaiekh conjecture and some of its applications to

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existences of equilibrium points and maximal elements of an abstract economy. Consequently, we obtain generalizations of some known results.

In Sect. 2, some historical background of our theme in this paper is given and one of our recent fixed point theorems is introduced. Section 3 deals with a fixed point theorem which is a partial resolution of the Ben-El-Mechaiekh conjecture. From the main result we obtain a general collectively fixed point theorem and an equivalent intersection theorem. In Sect. 4, our main result is applied to existences of equilibrium points and maximal elements of an abstract economy.

2 Preliminaries

A t.v.s. means a topological vector space.

The following is the well-known Schauder conjecture raised in 1935; see *The Scottish Book* [16], Problem 54.

Conjecture 1 (Schauder [16]) *Every nonempty compact convex subset X of a (metrizable) t.v.s. E has the fixed point property; that is, every continuous map $f : X \to X$ has a point $x_0 \in X$ such that $x_0 = f(x_0)$.*

This conjecture was motivated by the Brouwer fixed point theorem in 1912 for Euclidean spaces and Schauder's fixed point theorems in 1930 for Banach spaces. Earlier partial resolutions of Conjecture 1 were due to Tychonoff in 1935 for locally convex Hausdorff t.v.s. E and to Fan in 1964 for a t.v.s. E on which its topological dual E^* separates points. For the history of partial resolutions of Conjecture 1, see [10, 12, 16, 20, 21].

A map or a multimap is said to be *compact* if its range is contained in a compact subset of its codomain. The following was also raised; see [17].

Conjecture 2 *For every nonempty convex subset X of a Hausdorff t.v.s. E, a compact continuous map $f : X \to X$ has a fixed point.*

The following are examples of the affirmative partial cases of Conjecture 2:

(1) (Schauder) E is a normed vector space.
(2) (Hukuhara) E is locally convex.
(3) (Klee) X is admissible (in the sense of Klee).
(4) (Idzik) $\overline{f(X)}$ is convexly totally bounded (simply, c.t.b.).
(5) (Nhu and Arandelović) X is compact and weakly admissible.

For the literature, see [10, 12, 21, 24]. In 2001, Cauty [6] claimed to resolve Conjecture 2 affirmatively, but later, it was known that his proof had a gap.

Let us say that a subset X of a t.v.s. has the *compact convex fixed point property* (c.c.f.p.p.) if for any nonempty convex subset C of X, a compact continuous map $f : C \to C$ has a fixed point. Note that the above (1)–(5) give examples of sets having the c.c.f.p.p.

A *polytope* P in a subset X of a t.v.s. E is a homeomorphic image of a standard simplex. A nonempty subset K of E is said to be *Klee approximable* if for any neighborhood V of 0 in E, there exists a continuous map $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope of E. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

We give some examples of Klee approximable sets as in [24, 25]:

(1) A subset X of E is admissible (in the sense of Klee) iff every compact subset K of X is Klee approximable into E.
(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.
(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.
(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.
(5) Any compact subset K of an admissible convex subset X of a t.v.s. is Klee approximable into X.
(6) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.

Note that (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3).

The following recent result contains a large number of known theorems:

Theorem 0 (Park [23]) *Let X be a subset of a Hausdorff t.v.s. E and $f : X \to X$ a compact continuous map. If $f(X)$ is Klee approximable into X, then f has a fixed point.*

Recall that a *Browder map* is a multimap with nonempty convex values and open fibers.

In 1968, Browder obtained the following:

Theorem 1 (Browder [5]) *If X is a compact convex set in a t.v.s. E and $T : X \multimap X$ is a Browder map, then T has a fixed point $x_0 \in X$, that is, $x_0 \in T(x_0)$.*

Browder proved this by applying the partition of unity argument subordinated to a finite open cover and the Brouwer fixed point theorem. Actually, he had to assume the Hausdorffness of E (which is redundant) and obtained a continuous selection $f : X \to X$ of T. The selection method is generalized as follows:

Lemma *Let X be a normal topological space, Y a convex subset of a t.v.s. E, and $S : X \multimap Y$ a multimap such that*

$$X = \bigcup_{i=1}^{n+1} \operatorname{Int} S^-(y_i) \quad \textit{for some } N = \{y_1, y_2, \ldots, y_{n+1}\} \subset Y.$$

Then there exists a continuous map $s : X \to Y$ such that $s(x) \in \operatorname{co} S(x)$ for all $x \in X$ and $s = \phi_N \circ p$, where $p : X \to \Delta_n$ and $\phi_N : \Delta_n \to \operatorname{co} N$ are continuous maps and Δ_n is an n-simplex.

