INTERNATIONAL PUBLICATIONS (USA)

PanAmerican Mathematical Journal
Volume 18(2008), Number 1, 21–34

Comments on Fixed Point and Coincidence Theorems for Families of Multimaps

Sehie Park
The National Academy of Sciences, Republic of Korea;
Seoul National University
Department of Mathematical Sciences
Seoul 151–747, Korea
shpark@math.snu.ac.kr

Communicated by the Editors
(Received September 2007; Accepted October 2007)

Abstract

Using recent results in analytical fixed point theory, some known basic fixed point and coincidence theorems for families of multimaps are generalized and improved by removing some redundant restrictions. Especially, we are mainly concerned with the class of locally selectionable multimaps having convex values instead of the Fan-Browder maps, which played main role in a number of previous works.

AMS (MOS) Subject Classification: 47H10, 46N10, 54H25, 91A13, 91B50.

Key words: Browder map, Fan-Browder map, locally selectionable map, Kakutani map, acyclic map, admissible set, Klee approximable set, the better admissible class of multimaps.

1 Introduction

Fixed point and coincidence theorems are basis of various equilibrium theorems, for example, the classical von Neumann minimax theorem, the Nash equilibrium theorem, the Ky Fan minimax inequality, and many others. In this field, the pioneering works of Fan and Browder are quite well-known; see [3,15]. Motivated by a work of Fan, various forms of the intersection theorem for a family of sets having convex sections have appeared and been applied to many problems. Later those intersection theorems were known to be equivalent to collectively fixed point theorems for a family of Fan-Browder type multimaps; see [1,2,7,14,21].

Recently, in a sequence of papers, Lin et al. [8-11] further generalized the above-mentioned results to some collective coincidence theorems for two families of multimaps within various circumstances. As applications of those new results, for example, they introduced in [10] an equilibrium problem in our real life with m families of players and $2m$ families of constraints on strategy sets; and obtained in [9] existence theorems of equilibria of such problems and existence theorem of equilibria of abstract economies with two families of players.

In the present paper, all of the basic theorems for a family of multimaps and the basic coincidence theorems for two families of multimaps are reexamined and generalized. In fact, the basic results in [8-11] do not reflect recent progress in analytical fixed point theory, and are mainly concerned with the Fan-Browder type multimaps and acyclic maps, which can be replaced by more general locally selectionable multimaps having convex values and the better admissible multimaps, respectively.

Section 2 is for a preliminary and deals with new fixed point theorems and continuous selection theorems. In Section 3, we introduce new versions of collectively fixed point theorems for a family of multimaps in [1,2,7,11]. Section 4 deals with generalizations of the basic coincidence theorems on two families of multimaps in [8,9,10]. We remove some redundant restrictions on such basic theorems. In Section 5, further comments on applications of the basic theorems in [1,7-11] are given.

2 Selection theorems and fixed point theorems

In this paper, all topological spaces are assumed to be Hausdorff unless explicitly stated otherwise, and t.v.s. means topological vector spaces.

A *multimap* (or simply, *a map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of the set Y; that is, a function with *values* $F(x) \subset Y$ for $x \in X$ and *fibers* $F^{-}(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$. A map is said to be *compact* if its range is contained in a compact subset of its codomain.

The following is known:

Theorem 2.1. *For every nonempty convex subset X of a t.v.s. E, a compact continuous function $f : X \to X$ has a fixed point if one of the following holds:*

(1) (Schauder) *E is a normed vector space;*
(2) (Hukuhara) *E is locally convex;*
(3) (Klee) *X is admissible (in the sense of Klee);*
(4) (Idzik) *$\overline{f(X)}$ is convexly totally bounded (simply, c.t.b.); and*
(5) (Nhu and Arandelović) *X is compact and weakly admissible.*

For the literature, see [5,6,15,19].

Let us say that a subset X of a t.v.s. has the *compact convex fixed point property* (c.c.f.p.p) if for any nonempty convex subset C of X, a compact continuous function $f : C \to C$ has a fixed point.

A *polytope* P in a t.v.s. E is a homeomorphic image of a standard simplex. A nonempty subset K of a t.v.s. E is said to be *Klee approximable* if for any neighborhood V of the origin 0 in E, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope of E. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

We give some examples of Klee approximable sets as in [19,20]:

(1) A subset X of E is admissible (in the sense of Klee) iff every compact subset K of X is Klee approximable into E.

(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.

(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.

(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.

(5) Any compact subset K of an admissible convex subset X of a t.v.s. is Klee approximable into X.

(6) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.

Note that (6)⇒(5)⇒(4)⇒(3).

The following recent result contains a large number of known theorems:

Theorem 2.2. (Park [18]) *Let X be a subset of a t.v.s. E and $f : X \to X$ a compact continuous function. If $f(X)$ is Klee approximable into X, then f has a fixed point.*

Recall that a *Browder map* is a multimap with nonempty convex values and open fibers. In 1968, Browder obtained the following Fan-Browder fixed point theorem:

Theorem 2.3. (Browder [3]) *If X is a compact convex set in a t.v.s. E and $T : X \multimap X$ is a Browder map, then T has a fixed point $x_0 \in X$, that is, $x_0 \in T(x_0)$.*

Browder proved this by applying the partition of unity argument subordinated to a finite open cover and the Brouwer fixed point theorem. Actually, he had to assume the Hausdorffness of E (which was shown to be redundant; see [16,17]) and obtained a continuous selection $f : X \to X$ of T. The selection method is generalized as follows:

Theorem 2.4. *Let X be a normal topological space, Y a convex subset of a t.v.s. E, and $S : X \multimap Y$ a map such that*

$$X = \bigcup_{i=1}^{n+1} \operatorname{Int} S^{-}(y_i) \text{ for some } N = \{y_1, y_2, \ldots, y_{n+1}\} \subset Y.$$

Then there exists a continuous function $s : X \to Y$ such that $s(x) \in \operatorname{co} S(x)$ for all $x \in X$ and $s = \phi_N \circ p$, where $p : X \to \Delta_n$ and $\phi_N : \Delta_n \to \operatorname{co} N$ are continuous functions and Δ_n is an n-simplex.

