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Applications of Michael's selection theorems to fixed point theory

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Abstract

Applying some of Ernest Michael's selection theorems, from recent fixed point theorems on u.s.c. multimaps, we deduce generalizations of the classical Bolzano theorem, several fixed point theorems on multimaps defined on almost convex sets, almost fixed point theorems, coincidence theorems, and collectively fixed point theorems. These results are related mainly to Michael maps, that is, l.s.c. multimaps having nonempty closed convex values.

Keywords: t.v.s.; Multimap (map); Selection; Almost convex set; Almost fixed point property; Multimap class $\mathfrak{B}$

1. Introduction

The fixed point theory of multimaps in topological vector spaces has numerous applications in many fields in mathematical sciences. This theory began with the celebrated Kakutani fixed point theorem in 1941 and, until recently, was mainly concerned with the class of upper semicontinuous multimaps with closed convex values (usually called *Kakutani maps*).

Recall that Ernest Michael's groundbreaking theory of continuous selections on multimaps began in 1956 and was mainly concerned with the class of lower semicontinuous multimaps with closed convex values (which will be called *Michael maps*). However, there have appeared only relatively fewer fixed point theorems on this class of multimaps. For the details, we can consult with the monograph [23].

In our previous works [14,17], we unified fixed point theorems for the class of convex-valued upper semicontinuous (or more general) multimaps defined mainly on convex subsets of topological vector spaces. Recently, there have appeared a number of fixed point theorems for new classes of multimaps defined on convex subsets of topological vector spaces. Moreover, in [20], we obtained a new fixed point theorem for the 'better' admissible class $\mathfrak{B}^p$ defined on almost convex subsets of topological vector spaces.

In the present paper, we obtain new results mainly on Michael maps by combining some of Michael's selection theorems with a number of recent results in the fixed point theory of multimaps in topological vector spaces. Consequently, we deduce generalizations of the classical Bolzano theorem, several fixed point theorems on Michael maps

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defined on almost convex sets, almost fixed point theorems, coincidence theorems, and collectively fixed point theorems. These results are related mainly to Michael maps.

In Section 2, we introduce three principal selection theorems of Michael in [10–12], which we will need in this paper, and obtain a simple generalization of Bolzano's theorem by applying one of Michael's theorems. Section 3 deals with a unified fixed point theorem on convex-valued upper hemicontinuous multimaps and some of its consequences. We obtain another multi-valued version of Bolzano's theorem and a fixed point theorem on Michael maps.

In Sections 4 and 5, we deduce fixed point theorems on Michael maps defined on almost convex sets and some almost fixed point theorems. Section 6 deals with existence theorems of coincidence points of multimaps in the class $\mathfrak{B}$ of multimaps with continuous functions or Michael maps. Finally in Section 7, we obtain some collectively fixed point theorems for families of Michael maps.

2. When Bolzano meets Michael

A *multimap* $F : X \multimap Y$ is a function from a set X into the set 2^Y of *nonempty* subsets of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. We use the term *map* instead of multimap. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$. For any $B \subset Y$, the (*lower*) *inverse* of B under F is defined by

$$F^-(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

For topological spaces X and Y, a map $F : X \multimap Y$ is said to be *closed* if its graph

$$\mathrm{Gr}(F) := \{(x, y) \mid y \in F(x), x \in X\}$$

is closed in $X \times Y$, and *compact* if $F(X)$ is contained in a compact subset of Y.

A map $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X; *lower semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, $F^-(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c.

If $F : X \multimap Y$ is u.s.c. with closed values and if Y is regular, then F is closed. The converse is true whenever Y is compact.

Usually, a *Kakutani map* is an u.s.c. map with closed convex values. In parallel, a l.s.c. map with closed convex values will be called a *Michael map* in this paper.