Recall that $S^{-}(y) := \{x \in X \mid y \in S(x)\}$ is the fiber of the multimap $S : X \multimap Y$ at $y \in Y$.

Lemma can be proved by following the proof of [19, Lemma 1] and by the fact that, for each locally finite open cover of a normal space, there is a partition of unity subordinated to it. (In this paper, a normal space or a paracompact space is always assumed to be Hausdorff.) Note also that Lemma works for a generalized convex space $(Y, D; \Gamma)$ instead of Y in Lemma; see [19].

Note that, in Lemma, if X is a subset of E containing $\mathrm{co}\, N$, then $s|_{\mathrm{co}\, N} : \mathrm{co}\, N \to \mathrm{co}\, N$ has a fixed point by the Brouwer fixed point theorem. This is the essential nature of the Browder Theorem 1.

For a topological space X and a subset Y of a t.v.s., a multimap $T : X \multimap Y$ is called a *Φ-map* or a *Fan-Browder map* provided that there exists a multimap $S : X \multimap Y$ such that

(1) for each $x \in X$, $\mathrm{co}\, S(x) \subset T(x)$; and
(2) $X = \bigcup\{\mathrm{Int}\, S^{-}(y) \mid y \in Y\}$.

Observe that a Browder map is a Φ-map.

For topological spaces X and Y, a multimap $T : X \multimap Y$ is said to be *locally (continuously) selectionable* if for each $x_0 \in X$, there exists an open neighborhood V_0 of x_0 and a continuous map $f : V_0 \to Y$ such that $f(x) \in T(x)$ for all $x \in V_0$; see [19].

The following is given [19, Theorems 4 and 5]:

Theorem 2 (Park [19]) *Let X be a paracompact topological space and Y a convex subset of a t.v.s. E. Then*

(1) *any Φ-map $T : X \multimap Y$ is locally selectionable*; *and*
(2) *any locally selectionable multimap $A : X \multimap Y$ having nonempty convex values has a continuous selection $s : X \to Y$.*

Corollary (Ben-El-Mechaiekh et al. [1, 2]) *Let X be a paracompact topological space*, *Y a convex subset of a t.v.s. E*, *and $A : X \multimap Y$ a Browder map. Then there exists a continuous selection $s : X \to Y$ of A.*

For the Browder Theorem 1, it is natural to ask whether the theorem still holds under the compactness of the map T instead of the compactness of the domain X. In fact, in 1990, Ben-El-Mechaiekh raised the following:

Conjecture 3 (Ben-El-Mechaiekh [3, 4]) *For a nonempty convex subset X of a t.v.s. E*, *a compact Browder map $T : X \multimap X$ has a fixed point.*

Of course, if X itself is compact, then Conjecture 3 reduces to the Browder Theorem 1. Hence, we assume that T is not surjective in Conjecture 3.

Some partial resolutions of Conjecture 3 were known by Ben-El-Mechaiekh et al. [2–4] and Park [20]. Moreover, we noticed in [20] that, in a sense, Conjecture 2 implies Conjecture 3 as follows:

Theorem 3 (Park [20]) *Let X be a nonempty convex subset of a Hausdorff t.v.s. E having the c.c.f.p.p. and $T : X \multimap X$ a Browder map. If T is compact, then T has a fixed point.*

3 Fixed point theorems

In this section, we obtain a fixed point theorem and a collectively fixed point theorem.

Motivated by Theorem 3, we obtain the following:

Theorem 4 *Let X be a convex subset of a Hausdorff t.v.s. E and $T : X \multimap X$ a locally selectionable compact multimap with convex values. If X has the c.c.f.p.p. or $T(X)$ is Klee approximable into* $\mathrm{co}\,\overline{T(X)}$, *then T has a fixed point.*