Theorem 2.4 can be proved by following the proof of [14, Lemma 1] and by the fact that, for each locally finite open cover of a normal space, there is a partition of unity subordinated to it.

For a topological space X and a subset Y of a t.v.s., a map $T : X \multimap Y$ is called a Φ-*map* or a *Fan-Browder map* provided that there exists a (companion) map $S : X \multimap Y$ such that

(1) for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$; and

(2) $X = \bigcup\{\operatorname{Int} S^{-}(y) \mid y \in Y\}$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *locally* (*continuously*) *selectionable* if for each $x_0 \in X$, there exists an open neighborhood V_0 of x_0 and a continuous function $f : V_0 \to Y$ such that $f(x) \in T(x)$ for all $x \in V_0$; see [14]. The following is given [14, Theorems 4 and 5]:

Theorem 2.5. *Let X be a paracompact topological space and Y a convex subset of a t.v.s. E. Then*

(1) *any Φ-map $T : X \multimap Y$ is locally selectionable; and*

(2) *any locally selectionable map $A : X \multimap Y$ having convex values has a continuous selection $s : X \to Y$.*

Every continuous function is locally selectionable and hence the converse of Theorem 2.5(1) is not true. For a Browder map, Theorem 2.5(2) is due to Ben-El-Mechaiekh et al. and for a Φ-map to Horvath; see [14].

The following selection theorem is classical:

Theorem 2.6. (Michael [12]) *Let X be a paracompact space and M a metrizable subset of a complete locally convex t.v.s. E. Let $\phi : X \multimap M$ be a lower semicontinuous (l.s.c.) map such that, for some metric on M, every $\phi(x)$ is complete. Then there exists a continuous function $f : X \to E$ such that $f(x) \in \overline{\operatorname{co}}\ \phi(x)$ for every $x \in X$.*

From Theorems 2.1(2) and 2.6, we have the following:

Corollary. *Let X be a metrizable convex subset of a locally convex t.v.s. E, and $\phi : X \multimap X$ a l.s.c. map with nonempty closed convex values. If ϕ is compact, then ϕ has a fixed point.*

Proof. Let M be a compact subset of X such that $\phi(X) \subset M$. Then M is regarded as a subset of the completion of E. Then we can apply the Michael Theorem 2.6 to ϕ and get a continuous selection $f : X \to M$ of ϕ. Hence the conclusion follows from Theorem 2.1(2).

Recall that the convex hull of a nonempty compact subset of a t.v.s. is always paracompact by the Fournier-Granas argument; see [4]. From this fact, we obtain the following in [18]:

Theorem 2.7. *Let X be an admissible convex subset of a t.v.s. E and $T : X \multimap X$ a locally selectionable map having convex values. If T is compact, then T has a fixed point.*

Note that, since any continuous function is locally selectionable, Theorems

2.1(3) and 2.7 are equivalent. As a consequence of Theorem 2.7, we obtain the following partial solution to the Ben-El-Mechaiekh conjecture raised in 1990; see [18]:

Corollary. *Let X be an admissible convex subset of a t.v.s. E and $T : X \multimap X$ a Φ-map or a Browder map. If T is compact, then T has a fixed point.*

Recall that a *Kakutani map* is an upper semicontinuous (u.s.c.) map having nonempty closed convex values.

A topological space is said to be *acyclic* if all of its reduced Čech homology groups over the rational field vanish. A u.s.c. map is said to be *acyclic* if it has compact acyclic values. The following is known in 1998; see [13]:

Theorem 2.8. *Let X be an admissible convex subset of a t.v.s. E and $T : X \multimap X$ a compact acyclic map. Then T has a fixed point.*

For a subset X of a t.v.s. E, we defined the *"better" admissible class* $\mathfrak{B}$ of multimaps as follows in [13]:

$T \in \mathfrak{B}(X, Y) \iff T : X \multimap Y$ is a map such that for any polytope P in X and any continuous function $f : T(P) \to P$, the composition $f(T|_P) : P \multimap P$ has a fixed point.

Note that the class $\mathfrak{B}$ has a large number of examples and an acyclic map is one of them; see [13].

The following fixed point theorem was obtained in [13]:

Theorem 2.9. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact closed map $T \in \mathfrak{B}(X, X)$ has a fixed point.*

In [13], it was shown that Theorem 2.9 subsumes more than sixty known or possible particular cases and generalizes them in terms of the involving spaces and maps as well.

3 Collectively fixed point theorems

Let $\{X_i\}_{i\in I}$ be a family of sets. For a given $i \in I$, let

$$X := \prod_{j\in I} X_j \quad \text{and} \quad X^i := \prod_{j\in I\setminus\{i\}} X_j.$$

For $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the jth coordinate of x^i. For $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its ith coordinate is x_i and, for $j \neq i$, the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where $x^i = \pi^i(x)$ and $x_i = \pi_i(x)$ denote the projection of x in X^i and X_i, respectively.