The following are principal theorems of [10–12]:

Michael's Theorem 3.2″. (1956, [10]) *The following properties of a T_1-space X are equivalent*:

(a) *X is paracompact.*
(b) *If Y is a Banach space, then every lower semi-continuous carrier* (*map*) *for X to the family of nonempty, closed, convex subsets of Y admits a continuous selection.*

Michael's Theorem 1.2. (1966, [11]) *Let X be paracompact, and M a metrizable subset of a complete locally convex space E. Let $\phi : X \to 2^M$ be l.s.c. and such that, for some metric on M, every $\phi(x)$ is complete. Then there exists a continuous $f : X \to E$ such that $f(x) \in \Gamma_E\phi(x)$ for every $x \in X$.*

Here, $\Gamma_E A$ denotes the closed convex hull $\overline{\mathrm{co}}_E A$ of A in E. Recall that the completeness can be replaced by the compactness of $\Gamma_E K$ for every compact $K \subset M$.

If S is a topological space and $A \subset S$, $\dim_S A \leqslant 0$ means that the covering dimension of T is $\leqslant 0$ for every set $T \subset A$ which is closed in S; see Hurewicz and Wallman [6].

Michael's Theorem 7.1. (1981, [12]) *Let X be a paracompact space, Y a Fréchet space, $Z \subset X$ with $\dim_X Z \leqslant 0$, $C \subset X$ countable, and $\varphi : X \to 2^Y$ l.s.c. such that $\varphi(x)$ is closed in Y for $x \notin C$ and $\overline{\varphi}(x)$ is convex when $x \notin Z$. Then φ has a continuous selection $f : X \to Y$.*

In this theorem, we adopted the version of Ben-El-Mechaiekh and Oudadess [1, Corollary 6].

In closing this section, in order to illustrate the usefulness of Michael's works, we state a generalization of the following:

Bolzano's Theorem. (1817, [13]) *Let $f : [-r, r] \to \mathbb{R}$ be a continuous function satisfying the following boundary condition*:

$$x \cdot f(x) > 0 \quad \text{for } |x| = r.$$

Then there exists at least one solution $x_0 \in [-r, r]$ of the equation $f(x) = 0$.

Note that actually $x_0 \in]-r, r[$.

Combining Michael's Theorem 1.2 [11] and Bolzano's Theorem, we immediately obtain

Theorem 2.1. *Let $\alpha > 0$ and $F : [-\alpha, \alpha] \multimap \mathbb{R}$ be a Michael map satisfying the following boundary condition*:

$$x \cdot y > 0 \quad \text{for } |x| = \alpha \text{ and } y \in F(x).$$

Then there exists at least one solution $x_0 \in]-\alpha, \alpha[$ of the inclusion $0 \in F(x)$.

Similarly, some other results in [13] or other works can be stated for Michael maps.

3. Descendants of Bolzano

A t.v.s. means a Hausdorff topological vector space E. Let E^* denote the topological dual of E. Kakutani's convex-valued u.s.c. multimaps are further extended as follows: For a subset X of a t.v.s. E, a map $F : X \multimap E$ is said to be

(i) *upper demicontinuous* (u.d.c.) if for each $x \in X$ and open half-space H in E containing $F(x)$, there exists an open neighborhood N of x in X such that $F(N) \subset H$;
(ii) *upper hemicontinuous* (u.h.c.) if for each $h \in E^*$ and for any real α, the set $\{x \in X \mid \sup \operatorname{Re} hF(x) < \alpha\}$ is open in X; and
(iii) *generalized u.h.c.* if for each $p \in \{\operatorname{Re} h \mid h \in E^*\}$, the set $\{x \in X \mid \sup pF(x) \geqslant p(x)\}$ is compactly closed in X.

For the literature, see [16].

In our earlier works [14,17], we unified a large number of generalizations of the Kakutani theorem to maps of the above-mentioned types. In this section, combining Michael's theorems and the main fixed point theorem in [14,17], we deduce another generalizations of the Bolzano theorem and new fixed point theorems.

According to Lassonde [9], a *convex space* X is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $N \subset X$ there is a compact convex set $L_N \subset X$ such that $L \cup N \subset L_N$.

Let $cc(E)$ denote the set of nonempty closed convex subsets of a t.v.s. E and $kc(E)$ the set of nonempty compact convex subsets of E.