Proof We show that the convex hull C of the compact set $K := \overline{T(X)}$ is a regular σ-compact space (For details, see Lassonde [15] and Fournier-Granas [9]), and hence it is Lindelöf and so paracompact. Indeed, C can be expressed as the countable union $\bigcup_{n=1}^{\infty} K_n$ of compact sets $K_n = \Psi_n(\Delta_n \times K^{n+1})$, where $\Delta_n = \mathrm{co}\{e_i\}_{i=1}^{n+1}$ is a standard n-simplex, $K^{n+1} = K \times K \times \cdots \times K$ the $(n+1)$-fold of K, and

$$\Psi_n : \Delta_n \times K^{n+1} \to C \quad \text{defined by } \Psi_n\left(\sum_{i=1}^{n+1} \alpha_i e_i,\ (x_i)_{1 \le i \le n+1}\right) = \sum_{i=1}^{n+1} \alpha_i x_i$$

is a continuous map. Note that the convex set X contains the paracompact set C. Now by Theorem 2(2), the restriction $T|_C : C \multimap \overline{T(X)} \subset C$ has a compact continuous selection $s : C \to C$. Since X has the c.c.f.p.p. or $s(C)$ is Klee approximable into C, by Theorem 0, there exists a point $x_0 \in C \subset X$ such that $x_0 = s(x_0) \in T(x_0)$. This completes our proof. □

The following is a particular resolution of the Ben-El-Mechaiekh conjecture:

Corollary 4.1 *Let X be a convex subset of a Hausdorff t.v.s. E and $T : X \multimap X$ be a compact Φ-map. If X has the c.c.f.p.p. or $T(X)$ is Klee approximable into* $\mathrm{co}\,\overline{T(X)}$, *then T has a fixed point.*

Note that Corollary 4.1 generalizes Yannelis and Prabhakar [29, Theorem 3.2].

From Corollary 4.1, we easily obtain the following existence result on maximal elements:

Corollary 4.2 *Let X be a convex subset of a Hausdorff t.v.s. E and $P : X \multimap X$ be a compact multimap such that* (1) *$P(x)$ is convex (possibly empty) and $x \notin P(x)$ for each $x \in X$; and* (2) *$P^-(y)$ is open for each $y \in X$. If X has the c.c.f.p.p. or $P(X)$ is Klee approximable into* $\mathrm{co}\,\overline{P(X)}$, *then there exists an $\overline{x} \in X$ such that $P(\overline{x}) = \emptyset$.*

It is clear from Theorem 1 that if X itself is compact, then the Hausdorffness of E is redundant. Note that Corollary 4.2 generalizes Yannelis and Prabhakar [29, Theorem 5.3].

Let $\{X_i\}_{i\in I}$ be a family of sets. For a given $i \in I$, let

$$X := \prod_{j\in I} X_j \quad \text{and} \quad X^i := \prod_{j\in I\setminus\{i\}} X_j.$$

For $x^i \in X^i$ and $j \in I\setminus\{i\}$, let x^i_j denote the jth coordinate of x^i. For $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its ith coordinate is x_i and, for $j \neq i$, the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where $x^i = \pi^i(x)$ and $x_i = \pi_i(x)$ denote the projection of x in X^i and X_i, respectively.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A\} \quad \text{and} \quad A(x_i) := \{y^i \in X^i \mid [y^i, x_i] \in A\}.$$

For a family $\{E_i\}_{i\in I}$ of t.v.s., let $E := \prod_{i\in I} E_i$. Similarly, $X := \prod_{i\in I} X_i$ and $K := \prod_{i\in I} K_i$ for subsets X_i and K_i of E_i for $i \in I$.

From Theorems 0 and 2, we deduce the following collectively fixed point theorem:

Theorem 5 *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a Hausdorff t.v.s. E_i, and K_i a nonempty compact subset of X_i. For each $i \in I$, let $S_i, T_i : X \multimap K_i$ be multimaps satisfying*

(1) *for each $x \in X$, $\operatorname{co} S_i(x) \subset T_i(x)$; and*
(2) $D := \operatorname{co} K \subset \bigcup_{y\in K_i} \operatorname{Int}_X S_i^-(y)$.

If X has the c.c.f.p.p. or K is Klee approximable into D, then there exists an $\overline{x} \in K$ such that $\overline{x}_i \in T_i(\overline{x})$, where $\overline{x}_i$ is the projection of $\overline{x}$ in X_i, for each $i \in I$.