For a family $\{E_i\}_{i\in I}$ of t.v.s., let $E := \prod_{i\in I} E_i$. Similarly, $X := \prod_{i\in I} X_i$ and $K := \prod_{i\in I} K_i$ for subsets X_i and K_i of E_i for $i \in I$.

From Theorems 2.1 and 2.2, we deduce the following collectively fixed point theorem for a family of multimaps:

Theorem 3.1. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a t.v.s. E_i, and K_i a nonempty compact subset of X_i. For each $i \in I$, let $T_i : X \multimap K_i$ be a locally selectionable map having convex values. If X has the c.c.f.p.p. or K is Klee approximable into $\operatorname{co} K$, then there exists an $\overline{x} \in K$ such that $\overline{x}_i \in T_i(\overline{x})$ for each $i \in I$.*

Proof. Since K is compact in X, by the Fournier-Granas argument, $\operatorname{co} K$ is paracompact in X. Since each T_i is locally selectionable with convex values, by Theorem 2.5, $T_i|_{\operatorname{co} K}$ has a continuos selection $f_i : \operatorname{co} K \to K_i \subset X_i$. Define $f : \operatorname{co} K \to K \subset X$ by $f(x) := (f_i(x))_{i\in I}$ for each $x \in \operatorname{co} K$. Since f is a compact continuous selfmap on $\operatorname{co} K$, by the c.c.f.p.p. or by Theorem 2.2, f has a fixed point $\overline{x} \in K$, that is, $\overline{x} = f(\overline{x})$ and $\overline{x}_i = f_i(\overline{x}) \in T_i(\overline{x})$ for each $i \in I$.

Remarks. 1. For Φ-maps, Theorem 3.1 reduces to [21, Theorem 5], which has a number of particular forms previously obtained by other authors.

2. When I is a singleton, Theorem 3.1 reduces to Theorem 2.7.

From Theorem 2.8, we deduce the following:

Theorem 3.2. *Let I be a finite index set, $\{X_i\}_{i\in I}$ a family of convex sets, each in a t.v.s. E_i, and K_i a compact subset of X_i. For each $i \in I$, let $T_i : X \multimap K_i$ be an acyclic map. If X is admissible, then there exists an $\overline{x} \in K$ such that $\overline{x}_i \in T_i(\overline{x})$ for each $i \in I$.*

Proof. Define $T := \prod_{i\in I} T_i : X \multimap K \subset X$. Then T is an acyclic map by the Künneth theorem. Since X is an admissible convex subset of the t.v.s. $E := \prod_{i\in I} E_i$ and T is compact, by applying Theorem 2.8, we have an $\overline{x} \in K$ such that $\overline{x} \in T(\overline{x}) = \prod_{i\in I} T_i(\overline{x})$, that is, $\overline{x}_i \in T_i(\overline{x})$ for each $i \in I$.

Similarly, for Kakutani maps, we have the following:

Theorem 3.3. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a t.v.s. E_i, and $T_i : X \multimap X_i$ a compact Kakutani map. If X is admissible, then there exists an $\overline{x} \in X$ such that $\overline{x}_i \in T_i(\overline{x})$ for each $i \in I$.*

The following is essentially given in [1, Theorem 1] and [11, Theorem 3.1] with two page proofs:

Theorem 3.4. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a t.v.s. E_i, and for each $i \in I$, $T_i : X \multimap X_i$ a Φ-map with the companion map $S_i : X \multimap X_i$. Suppose that for each $i \in I$,*

(a) *there exists a nonempty compact subset K of X;*

(b) *if $X \neq K$, for each nonempty finite subset N_i of X_i, there exists a compact convex subset L_{N_i} of X_i containing N_i such that, for $L := \prod_{i\in I} L_{N_i}$, we have*

$$L \setminus K \subset \bigcup \{\operatorname{Int}_X S_i^-(z_i) \mid z_i \in L_{N_i}\}.$$

Then there exists a point $x \in X$ such that $x \in T(x) := \prod_{i\in I} T_i(x)$.

Remarks. 1. When I is a singleton, Theorem 3.4 reduces to a well-known generalization of the Fan-Browder fixed point theorem in [14-17, 21].

2. Note that each E_i is not necessarily Hausdorff.

3. For the case $X = K$, put $L_{N_i} := X_i$ and $L := X$. Then we have the same conclusion.

4. Note that many particular forms of Theorem 3.4 have appeared by other authors; see [11].

From the above theorem, we immediately obtain the following:

Theorem 3.5. Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a t.v.s. E_i, and for each $i \in I$, $T_i : X^i \multimap X_i$ a Φ-map with the companion map $S_i : X^i \multimap X_i$. Suppose that for each $i \in I$,

(a) there exists a nonempty compact subset $K(i)$ of X^i;

(b) if $X^i \neq K(i)$, for each nonempty finite subset N_i of X_i, there exists a compact convex subset L_{N_i} of X_i containing N_i such that, for $L^i := \prod_{j\in I, i\neq j} L_{N_j}$, we have

$$L^i \backslash K(i) \subset \bigcup \{\mathrm{Int}_{X^i} S_i^-(z_i) \mid z_i \in L_{N_i}\}.$$

Then there exists a point $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ such that $\overline{x}_i \in T_i(\overline{x}^i)$ for all $i \in I$.

Proof. For each $i \in I$, let $G_i : X = X^i \times X_i \multimap X_i$ defined by $G_i[x^i, x_i] = T_i(x^i)$ for each $x = [x^i, x_i] \in X$. Then each G_i is a Φ-map with the companion map $H_i : X \multimap X_i$ defined by $H_i[x^i, x_i] = S_i(x^i)$. Therefore, by Theorem 3.4, there exists $\overline{x} \in X$ such that $\overline{x} \in G(\overline{x}) := \prod_{i\in I} G_i(\overline{x})$; that is, $\overline{x}_i \in G_i(\overline{x}) = T_i(\overline{x}^i)$ for all $i \in I$.