Let X be a nonempty convex subset of a vector space E. The *algebraic boundary* $\delta_E(X)$ of X in E is the set of all $x \in X$ for which there exists $y \in E$ such that $x + ry \notin X$ for all $r > 0$. If E is a t.v.s., the *topological boundary* $\operatorname{Bd} X = \operatorname{Bd}_E X$ of X is the complement of $\operatorname{Int}_E X$ in the closure $\overline{X}$. It is known that $\delta_E(X) \subset \operatorname{Bd} X$ and in general $\delta_E(X) \neq \operatorname{Bd} X$.

Let $X \subset E$ and $x \in E$. The *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, resp., are defined as follows:

$$I_X(x) := x + \bigcup_{r>0} r(X - x), \qquad O_X(x) := x + \bigcup_{r<0} r(X - x).$$

For $p \in \{\operatorname{Re} h \mid h \in E^*\}$ and $U, V \subset E$, let

$$d_p(U, V) = \inf\{|p(u - v)|:\ u \in U, v \in V\}.$$

The following is the main theorem in [14,17]:

Theorem 3.1. *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and F a map satisfying either*

(A) *E^* separates points of E and $F : X \to kc(E)$, or*

(B) *E is locally convex and $F : X \to cc(E)$.*

(I) *Suppose that for each* $p \in \{\mathrm{Re}\, h \mid h \in E^*\}$,

(0) $p|_X$ *is continuous on* X;

(1) $X_p := \{x \in X \mid \inf pF(x) \leqslant p(x)\}$ *is compactly closed in* X;

(2) $d_p(F(x), \overline{I_X(x)}) = 0$ *for every* $x \in K \cap \delta_E(X)$; *and*

(3) $d_p(F(x), \overline{I_L(x)}) = 0$ *for every* $x \in X \setminus K$.

Then there exists an $x \in X$ *such that* $x \in F(x)$.

(II) *Suppose that for each* $p \in \{\mathrm{Re}\, h \mid h \in E^*\}$,

$(0)'$ $p|_X$ *is continuous on* X;

$(1)'$ $X_p := \{x \in X \mid \sup pF(x) \geqslant p(x)\}$ *is compactly closed in* X;

$(2)'$ $d_p(F(x), \overline{O_X(x)}) = 0$ *for every* $x \in K \cap \delta_E(X)$; *and*

$(3)'$ $d_p(F(x), \overline{O_L(x)}) = 0$ *for every* $x \in X \setminus K$.

Then there exists an $x \in X$ *such that* $x \in F(x)$. *Further, if* F *is u.h.c., then* $F(X) \supset X$.

Remarks. 1. In Theorem 3.1, we do not require any concrete connection between topologies of X and E except (0). This is why there have appeared fixed point theorems on maps whose domains and ranges have different topologies.

2. If F is u.h.c. on each nonempty compact subset C of X, then F satisfies the "continuity" condition (1) for all $p \in \{\mathrm{Re}\, h \mid h \in E^*\}$, but not conversely. Any map F satisfying (1) is generalized u.h.c.

3. The "boundary" condition (2) is equivalent to the following:

$(2)''$ $x \in K$ and $p(x) = \max p(\overline{I_X(x)})$ implies $x \in X_p$.

In fact, $p(x) = \max p(X)$ is equivalent to $p(x) = \max p(\overline{I_X(x)})$.

Moreover, the "boundary" condition $(2)''$ is equivalent to the following:

$(2)'''$ $x \in K \cap \delta_E(X)$ and $p(x) = \max p(\overline{I_X(x)})$ implies $x \in X_p$.

4. The "coercivity" or "compactness" condition (3) is equivalent to the following:

$(3)''$ $x \in X \setminus K$ and $p(x) = \max p(\overline{I_L(x)})$ implies $x \in X_p$.

5. For conditions $(0)'$–$(3)'$, facts similar to 1–4 hold. The property $F(X) \supset X$ is called the *surjectivity* of F.

Recall that Theorem 3.1 unifies, improves and generalizes historically well-known fixed point theorems published in nearly 50 papers; see the diagrams and references in [14,16,17].

