REMARKS ON CONCEPTS OF GENERALIZED CONVEX SPACES

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Abstract. We are mainly concerned with various modifications of the concept of generalized convex spaces. We discuss the nature of those spaces and our proofs of basic theorems in our KKM theory. Finally, we criticize recently obtained generalizations of our works due to other authors.

1. Introduction

In 1993, the author introduced the concept of generalized convex spaces (or G-convex spaces) as a far-reaching generalization of various general convexities without linear structures due to a large number of other authors. We established within such a frame the foundations of the KKM theory initiated by Knaster, Kuratowski, and Mazurkiewicz, as well as fixed point theorems, and many other equilibrium results for multimaps. This direction of study has been followed by several other authors.

In some cases, certain results of these authors are meaningful generalizations of known theorems related to topological vector spaces. However, it is regretful to say that many of their results are mere imitations, sometimes incorrect, of our works. This is mainly because of that they misunderstood the abstract or topological nature of generalized convex spaces, and that, in order to claim their own results are more general than known ones, they introduce unnecessarily artificial concepts or terminology and pretend to obtain new results. In many cases, they quote the present author's works for only borrowing basic ideas and they do not check or compare theirs with known results. Moreover, it is rather

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surprising that they could not supply any single proper example of generalized convex spaces other than the spaces listed by the present author.

In our previous work [P7], we could destroy many of artificial terminology adopted by other authors in the KKM theory, by giving finer topology (or extensions of given topology) on the underlying space. Moreover, in a forthcoming work [P10], we showed that a number of fixed point theorems or other results related to generalized convex spaces appeared in other author's works are simple consequences of known results due to the present author.

In this paper, we are mainly concerned with the concept of generalized convex spaces. In Section 2, we discuss the nature of those spaces and, in Section 3, show that basic theorems in our KKM theory for those spaces were proved implicitly by replacing $\Gamma_A = \Gamma(A)$ by corresponding subsets $\phi_A(\Delta_n)$ whenever $|A| = n + 1$. Section 4 deals with Saveliev's example [S] of the Michael convex structure [M], which is shown to be a generalized convex space, contrary to his claim. Sections 5, 6, and 7 are devoted to discuss recent works in [SL], [DX], and [F], respectively. Those authors' claims to obtain generalizations of our works are essentially not proper.

2. Preliminaries

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$; and for any $B \in Y$, $F^-(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}$. Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with its cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A = \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J = \Gamma(J)$.

Note that Δ_n is an n-simplex with vertices $v_0, v_1, \cdots, v_n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$.

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$ (or $\phi_N(\Delta_{|N|-1}) \subset Y$, if you prefer).

Examples of G-convex spaces can be found in [P3,5,6,9, PK1] and references therein.

Now, we discuss on the nature of generalized convex spaces, and, more precisely, we give remarks on the relation $\phi_A(\Delta_J) \subset \Gamma_J = \Gamma(J)$ in the concept of generalized convex spaces as follows:

(I) It should be noted that ϕ_A depends on $A \in \langle D \rangle$. Hence, for example, for any A, J in $\langle D \rangle$, if $A \supset J$ as above, $\phi_A|_{\Delta_J} : \Delta_J \to \Gamma(J)$ might be different from $\phi_J : \Delta_k \multimap \Gamma(J)$. In case when the family $\{\phi_A\}_{A \in \langle D \rangle}$ can be so chosen that for any $A \supset J$ as above, we have

$$\phi_A\left(\sum_{j=0}^{k} \lambda_j v_{i_j}\right) = \phi_J\left(\sum_{j=0}^{k} \lambda_j v_j\right) \text{ for any } \lambda_j \geq 0, \ \sum_{j=0}^{k} \lambda_j = 1,$$

then we can put $\Gamma(A) = \phi_A(\Delta_n)$ for all $A \in \langle D \rangle$.

We give some examples of Case (I) as follows; see [PK1, P6].