Remarks. 1. Each E_i is not necessarily Hausdorff in Theorem 3.5. Note that particular forms of Theorem 3.5 have appeared in [2,7], and others.

2. Recall that, as in the generalized Fan-Browder type fixed point theorem appeared in [15-17, 21]; and moreover, X_i's in them can be assumed to be convex spaces in the sense of Lassonde.

4 Coincidence theorems for families of multimaps

In this section, we begin to derive new versions of results of Section 2 of [9]. Throughout this section, for a family $\{Y_j\}_{j\in J}$ of sets, each in a t.v.s. F_j, let $Y := \prod_{j\in J} Y_j$. Similarly, $L := \prod_{j\in J} L_j$ for subsets L_j of Y_j for each $j \in J$.

Theorem 4.1. *Let $\{X_i\}_{i\in I}$ be a family of convex subsets, each in a t.v.s. E_i, $\{Y_j\}_{j\in J}$ a family of convex subsets, each in a t.v.s. F_j, and $K_i \subset X_i$ compact subsets. Let $G_j : X \multimap Y_j$ and $H_i : Y \multimap K_i$ multimaps satisfying the following conditions:*

(1) *each G_j is a locally selectionable map with convex values;*

(2) *each H_i is a locally selectionable map with convex values.*

If X has the c.c.f.p.p. or $K := \prod_{i\in I} K_i$ is Klee approximable into $D := \operatorname{co} K$, then there exist $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ and $\overline{y} = (\overline{y}_j)_{j\in J} \in Y$ such that $\overline{y}_j \in G_j(\overline{x})$ and $\overline{x}_i \in H_i(\overline{y})$ for each $i \in I$ and $j \in J$.

Proof. Since D is paracompact by the Fournier-Granas argument [4], and hence, by Theorem 2.5(2), $G_j|_D$ has a continuous selection $g_j : D \multimap Y_j$ for each $j \in J$. Define $g : D \to Y$ by $g(x) := (g_j(x))_{j\in J}$ for $x \in D$. Then each $H_i g : D \multimap K_i$ is locally selectionable with convex values. Since D is paracompact, by Thorem 2.5(2) again, $H_i g$ has a continuous selection $f_i : D \to K_i$ for each $i \in I$. Define $f : D \to K \subset D$ by $f(x) = (f_i(x))_{i\in I}$ for $x \in D$. Then, by the c.c.f.p.p. or by Theorem 2.2, f has a fixed point $\overline{x} = f(\overline{x}) \in K$. Let $\overline{y} = g(\overline{x}) = (g_j(\overline{x}))_{j\in J}$. Then $\overline{y}_j = g_j(\overline{x}) \in G_j(\overline{x})$ for each $j \in J$; and $\overline{x}_i = f_i(\overline{x}) \in H_i g(\overline{x}) = H_i(\overline{y})$ for each $i \in I$. This completes our proof.

Remarks. 1. Note that Theorem 4.1 properly generalizes [9, Theorem 3.1] where (1) each E_i and F_j are locally convex and (2) each G_j and H_i are Φ-maps. Moreover, Theorem 4.1 generalizes Theorem 3.1. In fact, for the case $I = J$, $X_i = Y_i$, and $G_j = \pi_j : X \to X_j$ (the jth projection), Theorem 4.1 reduces to Theorem 3.1.

2. When I and J are singletons, Theorem 4.1 reduces to a new coincidence theorem. This remark works for other theorems in this section.

Similarly, [9, Theorems 3.4] can be improved as follows:

Theorem 4.2. *Let $\{E_i\}_{i\in I}$ be a family of t.v.s. and $\{F_j\}_{j\in J}$ a family of locally convex t.v.s. For each $i \in I$ and $j \in J$, let X_i and Y_j be nonempty convex subsets of E_i and F_j, resp., and L_j a nonempty compact metrizable subset of Y_j. For each $i \in I$ and $j \in J$, let $G_j : X \multimap L_j$ and $H_i : Y \multimap X_i$ be multimaps satisfying the following conditions:*

(1) *each G_j is a l.s.c. map with nonempty closed convex values;*

(2) *each H_i is a compact locally selectionable map with convex values.*

If X has the c.c.f.p.p., then there exist $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ and $\overline{y} = (\overline{y}_j)_{j\in J} \in L$ such that $\overline{y}_j \in G_j(\overline{x})$ and $\overline{x}_i \in H_i(\overline{y})$ for each $i \in I$ and $j \in J$.

Proof. Since each H_i is compact, there exists a compact subset K_i of X_i such that $H_i(Y) \subset K_i \subset X_i$. Since $\operatorname{co} K$ is paracompact, by the Michael Theorem 2.6 (L_j being a compact metrizable subset of the completion of F_j), each $G_j|_{\operatorname{co} K} : \operatorname{co} K \multimap L_j$ has a continuous selection $g_j : \operatorname{co} K \to L_j$. Define $g : \operatorname{co} K \to L$ by $g(x) := (g_j(x))_{j\in J}$ for each $x \in \operatorname{co} K$. Since $\operatorname{co} L$ is a paracompact subset of Y, $H_i|_{\operatorname{co} L} : \operatorname{co} L \multimap K_i$ has a continuous selection $h_i : \operatorname{co} L \to K_i$ by Theorem 2.5. Define $h : \operatorname{co} L \to K$ by $h(y) := (h_i(y))_{i\in I}$ for $y \in \operatorname{co} L$. Then $hg : \operatorname{co} K \to K$ is a compact continuous function. Note that $\operatorname{co} K$ is a nonempty convex subset of the t.v.s. E. Therefore, by the c.c.f.p.p., hg has a fixed point $\overline{x} \in \operatorname{co} K \subset X$, that is, $\overline{x} = h(g(\overline{x}))$. Let $\overline{y} := g(\overline{x}) \in L$. Then $\overline{x} = h(\overline{y}) = (h_i(\overline{y}))_{i\in I}$ with $\overline{x}_i = h_i(\overline{y}) \in H_i(\overline{y})$ for each $i \in I$; and $\overline{y}_j = g_j(\overline{x}) \in G_j(\overline{x})$ for each $j \in J$. This completes our proof.