For a Kakutani map, Theorem 3.1 reduces to the following:

Corollary 3.2. *Let* X *be a convex subset of a t.v.s.* E, L *a c-compact subset of* X, K *a nonempty compact subset of* X, *and* F *an u.s.c. map satisfying either*

(A) E^* *separates points of* E *and* $F : X \to kc(E)$, *or*

(B) E *is locally convex and* $F : X \to cc(E)$.

(I) *Suppose that*

(i) $F(x) \cap \overline{I_X(x)} \neq \emptyset$ *for every* $x \in K \cap \delta_E(X)$; *and*

(ii) $F(x) \cap \overline{I_L(x)} \neq \emptyset$ *for every* $x \in X \setminus K$.

Then there exists an $x \in X$ *such that* $x \in F(x)$.

(II) *Suppose that*

$(\mathrm{i})'$ $F(x) \cap \overline{O_X(x)} \neq \emptyset$ *for every* $x \in K \cap \delta_E(X)$; *and*

$(\mathrm{ii})'$ $F(x) \cap \overline{O_L(x)} \neq \emptyset$ *for every* $x \in X \setminus K$.

Then there exists an $x \in X$ *such that* $x \in F(x)$ *and* $F(X) \supset X$.

For a single-valued map defined on a convex subset of a locally convex t.v.s., Theorem 3.1 reduces to the following:

Corollary 3.3. *Let E be a locally convex t.v.s., X a convex subset of E, L a c-compact subset of X, K a nonempty compact subset of X, and $f : X \to E$ a continuous function.*

(I) *Suppose that*
  (i) $f(x) \in \overline{I_X(x)}$ *for every* $x \in K \cap \delta_E(X)$; *and*
  (ii) $f(x) \in \overline{I_L(x)}$ *for every* $x \in X \setminus K$.
  Then there exists an $x \in X$ *such that* $x = f(x)$.

(II) *Suppose that*
  (i)′ $f(x) \in \overline{O_X(x)}$ *for every* $x \in K \cap \delta_E(X)$; *and*
  (ii)′ $f(x) \in \overline{O_L(x)}$ *for every* $x \in X \setminus K$.
  Then there exists an $x \in X$ *such that* $x = f(x)$ *and* $f(X) \supset X$.

The following is another multi-valued version of Bolzano's theorem:

Theorem 3.4. *Let* $G : [-\alpha, \alpha] \multimap \mathbb{R}$ *be a Kakutani map satisfying the following boundary condition*:

$$\text{if } |x| = \alpha, \text{ then } x \cdot y > 0 \quad \text{for some } y \in G(x).$$

Then there exists at least one solution $x_0 \in [-\alpha, \alpha]$ *of the inclusion* $0 \in G(x)$.

Proof. Let $F(x) := G(x) + x$ for each $x \in [-\alpha, \alpha]$. It suffices to show that F has a fixed point. We use Corollary 3.2(II) with $E = \mathbb{R}$ and $X = L = K = [-\alpha, \alpha]$. Note that condition (ii)′ holds trivially. For $x = \alpha$, $\alpha \cdot y > 0$ implies $y > 0$ for some $y \in G(\alpha)$, and hence

$$F(\alpha) \cap O_X(\alpha) = \big(G(\alpha) + \alpha\big) \cap]\alpha, \infty[\neq \emptyset.$$

Similarly, we have $y < 0$ for some $y \in G(-\alpha)$, and hence

$$F(-\alpha) \cap O_X(-\alpha) = \big(G(-\alpha) - \alpha\big) \cap]-\infty, -\alpha[\neq \emptyset.$$

Hence condition (i)′ holds. Therefore the conclusion follows from Corollary 3.2(II). □

Remark. Bolzano's theorem follows from Theorem 3.4, and hence from Theorem 3.1. Now, instead of the Brouwer fixed point theorem in 1912, Bolzano's theorem in 1817 can be regarded as the foremost ancestor of Theorem 3.1. We will meet another descendants of Bolzano's.