(1) For any convex subset X of a t.v.s., Lassonde's convex space X, and Park's convex space (X, D), it is assumed that

$$\Gamma(A) = \text{co } A := (\text{the convex hull of } A)$$

for all $A \in \langle X \rangle$ or $A \in \langle D \rangle$. If $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to$ co A such that $\Gamma_A = \text{co } A = \phi_A(\Delta_n)$ and $\phi_A(\Delta_J) = \Gamma_J = \Gamma(J)$ for $J \in \langle A \rangle$.

(2) Similarly, the following type of G-convex spaces are defined, from the beginning, to satisfy as above as in (I): Komiya's convex space, Bielawski's simplicial convexity (which includes Michael's convex structure), Joó's pseudoconvexity, Kulpa's simplicial space, $P_{1,1}$-space of Forgo and Joó, and Llinares' MC-space.

(3) If D is finite in a G-convex pace $(X, D; \Gamma)$, then, by defining

$$\Gamma'(A) := \phi_D(\Delta_A) \subset \Gamma(A) \quad \text{for all} \quad A \in \langle D \rangle,$$

$(X, D; \Gamma')$ becomes a new G-convex space satisfying the requirements of (I).

(II) Horvath's c-space or H-space $(X; F)$ is a very unique G-convex space having a large number of examples where $F(A)$ is contractible (or more generally, ω-connected) for $A \in \langle X \rangle$; for the literature, see [PK1, P6] and references therein. Moreover, Park's H-space $(X, D; F)$, any homeomorphic image of a convex space, and any continuous image of a c-space are G-convex spaces. For such a H-space we have

$$\phi_A(\Delta_J) \subset F_J = F(J) \quad \text{for} \quad J \in \langle A \rangle$$

and $\phi_A(\Delta_J) \neq F(J)$ in general, since any continuous image of a simplex is not necessarily contractible (nor ω-connected). However, for any H-space $(X, D; F)$ and any $N \in \langle D \rangle$, we have a new G-convex space $(X, N; \Gamma)$ defined by

$$\Gamma_J = \Gamma(J) := \phi_N(\Delta_J) \subset F(J) \quad \text{for each} \quad J \in \langle N \rangle.$$

(III) We should recognize that, in the KKM theory of G-convex spaces, every argument is related to the finite intersection property of functional values of KKM maps, in other words, related to some $N \in \langle D \rangle$, in $(X, D; \Gamma)$. Therefore, the argument can be switched to the one for $(X, N; \Gamma')$ where

$$\Gamma'_J = \Gamma'(J) := \phi_N(\Delta_J) \subset \Gamma_J \quad \text{for all} \quad J \in \langle N \rangle.$$

Recently, there have appeared some authors who tried to rewrite our works on G-convex spaces by replacing $\Gamma(A)$ by $\phi_A(\Delta_n)$ everywhere and claimed to obtain generalizations.

We will give more details in Sections 3 and 5–7.

3. Basic theorems in KKM theory

In our KKM theory on generalized convex spaces, there exist some basic theorems from which we can deduce several equivalent formulations that can be used to applications. In this section, we show that such basic theorems were proved implicitly by replacing $\Gamma_A = \Gamma(A)$ by corresponding subsets $\phi_A(\Delta_n)$ whenever $|A| = n + 1$.

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if, for each $N \in \langle D \rangle$, we have

$$\Gamma_N \subset F(N).$$

The following is our basic KKM theorem for G-convex spaces [P5, PL]:

Theorem A. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a multimap such that*

(1) *F has closed* [*resp. open*] *values*; *and*

(2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property. Furthermore, if

(3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have

$$\bigcap_{z \in D} \overline{F(z)} \neq \emptyset.$$

It is clear that Theorem A is equivalent to the following:

Theorem A$'$. *Let $(X, N; \Gamma)$ be a G-convex space with $|N| = n+1$ and $F : N \multimap X$ a multimap such that*

(1) *F has closed [resp. open] values; and*
(2) *F is a KKM map.*

Then we have $\bigcap_{z\in N} F(z) \neq \emptyset$.