Remark. For the case when (1) each E_i is locally convex and (2) each H_i is a compact Φ-map, Theorem 3.2 reduces to [9, Theorem 3.4].

Theorem 4.3. *Let I and J be any index sets. For each $i \in I$ and $j \in J$, let X_i and Y_j be nonempty convex subsets of locally convex t.v.s. E_i and F_j, resp., and K_i and L_j be nonempty compact metrizable subsets of X_i and Y_j, resp. For each $i \in I$ and $j \in J$, let $G_j : X \multimap L_j$ and $H_i : Y \multimap K_i$ be multimaps satisfying the following conditions:*

(1) *each G_j is a l.s.c. map with nonempty closed convex values;*

(2) *each H_i is a l.s.c. map with nonempty closed convex values.*

Then there exist $\overline{x} = (\overline{x}_i)_{i \in I} \in K$ and $\overline{y} = (\overline{y}_j)_{j \in J} \in L$ such that $\overline{y}_j \in G_j(\overline{x})$ and $\overline{x}_i \in H_i(\overline{y})$ for each $i \in I$ and $j \in J$.

Proof. Since $\mathrm{co}\, K$ is paracompact, each $G_j|_{\mathrm{co}\, K}$ has a continuous selection g_j: $\mathrm{co}\ K \to L_j$ by Theorem 2.6. Define g: $\mathrm{co}\, K \to L$ by $g(x) = (g_j(x))_{j \in J}$ for each $x \in \mathrm{co}\, K$. Moreover, since $\mathrm{co}\, L$ is paracompact, each $H_i|_{\mathrm{co}\, L}$ has a continuous selection $h_i : \mathrm{co}\, L \to K_i$ by Theorem 2.6. Define $h : \mathrm{co}\, L \to K$ by $h(y) = (h_i(y))_{i \in I}$ for each $y \in \mathrm{co}\, L$. Then $hg : \mathrm{co}\, K \to K$ is a compact continuous function on a convex subset $\mathrm{co}\, K$ of the locally convex t.v.s. E, and hence, hg has a fixed point $\overline{x} \in K \subset X$ by Theorem 2.1(2). Then we have the conclusion as in the end of the proof of Theorem 4.2.

Remark. By putting $G_j = \overline{\mathrm{co}}\, S_j$ and $H_i = \overline{\mathrm{co}}\, F_i$, [9, Theorem 3.5] follows from Theorem 4.3. Our proof is quite different from that in [9].

Theorem 4.4. *Let I and J be finite index sets, $\{X_i\}_{i \in I}$ a family of nonempty convex subsets, each in a t.v.s. E_i, and $\{Y_j\}_{j \in J}$ a family of nonempty convex subsets, each in a t.v.s. F_j. For each $i \in I$ and $j \in J$, let $G_j : X \multimap Y_j$ and $H_i : Y \multimap X_i$ be multimaps satisfying the following conditions:*

(1) *each G_j is a compact acyclic map;*

(2) *each H_i is a locally selectionable map with convex values.*

If Y is admissible, then there exist $\overline{x} = (\overline{x}_i)_{i \in I} \in X$ and $\overline{y} = (\overline{y}_j)_{j \in J} \in Y$ such that $\overline{y}_j \in G_j(\overline{x})$ and $\overline{x}_i \in H_i(y)$ for each $i \in I$ and $j \in J$.

Proof. Since each G_j is compact, there exists a compact subset L_j of Y_j such that $G_j(X) \subset L_j$. Since L is compact and $\mathrm{co}\, L$ is paracompact, by Theorem 2.5(2), each $H_i|_{\mathrm{co}\ L}$ has a continuous selection $h_i : \mathrm{co}\, L \multimap X_i$. Let $h : \mathrm{co}\, L \multimap X$ be defined by $h(y) = (h_i(y))_{i \in I}$ for all $y \in Y$. Then h is continuous and each $G_j h : \mathrm{co}\, L \multimap L_j$ is a compact acyclic map. Therefore, by Theorem 3.2, there exists a $\overline{y} \in \mathrm{co}\, L$ such that $\overline{y}_j \in (G_j h)(\overline{y})$ for each $j \in J$. Let $\overline{x} := h(\overline{y}) \in X$. Then $\overline{y}_j \in G_j(\overline{x})$ and $\overline{x}_i = h_i(\overline{y}) \in H_i(\overline{y})$ for all $i \in I$ and $j \in J$. This completes our proof.

Remarks. 1. Note that [9, Theorem 3.7] is a particular form of Theorem 4.4 when (1) each F_j is locally convex, (2) each Y_j is compact, and (3) each H_i is a Φ-map. As is noted in [9], Theorem 4.4 contains the original Fan-Browder fixed point theorem and the Browder coincidence theorem.