In view of Michael's Theorem 1.2 [11], we immediately deduce the following from Corollary 3.3:

Theorem 3.5. *Let E be a completely metrizable locally convex t.v.s., X a convex subset of E, L a compact convex subset of X, K a nonempty compact subset of X, and $F : X \multimap E$ a Michael map.*

(I) *Suppose that*
  (i) $F(x) \subset \overline{I_X(x)}$ *for every* $x \in K \cap \delta_E(X)$; *and*
  (ii) $F(x) \subset \overline{I_L(x)}$ *for every* $x \in X \setminus K$.
  Then there exists an $x \in X$ *such that* $x \in F(x)$.

(II) *Suppose that*
  (i)′ $F(x) \subset \overline{O_X(x)}$ *for every* $x \in K \cap \delta_E(X)$; *and*
  (ii)′ $F(x) \subset \overline{O_L(x)}$ *for every* $x \in X \setminus K$.
  Then there exists an $x \in X$ *such that* $x \in F(x)$ *and* $F(X) \supset X$.

Recall that particular forms of Theorem 3.5 are due to Reich [22, Theorem 1.9] and Dugundji–Granas [2, Theorem 5.11.6]; see also [23].

4. Michael maps on almost convex sets

Let E be a t.v.s. with a base $\mathcal{V}$ of neighborhoods of 0. According to Himmelberg [4], a subset X of a t.v.s. E is said to be *almost convex* if for any $V \in \mathcal{V}$ and for any finite subset $A := \{x_1, x_2, \dots, x_n\}$ of X, there exists a subset $B := \{y_1, y_2, \dots, y_n\}$ of X such that $y_i - x_i \in V$ for each $i = 1, 2, \dots, n$ and $\operatorname{co} B \subset X$.

Here we give a well-known result due to Idzik [7]; see also [21]:

Theorem 4.1. *Let X be an almost convex subset of a locally convex t.v.s. E and $F : X \multimap X$ a compact Kakutani map. Then F has a fixed point.*

Remark. Theorem 4.1 for a convex X is due to Himmelberg [4] which contains results due to Brouwer, Schauder, Tychonoff, Hukuhara, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, and others; see [16].

Combining Michael's Theorem 1.2 [11] with Theorem 4.1 for a single-valued function F and a convex subset X, we obtain the following:

Theorem 4.2. *Let X be a convex subset of a locally convex t.v.s. E, and Y a compact metrizable subset of X. Then a Michael map $S : X \multimap Y$ has a fixed point.*

Proof. Since Y is compact, by the well-known argument of Fournier and Granas [3], $\operatorname{co} Y$ is a σ-compact subset of X and hence $\operatorname{co} Y$ is Lindelöf. Since $\operatorname{co} Y$ is regular as a subset of a t.v.s., we know that $\operatorname{co} Y$ is paracompact. Then, by Michael's Theorem 1.2 [11], there exists a continuous function $f : \operatorname{co} Y \to \overline{E}$, where $\overline{E}$ is the completion of E, such that $f(x) \in S(x) \subset Y \subset \operatorname{co} Y$ for all $x \in \operatorname{co} Y$. Note that $f : \operatorname{co} Y \to \operatorname{co} Y$ and f is compact. Therefore, by Theorem 4.1, f has a fixed point $x_0 \in \operatorname{co} Y \subset X$; that is, $x_0 = f(x_0) \in S(x_0)$. This completes our proof. □

Remarks. 1. Theorem 4.2 is due to Wu [24, Corollary 3] with slightly different proof. When X itself is compact and metrizable, Theorem 4.2 reduces to Himmelberg et al. [5, Theorem 3].

2. Some particular forms of Theorem 4.2 are due to Kim and Lee, Zheng, and Zhang; see [15].

Similarly, we immediately deduce the following new result:

Theorem 4.3. *Let E be a locally convex t.v.s. and X an almost convex metrizable subset of E. Then any compact Michael map $F : X \multimap X$ has a fixed point.*

Proof. Since $\overline{F(X)}$ is a compact subset of X, we may regard it a compact metrizable subset of the completion $\overline{E}$ of E. Since X is a paracompact subset of $\overline{E}$, by Michael's Theorem 1.2 [11], F has a continuous selection $f : X \multimap \overline{F(X)} \subset X$. Since f is compact, by Theorem 4.1, f has a fixed point $x_0 = f(x_0) \in F(x_0)$. This completes our proof. □

Remark. When X is compact and convex, Theorem 4.3 reduces to Himmelberg et al. [5, Theorem 3].