In fact, we can put $\Gamma_J = \phi_N(\Delta_J)$ for each $J \in \langle N\rangle$, and F is KKM means $\phi_N(\Delta_J) \subset F(J)$ for each $J \in \langle N\rangle$.

Moreover, replacing X by $\phi_N(\Delta_n)$ and F by $F' : N \multimap \phi_N(\Delta_n)$ defined by $F'(z) := F(z) \cap \phi_N(\Delta_n)$, we have the conclusion

$$\bigcap_{z\in N} F'(z) = \bigcap_{z\in N} F(z) \cap \phi_N(\Delta_n) = \phi_N(\Delta_n) \cap \bigcap_{z\in N} F(z) \neq \emptyset.$$

In our previous work [P5], Theorem A is used to obtain more than 17 equivalent formulations [P5, Theorems 1-17 and Corollaries] which consist of elements of KKM theory, fixed point theory, equilibria theory, and others. Note that, in proofs of each formulations, we used certain "Γ_J-argument" except in Theorem A [P5, Theorem 1] for which we used "$\phi_N(\Delta_J)$-argument."

From Theorem A, we have the following Fan-Browder type fixed point theorem:

Theorem B. *Let $(X, N; \Gamma)$ be a G-convex space, N a finite set, and $S : N \multimap X$, $T : X \multimap X$ two maps satisfying*

(1) *for each $z \in N$, $S(z)$ is open [resp. closed];*
(2) *for each $y \in X$, $J \in \langle S^-(y)\rangle$ implies $\Gamma_J \subset T^-(y)$; and*
(3) *$X = S(N)$.*

Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

In view of Theorem A$'$, Γ_J in Theorem B can be replaced by $\phi_N(\Delta_J)$ without affecting its conclusion.

Theorem B is essentially obtained early in 1997 [P2] and applied in [P8] to various forms of the Fan–Browder theorem, the Ky Fan intersection theorem, and the Nash equilibrium theorem for G-convex spaces.

From Theorem B, we deduced the following result [P8]:

Theorem B′. *Let $(X \supset N; \Gamma)$ be a G-convex space, $N = \{z_i\}_{i=0}^n$, and $A : X \multimap X$ a multimap such that $A(x)$ is Γ-convex for each $x \in X$. If there exist nonempty open [resp. closed] subsets $G_i \subset A^-(z_i)$ for each $i = 0, 1, 2, \cdots, n$ such that $X = \bigcup_{i=0}^n G_i$, then A has a fixed point.*

For a G-convex space $(X \supset D; \Gamma)$, a subset Y of X is called a *G-convex subspace* of $(X \supset D; \Gamma)$ if $(Y, Y \cap D; \Gamma')$ is a G-convex space where $\Gamma'_A := \Gamma_A \cap Y$ for $A \in \langle Y \cap D \rangle$. Note that for a G-convex space $(X; \Gamma)$, any nonempty Γ-convex subset of X is a G-convex subspace.

The following in [P7] is now a simple consequence of Theorem B′:

Theorem C. *Let $(X \supset D; \Gamma)$ be a G-convex space, K a nonempty subset of X, and $S : X \multimap D$, $T : X \multimap X$ multimaps. Suppose that*

(1) *for each $x \in X$, $J \in \langle S(x) \rangle$ implies $\Gamma_J \subset T(x)$;*
(2) *$K \subset \bigcup \{\text{Int}\, S^-(z) \,|\, z \in N\}$ for some $N \in \langle D \rangle$; and*
(3) *there exists a G-convex subspace L_N of X containing N such that*

$$L_N \backslash K \subset \bigcup \{\text{Int}\, S^-(z) \mid z \in M\}$$

for some $M \in \langle L_N \cap D \rangle$.

Then T has a fixed point in L_N.