Proof Since $K = \prod_{i\in I} K_i$ is compact in E, $D = \operatorname{co} K$ is σ-compact and hence Lindelöf. Since D is regular as a subset of a Hausdorff t.v.s. E, we know that D is paracompact as in the proof of Theorem 4. Consider $S_i|_D : D \multimap K_i$. Note that $D = \bigcup_{y\in K_i} (\operatorname{Int} S_i^-(y)) \cap D$ by (2) and $\operatorname{Int}_D (S_i|_D)^-(y) = (\operatorname{Int} S_i^-(y)) \cap D$ for $y \in K_i$. Therefore, by Theorem 2(2), $(\operatorname{co} S_i)|_D : D \multimap X_i$ has a continuous selection $s_i : D \multimap K_i$ such that $s_i(x) \in \operatorname{co} S_i(x) \subset T_i(x)$ for each $x \in D$. Define $s : D \to K$ by $(s(x))_i = s_i(x)$ for each $i \in I$ and $x \in D$. Then s is continuous. Since X has the c.c.f.p.p. or by Theorem 0, s has a fixed point $\overline{x} \in K$; that is, $\overline{x} \in s(\overline{x})$ and $\overline{x}_i = (s(\overline{x}))_i = s_i(\overline{x}) \in T_i(\overline{x})$. This completes our proof. □

Remark If each E_i is locally convex, then Theorem 5 reduces to Yannelis and Prabhakar [29, Theorem 3.2], Ding et al. [8, Theorem 2], Husain and Tarafdar [12, Theorem 2.2], and Wu and Shen [28, Theorem 2]. Note that if I is a singleton, Theorem 5 reduces to Corollary 4.1.

From Theorem 5, we have the following equivalent form:

Theorem 6 *Let $\{X_i\}_{i\in I}$ and $\{K_i\}_{i\in I}$ be the same as in Theorem 5. For each $i \in I$, A_i and B_i are subsets of X satisfying*

(1) *for each* $x^i \in X^i$, $\emptyset \neq \operatorname{co} A_i(x^i) \subset B_i(x^i) \subset K_i$; *and*
(2) *for each* $y \in K_i$, $A_i(y)$ *is open in* X^i.

If X *has the c.c.f.p.p. or* K *is Klee approximable into* $D := \operatorname{co} K$, *then we have* $\bigcap_{i \in I} B_i \neq \emptyset$.

Proof We apply Theorem 5 with S_i, $T_i : X \multimap K_i$ given by $S_i(x) := A_i(x^i)$ and $T_i(x) := B_i(x^i)$ for each $x \in X$. Then for each $i \in I$ we have the following:

(a) For each $x \in X$, we have $\operatorname{co} S_i(x) \subset T_i(x) \subset K_i$.
(b) For each $y \in K_i$, we have

$$x \in S_i^-(y) \Longleftrightarrow y \in S_i(x) \Longleftrightarrow (x, y) \in \operatorname{Gr}(S_i) \subset X \times K_i$$

and, on the other hand,

$$x \in S_i^-(y) \Longleftrightarrow y \in S_i(x) = A_i(x^i) \Longleftrightarrow (x^i, y) \in A_i.$$

Hence,

$$S_i^-(y) = \{x = [x^i, x_i] \in X \mid x^i \in A_i(y), x_i \in X_i\} = A_i(y) \times X_i.$$

Note that $S_i^-(y)$ is open in $X = X^i \times X_i$.

Therefore, by Theorem 5, there exists an $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x}) := \prod_{i \in I} T_i(\hat{x})$; that is, $\hat{x}_i \in T_i(\hat{x}) = B_i(\hat{x}^i)$ for all $i \in I$. Hence, $\hat{x} = [\hat{x}^i, \hat{x}_i] \in \bigcap_{i \in I} B_i \neq \emptyset$. This completes our proof. □

4 Equilibrium points and maximal elements

We apply Theorem 6 to existences of equilibrium points and maximal elements of an abstract economy.

An *abstract economy* $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ consists of an index set I of agents, a choice set X_i in a t.v.s., constraint correspondences $A_i, B_i : X = \prod_{i \in I} X_i \multimap X_i$, and a preference correspondence $P_i : X \multimap X_i$ for each $i \in I$. An *equilibrium point* $x = (x_i)_{i \in I} \in X$ is the one satisfying $x_i \in B_i(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for each $i \in I$. We say that $x \in X$ is a *maximal point* of the game $(X_i, P_i)_{i \in I}$ if $P_i(x) = \emptyset$ for each $i \in I$.

Theorem 7 *Let* $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ *be an abstract economy such that, for each* $i \in I$,

(1) X_i *is a convex subset of a Hausdorff t.v.s.* E_i, *and* K_i *is a nonempty compact subset of* X_i;
(2) *for each* $x \in X$, $\operatorname{co} A_i(x) \subset B_i(x) \subset K_i$;
(3) $D := \operatorname{co} K \subset \bigcup_{y \in K_i} \operatorname{Int}(A_i^-(y) \cap (P_i^-(y) \cup F_i))$, *where* $F_i := \{x \in X \mid A_i(x) \cap P_i(x) = \emptyset\}$; *and*
(4) *for each* $x = (x_i)_{i \in I} \in X$, $x_i \notin \operatorname{co} P_i(x)$.