Corollary 4.4. *Let E be a normed vector space and X an almost convex subset of E. Then any compact Michael map $F : X \multimap X$ has a fixed point.*

Combining Michael's Theorem 7.1 [12] with Theorem 4.1, we have the following:

Theorem 4.5. *Let X be an almost convex subset of a Fréchet space E, Y a compact subset of X, $Z \subset X$ with $\dim_X Z \leqslant 0$, and $C \subset X$ countable. Let $T : X \multimap Y$ be a l.s.c. map such that $T(x)$ is closed for $x \notin C$ and $T(x)$ is convex for $x \notin Z$. Then T has a fixed point.*

For normed vector spaces, we have the following form of Michael's Theorem 7.1 [12]:

Lemma 4.6. *Let X be a paracompact space, $Z \subset X$ with $\dim_X Z \leqslant 0$, $C \subset X$ countable, Y a normed vector space, and $T : X \multimap Y$ a l.s.c. map such that $T(x)$ is complete for $x \notin C$ and $T(x)$ is convex for $x \notin Z$. Then T has a continuous selection.*

Proof. Without loss of generality, we may assume that Y is complete (for the conditions on T remain unchanged in the completion of Y). Now by applying Michael's Theorem 7.1 [12], we have the conclusion. □

From Lemma 4.6 and Theorem 4.1, we have the following:

Theorem 4.7. *Let X be an almost convex subset of a normed vector space, $Z \subset X$ with $\dim_X Z \leqslant 0$, $C \subset X$ countable, and $T : X \multimap X$ a compact l.s.c. map such that $T(x)$ is closed for $x \notin C$ and $T(x)$ is convex for $x \notin Z$. Then T has a fixed point.*

Proof. Note that $T(x)$ is compact and hence complete for each $x \notin C$. Applying Lemma 4.6, T has a continuous selection $f : X \to X$. Since $f(X) \subset T(X)$ and T is compact, so is f. Therefore, by Theorem 4.1, f has a fixed point $x_0 = f(x_0) \in T(x_0) \subset X$. This completes our proof. □

For $Z = C = \emptyset$, Theorem 4.7 reduces to Corollary 4.4.

Some results related to this section can be found in [18].

5. Almost fixed points

The celebrated KKM principle due to Knaster, Kuratowski, and Mazurkiewicz [8] in 1929 is as follows:

KKM principle. *Let D be the set of vertices of a simplex S and $F : D \multimap S$ a map with closed [resp., open] values such that*

$$\operatorname{co} N \subset F(N) \quad \textit{for each } N \subset D.$$

Then $\bigcap_{z\in D} F(z) \neq \emptyset$.

The KKM principle follows from the Sperner combinatorial lemma appeared in 1928 and was used to obtain one of the most direct proofs of the Brouwer fixed point theorem. Later, it was known that those three theorems are mutually equivalent. In fact, those three theorems are regarded as a sort of mathematical trinity. All are extremely important and have many applications. Moreover, many important results in nonlinear functional analysis and other fields are known to be equivalent to those three theorems. For details, see [16].