2. Theorem 4.4 generalizes Theorem 3.2. In fact, for the case $I = J, K_i = Y_i$, and $H_i = \pi_i : K \to K_i$ (the ith projection), Theorem 4.4 reduces to Theorem 3.2.

Theorem 4.5. *Let I and J be finite index sets. For each $i \in I$, let X_i be a* *nonempty compact metrizable convex subset of a locally convex t.v.s. E_i. For each $j \in J$, let Y_j be a compact convex subset of a t.v.s. F_j. For each $i \in I$ and $j \in J$, let $G_j : X \multimap Y_j$ and $H_i : Y \multimap X_i$ be multimaps satisfying the following:*

(1) *each G_j is a acyclic map;*

(2) *each H_i is a l.s.c. map with nonempty closed convex values.*

If Y is admissible, then there exist $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ and $\overline{y} = (\overline{y}_j)_{j\in J} \in Y$ such that $\overline{x}_i \in H_i(\overline{y})$ for each $i \in I$ and $\overline{y}_j \in G_j(\overline{x})$ for each $j \in J$.

Proof. For each $i \in I$, since X_i is compact in E_i, we may replace each E_i by its completion. Since Y is compact and X_i is metrizable, by the Michael Theorem 2.6, H_i has a continuous selection $h_i : Y \to X_i$. Define $h : Y \to X$ by $h(y) := (h_i(y))_{i\in I} \in X$ for each $y \in Y$. Then for each $j \in J$, $G_j h : Y \multimap Y_j$ is an acyclic map and hence, by Theorem 3.2, there exists a $\overline{y} \in Y$ such that $\overline{y}_j \in (G_j h)(\overline{y})$. Let $\overline{x} := h(\overline{y}) \in X$. Then $\overline{y}_j \in G_j(\overline{x})$ and $\overline{x}_i = h_i(\overline{y}) \in H_i(\overline{y})$ for each $i \in I$ and $j \in J$. This completes our proof.

Remark. Note that [9, Theorem 3.9] is a particular form of Theorem 4.5 under the more strict restriction that each Y_j is a compact convex subset of a locally convex t.v.s. F_j.

By applying Theorem 2.8, we have the following:

Theorem 4.6. *Let I be a finite index set, $\{X_i\}_{i\in I}$ a family of admissible paracompact convex subsets, each in a t.v.s. E_i. Let $T_i : X^i \multimap X_i$ and $H_i : X_i \multimap X^i$ be two families of multimaps satisfying the following conditions:*

(1) *each T_i is a compact acyclic map;*

(2) *each H_i is a locally selectionable map with convex values.*

Then there exist $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ and $\overline{u} = (\overline{u}_i)_{i\in I} \in X$ such that $\overline{x}_i \in T_i(\overline{u}^i)$ and $\overline{u}^i \in H_i(\overline{x}_i)$ for each $i \in I$.

Proof. By Theorem 2.5, each H_i has a continuous selection $h_i : X_i \multimap X^i$. Then $T_i h_i : X_i \multimap X_i$ is a compact acyclic map, and hence, has a fixed point $\overline{x}_i \in X_i$ by Theorem 2.8, for each $i \in I$, that is, $\overline{x}_i \in (T_i h_i)(\overline{x}_i)$. Let $\overline{u}_i = h_i(\overline{x}_i) \in H_i(\overline{x}_i)$. Then $\overline{x}_i \in T_i(\overline{u}_i)$. By letting $\overline{x} := (\overline{x}_i)_{i\in I} \in X$ and $\overline{u} := (\overline{u}_i)_{i\in I} \in X$, we have the conclusion.

Remarks. 1. For the case when (1) each X_i is compact, (2) each H_i is Φ-maps, and (3) each E_i is locally convex, Theorem 4.6 reduces to [9, Theorem 3.11].

2. Theorem 4.6 can be generalized to a compact closed map $T_i \in \mathfrak{B}(X^i, X_i)$ in the better admissible class instead of a compact acyclic map; see [13,18].

From now on, we derive new versions of Section 3 of [10].

Theorem 4.7. *For each $i \in I$, let X_i be a nonempty convex subset of a t.v.s. E_i and let $G_i : X^i \multimap X_i$ and $H_i : X_i \multimap X^i$ be multimaps satisfying*

(a) *each G_i is a compact locally selectionable map with convex values;*

(b) *each H_i is a compact locally selectionable map with convex values.*

If X has the c.c.f.p.p., then there exist $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ and $\overline{u} = (\overline{u}_i)_{i\in I} \in X$ such that $\overline{x}_i \in G_i(\overline{u}^i)$ and $\overline{u}_i \in H_i(\overline{x}^i)$ for all $i \in I$.

Proof. Since each G_i is compact, there exists a compact subset $K_i \subset X_i$ such that $G_i(X^i) \subset K_i$. Let $D_i := \text{co } K_i$ be the paracompact subset of X_i. For each i, let $h_i : D_i \to X^i$ be a continuous selection of $H_i|_{D_i} : D_i \multimap X^i$. Note that $h_i(D_i)$ is contained in a compact subset $K(i)$ of X^i. Let $g_i : K(i) \to K_i$ be a continuous selection of $G_i|_{K(i)}$. Then, for each $i \in I$, the composition $g_i h_i : D_i \multimap K_i \subset D_i$ is a compact continuous function. Therefore, by the c.c.f.p.p., we have an $\overline{x}_i \in D_i \subset X_i$ such that $\overline{x}_i = g_i(h_i(\overline{x}_i))$. Let $\overline{u}^i := h_i(\overline{x}_i) \in H_i(\overline{x}_i)$. Then $\overline{x}_i = g_i(\overline{u}^i) \in G_i(\overline{u}^i)$. This completes our proof.