If X has the c.c.f.p.p. or K is Klee approximable into D, then Γ has an equilibrium point in K.

Proof Let

$$G_i := \{x \in X \mid A_i(x) \cap P_i(x) \neq \emptyset\} \quad \text{for each } i \in I.$$

For each $i \in I$, we define two multimaps $S_i, T_i : X \multimap K_i$ by

$$S_i(x) = \begin{cases} A_i(x) \cap \operatorname{co} P_i(x) & \text{if } x \in G_i, \\ A_i(x) & \text{if } x \in F_i, \end{cases}$$

$$T_i(x) = \begin{cases} B_i(x) \cap \operatorname{co} P_i(x) & \text{if } x \in G_i, \\ B_i(x) & \text{if } x \in F_i. \end{cases}$$

Then for each $i \in I$ and $x \in X$, we have $\operatorname{co} S_i(x) \subset T_i(x)$; and for each $y \in K_i$, we have

$$\begin{aligned} S_i^-(y) &= [(A_i^-(y) \cap (\operatorname{co} P_i)^-(y)) \cap G_i] \cup [A_i^-(y) \cap F_i] \\ &\supset [(A_i^-(y) \cap P_i^-(y)) \cap G_i] \cup [A_i^-(y) \cap F_i] \\ &= [A_i^-(y) \cap P_i^-(y)] \cup [A_i^-(y) \cap F_i] \\ &= A_i^-(y) \cap (P_i^-(y) \cup F_i), \end{aligned}$$

which implies $D = \operatorname{co} K \subset \bigcup_{y \in K_i} \operatorname{Int} S_i^-(y)$ by (3). Therefore, all of the requirements of Theorem 5 are satisfied. Hence, there exists an $\overline{x} \in K$ such that $\overline{x}_i \in T_i(\overline{x})$ for each $i \in I$. By (4), $\overline{x}_i \notin \operatorname{co} P_i(\overline{x})$. Therefore, $\overline{x}_i \in B_i(\overline{x})$ for each $i \in I$ by the definition of T_i and hence $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$. This shows that $\overline{x}$ is an equilibrium point of Γ. □

Remark We followed the proof of Tarafdar [26, Theorem 3.1]. If all of E_i's are locally convex, then Theorem 7 reduces to Ding et al. [8, Theorems 4 and 5] and Husain and Tarafdar [11, Theorem 3.1]. Moreover, condition (3) seems to be artificial and is implied by

(3)′ $D := \operatorname{co} K \subset \bigcup_{y \in K_i} \operatorname{Int}(A_i^-(y) \cap P_i^-(y))$.

In this case, for locally convex E_i's, Theorem 7 sharpens Wu and Shen [28, Theorem 10], which in turn extends results of Yannelis and Prabhakar [29], S.-Y. Chang [7], Tian [27], and Im et al. [13]. Similarly, [28, Theorems 11 and 12] can also be improved.

Theorem 8 *Let $\Gamma = (X_i, P_i)_{i \in I}$ be a game such that, for each $i \in I$,*

(1) *X_i is a convex subset of a Hausdorff t.v.s. E_i, and K_i is a nonempty compact convex subset of X_i;*
(2) *$K \subset \bigcup_{y \in K_i} \operatorname{Int}_X(P_i^-(y) \cup F_i)$, where $X = \prod_{i \in I} X_i$ and $F_i := \{x \in X \mid P_i(x) = \emptyset\}$; and*

(3) *for each* $x = (x_i)_{i \in I} \in X$, $x_i \notin \mathrm{co}\, P_i(x)$.

If X *has the c.c.f.p.p. or* K *is Klee approximable into* $D := \mathrm{co}\, K$, *then* Γ *has a maximal element in* K.

Proof For each $i \in I$, define a multimap $A_i : X \multimap K_i$ by $A_i(x) := K_i$ for $x \in X$. Now we can apply Theorem 7 with $A_i \equiv B_i$ and the conclusion follows. □

Remark If each E_i is locally convex, then Theorem 8 reduces to Husain and Tarafdar [11, Theorem 3.2]. Note also that Theorems 5–8 improve corresponding ones in our previous works [18] and [22]. Finally, we notice that Lemma 3 of Kim and Ding [14] is an incorrectly stated version of our Corollary 4.1 and they had to assume local convexity of the space E.

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