From the KKM principle, we obtained recently an almost fixed point theorem [19, Theorem 5.1], [20, Theorem 3.1] for u.s.c. or l.s.c. maps. The following is a particular form for close relatives of Kakutani maps or Michael maps:

Theorem 5.1. *Let X be a convex subset of a locally convex t.v.s. E and $T : X \multimap X$ an u.s.c. [resp., a l.s.c.] map with convex values such that $T(X)$ is totally bounded. Then T has the almost fixed point property; that is, for each $V \in \mathcal{V}$, there exists an $x_V \in X$ such that $T(x_V) \cap (x_V + V) \neq \emptyset$.*

Now we apply Theorem 5.1 to compact maps:

Theorem 5.2. *Let X be a convex subset of a locally convex t.v.s. E and $T : X \multimap X$ a compact u.s.c. [resp., l.s.c.] map with convex values. Then there exists a point $x_0 \in X$ such that $(x_0, x_0) \in \overline{\mathrm{Gr}(T)}$.*

Proof. For each element $V \in \mathcal{V}$, there exist $x_V, y_V \in X$ such that $y_V \in T(x_V)$ and $y_V \in x_V + V$. Since $T(X)$ is relatively compact in X, we may assume that the net y_V converges to some $x_0 \in \overline{T(X)}$. Then x_V also converges to x_0. Since $(x_V, y_V) \in \mathrm{Gr}(T)$, we have the conclusion. □

From Theorem 5.2, we have the following due to Himmelberg [4]:

Corollary 5.3. *Let X be a convex subset of a locally convex t.v.s. E and $T : X \multimap X$ a compact Kakutani map. Then T has a fixed point $x_0 \in T(x_0)$.*

Proof. Since T is u.s.c. with closed values and X is regular, the graph of T is closed in $X \times \overline{T(X)}$. By Theorem 5.2, there exists a point $x_0 \in X$ such that $(x_0, x_0) \in \overline{\mathrm{Gr}(T)} = \mathrm{Gr}(T)$, and hence we have $x_0 \in T(x_0)$. This completes our proof. □

6. Coincidence theorems

For maps $F : X \multimap Y$ and $G : Y \multimap X$, a *coincidence point* $(x_0, y_0) \in X \times Y$ is the one satisfying $y_0 \in F(x_0)$ and $x_0 \in G(y_0)$ [that is, $x_0 \in X$ is a fixed point of $G \circ F : X \multimap X$].

An equivalent definition is as follows: For maps $F : X \multimap Y$ and $G : X \multimap Y$, a *coincidence point* $x_0 \in X$ is the one satisfying $F(x_0) \cap G(x_0) \neq \emptyset$ [that is, $x_0 \in X$ is a fixed point of $G^- \circ F : X \multimap X$].

In this section, we derive existence theorems of coincidence points of maps in the class $\mathfrak{B}$ with continuous functions or Michael maps.

A *polytope* P in a subset X of a t.v.s. E is a homeomorphic image of a simplex.

In 1996, the author introduced the 'better' admissible class $\mathfrak{B}$ of maps defined on a subset X of a t.v.s. E into a topological space Y as follows:

$$F \in \mathfrak{B}(X, Y) \quad \Leftrightarrow \quad \begin{array}{l} F : X \multimap Y \text{ is a map such that, for each polytope } P \text{ in } X \text{ and for any continuous} \\ \text{function } f : F(P) \to P, \text{ the composition } f(F|_P) : P \multimap P \text{ has a fixed point.} \end{array}$$

Subclasses of $\mathfrak{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$, the Aronszajn maps $\mathbb{M}$ (u.s.c. with R_δ values), the acyclic map $\mathbb{V}$ (u.s.c. with compact acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the Fan–Browder maps (codomains are convex sets), locally selectionable maps having convex values (codomains are convex sets), the Simons maps $\mathbb{K}_c$, the approachable maps $\mathbb{A}$ (whose domains are uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}_c^\sigma$ of Lassonde, the class $\mathbb{V}_c^\sigma$ of Park et al., approximable maps of Ben-El-Mechaiekh and Idzik, and others. Those subclasses are examples of the admissible class $\mathfrak{A}_c^\kappa$ due to the author. Moreover, compact closed maps in the KKM class due to Chang and Yen and in the s-KKM class due to Chang, Huang, and Jeng also belong to the class $\mathfrak{B}$. For references, see [20].