Remark. For the case when (1) I is finite, (2) each X_i is compact, (3) each E_i is locally convex, and (4) G_i and H_i are Fan-Browder maps (with the same companion maps), Theorem 4.7 reduces to [10, Theorem 3.2].

Theorem 4.8. *Let I be an index set and for each $k \in I$, let J_k be an index set. For each $k \in I$ and $j \in J_k$, let X_{k_j} be a nonempty convex subset of a t.v.s. E_{k_j}. Let $Y_k = \prod_{j\in J_k} X_{k_j}, Y = \prod_{k\in I} Y_k, Y^k = \prod_{l\in I, l\neq k} Y_l$, and $Y = Y^k \times Y_k$. For each $k \in I$ and $j \in J_k$, let $G_{k_j} : Y^k \multimap X_{k_j}$ be a compact locally selectionable map with convex values. If X has the c.c.f.p.p., then there exists $\overline{y} = (\overline{y}_k)_{k\in I} \in Y$ such that $\overline{x}_{k_j} \in G_{k_j}(\overline{y}^k)$ for all $k \in I$ and $j \in J_k$, where $\overline{y}_k = (\overline{x}_{k_j})_{j\in J_k}$ and $\overline{y}^k = (\overline{y}_l)_{l\in I, l\neq k}$.*

Proof. For each $k \in I$, let $T_k : Y = Y^k \times Y_k \multimap Y_k$ by $T_k[y^k, y_k] = \prod_{j\in I_k} G_{k_j}(y^k)$ for each $[y^k, y_k] \in Y$. Then each T_k is a compact locally selectionable map with convex values. Then, by Theorem 3.1, there exists $\overline{y}_k \in T_k(\overline{y})$ for each $k \in I$. Let $\overline{y} := (\overline{y}_k)_{k\in I} \in Y$ and $\overline{y}_k := (\overline{x}_{k_j})_{j\in J_k} \in \prod_{j\in J_k} G_{k_j}(\overline{y}^k)$. Then $\overline{x}_{k_j} \in G_{k_j}(\overline{y}^k)$ and $\overline{y}^k = (\overline{y}_l)_{l\in I, l\neq k}$ for all $k \in I$ and $j \in J_k$. This competes our proof.

Remark. When (1) I and each J_k is finite, (2) each X_{k_j} is compact, and (3) each G_{k_j} is a Fan-Browder map (with the same companion map), Theorem 4.8 holds without assuming the c.c.f.p.p.; see [10, Theorem 3.3] whose proof is based on Theorem 3.5.

Finally, in this section, we give the following variant of the main theorem of [8, Theorem 3.1]:

Theorem 4.9. *Let $\{X_i\}_{i\in I}$ be a family of convex subsets, each in a t.v.s. E_i, and $S_i, F_i : X^i \multimap X_i$ and $H_i, T_i : X_i \multimap X^i$ multimaps for each $i \in I$ satisfying the following conditions:*

(1) *F_i is a Φ-map with the companion map S_i;*

(2) *if X^i is not compact, there exists a nonempty compact subset $K(i)$ of X^i such that, for each finite subset P_i of X_i, there exists a compact convex subset L_{P_i} of X_i containing P_i such that*

$$X^i \setminus K(i) \subset \bigcup \{\text{Int } S_i^-(y_i) \mid y_i \in L_{P_i}\};$$

(3) *T_i is a Φ-map with the companion map H_i;*

(4) *if X_i is not compact, there exists a nonempty compact subset $M(i)$ of X_i such that, for each finite subset Q^i of X^i, there exists a compact convex subset*

L_{Q^i} of X^i containing Q^i such that

$$X_i \setminus M(i) \subset \bigcup \{\text{Int } H_i^-(x^i) \mid x^i \in L_{Q^i}\}.$$

If each X_i has the c.c.f.p.p., then there exists $\overline{x} = (\overline{x}_i)_{i \in I} \in X$ and $\overline{y} = (\overline{y}_i)_{i \in I} \in X$ such that $\overline{y}_i \in F_i(\overline{x}^i)$ and $\overline{x}^i \in T_i(\overline{y}_i)$ for each $i \in I$.

In fact, by choosing P_i such that

$$K(i) \subset \bigcup \{\text{Int } S_i^-(y_i) \mid y_i \in P_i\},$$

condition (2) can be replaced by the following:

(2)′ there exists a compact convex subset $L_i := L_{P_i}$ of X_i such that

$$X^i = \bigcup \{\text{Int } S_i^-(y_i) \mid y_i \in L_i\}.$$

Similarly, by choosing Q^i such that

$$M(i) \subset \bigcup \{\text{Int } H_i^-(x^i) \mid x^i \in Q^i\},$$

condition (4) can be replaced by the following:

(4)′ there exists a compact convex subset $L^i := L_{Q^i}$ of X^i such that

$$X_i = \bigcup \{\text{Int } H_i^-(x^i) \mid x^i \in L^i\}.$$

Proof of Theorem 4.9 with (2)′ *and* (4)′. Note that $F_i(X^i) \subset L_i \subset X_i$ and $T_i(X_i) \subset L^i \subset X^i$, and hence F_i and T_i are compact. Since $F_i|_{L^i} : L^i \multimap L_i$ is a Φ-map, by Theorem 2.2, it has a continuous selection $f_i : L^i \to L_i$. Similarly, $T_i|_{L_i} : L_i \multimap L^i$ has a continuous selection $g_i : L_i \to L^i$. By the c.c.f.p.p., the composition $f_i g_i : L_i \to L_i$ has a fixed point $y_i \in L_i$, that is, $(f_i g_i)(y_i) = f_i(g_i(y_i)) = y_i$ for each $i \in I$. Let $x^i := g_i(y_i) \in L^i \subset X^i$ and $x := (x_i)_{i \in I} \in X$, $y := (y_i)_{i \in I} \in X$. Then $y_i = f_i(x^i) \in F_i(x^i)$ and $x^i = g_i(y_i) \in T_i(y_i)$ for each $i \in I$. This completes our proof.