The following is a particular case of [20, Theorem 3.7]:

Theorem 6.1. *Let X be an almost convex subset of a locally convex t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then F has a fixed point.*

From Theorem 6.1, we have the following:

Theorem 6.2. *Let X be an almost convex subset of a locally convex t.v.s. E and Y a compact space. Let $G \in \mathfrak{B}(X, Y)$ be a closed map and $f \in \mathbb{C}(Y, X)$. Then G and f have a coincidence point.*

Proof. We show that $f \circ G \in \mathfrak{B}(X, X)$. In fact, for any polytope P of X and any continuous $g : (f \circ G)(P) \to P$, the composition $g((f \circ G)|_P) = (g \circ f)(G|_P)$ has a fixed point because $G \in \mathfrak{B}(X, Y)$. Moreover, since f is compact and continuous and G is closed, $f \circ G$ is closed and compact. Now, by Theorem 6.1, $f \circ G$ has a fixed point. □

In view of Michael's Theorem 1.2 [11], we deduce the following from Theorem 6.2:

Theorem 6.3. *Let X be an almost convex metrizable subset of a complete locally convex t.v.s. E and Y a compact space. Let $F : Y \multimap X$ be a Michael map and $G \in \mathfrak{B}(X, Y)$ a closed map. Then F and G have a coincidence point.*

In view of Michael's Theorem 3.2″ [10], we immediately have the following:

Corollary 6.4. *Let X be an almost convex subset of a Banach space E and Y a compact space. Let $F : Y \multimap X$ be a Michael map and $G \in \mathfrak{B}(X, Y)$ a closed map. Then F and G have a coincidence point.*

When Michael meets Kakutani, we have the following:

Corollary 6.5. *Let X be an almost convex metrizable subset of a complete locally convex t.v.s. E_1 and Y a compact subset of a t.v.s. E_2. Let $F : Y \multimap X$ be a Michael map and $G : X \multimap Y$ a Kakutani map. Then F and G have a coincidence point.*

7. Collectively fixed points

In this section, we deduce some collectively fixed point theorems for families of Michael maps.

Let $\{X_i\}_{i\in I}$ be a family of nonempty sets, and let $i \in I$ be fixed. Let

$$X := \prod_{j\in I} X_j \quad \text{and} \quad X^i := \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its ith coordinate is x_i and, for $j \neq i$, the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A\} \quad \text{and} \quad A(x_i) := \{y^i \in X^i \mid [y^i, x_i] \in A\}.$$

The following are variants of known ones:

Theorem 7.1. *Let $\{X_i\}_{i\in I}$ be a family of almost convex sets, each in a locally convex t.v.s. E_i, and $T_i : X \multimap X_i$ a compact Michael map for each $i \in I$. If X is metrizable, then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

Proof. We may assume each E_i is complete (for the conditions on T_i remain unchanged in the completion of E_i). Now by applying Michael's Theorem 1.2 [11], each T_i has a continuous selection $f_i : X \to X_i$. Define $f : X \to X$ by $f(x) = \prod_{i\in I} f_i(x)$ for each $x \in X$. Then $f : X \to X$ is a compact continuous function. By Theorem 4.1, f has a fixed point $\hat{x} \in X$; that is, $\hat{x} = f(\hat{x})$ and hence $\hat{x}_i = f_i(\hat{x}) \in T_i(\hat{x})$ for each $i \in I$. □

Theorem 7.2. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a locally convex t.v.s. E_i, K_i a nonempty compact metrizable subset of X_i, and $T_i : X \multimap K_i$ a Michael map for each $i \in I$. Then there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

Proof. Since $K := \prod_{j\in I} K_j$ is compact, $\operatorname{co} K$ is a paracompact subset of X as in the proof of Theorem 4.2. Then, by Michael's Theorem 1.2 [11], there exists a continuous selection $f_i : \operatorname{co} K \to \overline{E}_i$ of $T_i|_{\operatorname{co} K}$, where $\overline{E}_i$ is the completion of E_i, such that $f_i(x) \in T_i(x) \subset K_i$ for all $x \in \operatorname{co} K$. Now follow the proof of Theorem 7.1. □

Remark. Theorem 7.2 is essentially due to Wu [24, Theorem 1] with much longer proof. Some equilibrium existence theorems for abstract economies are deduced in [24].

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