Remark. In view of conditions (1)-(4), the Φ-maps F_i and T_i in Theorem 4.9 are actually compact locally selectionable maps with convex values. Therefore, Theorem 4.9 is a simple consequence of Theorem 4.7.

5 Remarks on applications

Our new results in this paper are generalizations or improvements of corresponding ones in previous works of other authors. Therefore, their applications also can be improved. We simply indicate, in this section, some of such applications without repeating to state all the details.

In [7], a particular form of Theorem 3.5 is used to obtain (1) results on sets with convex sections and (2) an analytic formulation in the form of a family of inequalities.

In [1], Theorem 3.4 are used to obtain (1) an equilibrium existence theorem for a noncompact abstract economy with arbitrary number of commodities and an infinite number of agents and (2) an existence theorem for a system of variational inequalities (SVI).

In [8], Theorem 4.9 is used to obtain (1) existence theorems of systems of inequalities and (2) systems of minimax theorems.

In [10], particular forms of Theorems 3.5, 4.7, and 4.8 are applied to equilibrium problems with (1) finite families of players and finite families of constraints on strategy sets and (2) finite families of players and two finite families of constraints on strategy sets, under various circumstances.

In [9], particular forms of Theorems 4.1–4.6 are applied to (1) equilibrium problems with m families of players and $2m$ families of constraints on strategy sets which were introduced by Lin et al. [10] and to (2) abstract economies with two families of players.

In [11], a particular form of Theorem 3.4 is applied to (1) maximal element theorems for a family of multimaps, (2) general equilibrium existence results for generalized abstract economies with infinitely many commodities, infinitely many agents, and general preference correspondences, and (3) general minimax inequalities.

References

[1] Q. H. Ansari and J. C. Yao, A fixed point theorem and its applications to the system of variational inequalities, *Bull. Austral. Math. Soc.* **54** (1999), 433–442.

[2] Q. H. Ansari, A. Idzik, and J. C. Yao, Coincidence and fixed point theorems with applications, *Topol. Methods Nonlinear Anal.* **15** (2000), 191–202.

[3] F. Browder, The fixed point theory of multi-valued mappings in topological spaces, *Math. Ann.* **177** (1968), 283–301.

[4] G. Fournier and A. Granas, The Lefschetz fixed point theorem for some classes of non-metrizable spaces, *J. Math. pures et appl.* **52** (1973), 271–284.

[5] O. Hadžić, Fixed Point Theory in Topological Vector Spaces, Univ. of Novi Sad, Novi Sad, 1984, 337pp.

[6] A. Idzik, Almost fixed point theorems, *Proc. Amer. Math. Soc.* **104** (1988), 779–784.

[7] K. Lan and J. Webb, New fixed point theorems for a family of mappings and applications to problems on sets with convex sections, *Proc. Amer. Math. Soc.* **126** (1998), 1127–1132.

[8] L.-J. Lin, System of coincidence theorems with applications, *J. Math. Anal. Appl.* **285** (2003), 408–418.

[9] L.-J. Lin and H.I. Chen, Coicidence theorems for families of multimaps and their applications to equilibrium problems, *Abstr. Appl. Anal.* 2003:5 (2003), 295–309.

[10] L.-J. Lin, S.F. Cheng, X. Y. Liu, and Q. H. Ansari, On the constrained equilibrium problems with finite families of players, *Nonlinear Anal.* **54** (2003), 525–543.

[11] L.-J. Lin, Z.-T. Yu, Q. H. Ansari, and L.-P. Lai, Fixed point and maximal element theorems with applications to abstract economies and minimax inequalities, *J. Math. Anal. Appl.* **284** (2003), 656–671.

[12] E. Michael, A selection theorem, *Proc. Amer. Math. Soc.* **17** (1966), 1404–1406.

[13] S. Park, A unified fixed point theory of multimaps on topological vector spaces, *J. Korean Math. Soc.* **35** (1998), 803–829. Corrections, *ibid.* **36** (1999), 829–832.

[14] S. Park, Continuous selection theorems in generalized convex spaces, *Numer. Funct. Anal. Optimiz.* **20** (1999), 567–583.

[15] S. Park, Ninety years of the Brouwer fixed point theorem, *Vietnam J. Math.* **27** (1999), 193–232.

[16] S. Park, New topological versions of the Fan-Browder fixed point theorem, *Nonlinear Anal.* **47** (2001), 595–606.

[17] S. Park, The KKM principle implies many fixed point theorems, *Topology Appl.* **153** (2004), 197–206.

[18] S. Park, Fixed points of multimaps in the better admissible class, *J. Nonlinear Convex Anal.* **5** (2004), 369–377.

[19] S. Park, Remarks on recent results in analytical fixed point theory, *Nonlinear Anal. and Convex Anal.* (NACA 2005, Okinawa), 517–525, Yokohama Publ., Yokohama, 2007.

[20] S. Park, Fixed point theorems for better admissible multimaps on almost convex sets, *J. Math. Anal. Appl.* **329** (2007), 690–702.

[21] S. Park, Compact Browder maps and equilibria of abstract economies, to appear.