Note that, replacing X, N, A, G_i in Theorem B′ by L_N, $M \cup N$, T, Int $S^-(z_i)$, respectively, we obtain Theorem C. Note also that Γ_J in Theorem C can be replaced by $\phi_{M \cup N}(\Delta_J)$ without affecting its conclusion.

The following particular form of Theorem C is essentially known in 1996 [PK1]:

Theorem C′. *Let $(X \supset D; \Gamma)$ be a G-convex space, K a nonempty compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ multimaps. Suppose that*

(1) *for each $x \in X$, $J \in \langle S(x) \rangle$ implies $\Gamma_J \subset T(x)$;*
(2) *$K \subset \bigcup \{\text{Int } S^-(z) \mid z \in D\}$; and*
(3) *for each $N \in \langle D \rangle$, there exists a compact G-convex subspace L_N of X containing N such that*

$$L_N \backslash K \subset \bigcup \{\text{Int } S^-(z) \mid z \in L_N \cap D\}.$$

Then T has a fixed point.

Recall that Theorem C′ contains a large number of particular forms due to other authors and has been a main target of a large number of fake generalizations or imitations due to other authors.

The following was the basis of our theory in [PK1,2]:

Theorem D. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$ maps, and $F \in \mathfrak{A}_c^{\kappa}(X, Y)$. Suppose that*

(1) *for each $x \in D$, $S(x)$ is compactly open in Y;*
(2) *for each $y \in F(X)$, $J \in \langle S^-(y)\rangle$ implies $\Gamma_J \subset T^-(y)$;*
(3) *there exists a nonempty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*
(4) *either*
(i) *$Y \backslash K \subset S(M)$ for some $M \in \langle D\rangle$; or*
(ii) *$X \supset D$ and for each $N \in \langle D\rangle$, there exists a compact G-convex subspace L_N of X containing N such that $F(L_N)\backslash K \subset S(L_N \cap D)$.*

Then there exists an $\overline{x} \in X$ such that $F(\overline{x}) \cap T(\overline{x}) \neq \emptyset$.

Here $\mathfrak{A}_c^{\kappa}$ denotes the admissible class of multimaps due to Park [P1, PK1,2], and if F is single-valued, then Y is not necessarily Hausdorff in Theorem D. Note that, by closely examining the proof of Theorem D, we notice that, certain Γ_J in that proof can be replaced by $\phi_A(\Delta_J)$ with respect to some $A \in \langle D\rangle$. Moreover, in [PK2], there have appeared nine equivalent formulations of Theorem D [PK1,2, Theorem 1] which would be the basis of other applications.

Note that 'compactly open' in Theorem 5 can be replaced simply by 'open', by adopting the compactly generated extension of the original topology on Y.

In [PK2, p.556], the following was essentially given:

Corollary. *Let $(X, N; \Gamma)$ be a G-convex space, N a finite set, Y a Hausdorff space, $F \in \mathfrak{A}_c^{\kappa}(X, Y)$, and $H : N \multimap Y$ such that, for any $J \in \langle N\rangle$, $F(\Gamma_J) \subset H(J)$. Then we have*

$$\bigcap_{z\in N} \overline{H(z)} \neq \emptyset \quad [\textit{or } \phi_N(\Delta_n) \cap \bigcap_{z\in N} \overline{H(z)} \neq \emptyset].$$

The above corollary was the origin of considering the so-called KKM class $\mathfrak{K}(X, Y)$ of multimaps.

Theorem D can be simplified and generalized as follows [P8,9, PK3]:

Theorem D′. *Let $(X, N; \Gamma)$ be a G-convex space, N finite, Y a Hausdorff space, $S : N \multimap Y$, $T : X \multimap Y$, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1) *S has open [resp. closed] values;*
(2) *for each $y \in F(X)$, $J \in \langle S^-(y)\rangle$ implies $\Gamma_J \subset T^-(y)$; and*
(3) *$Y = S(N)$.*

Then there is a point $x_* \in X$ such that $F(x_*) \cap T(x_*) \neq \emptyset$.

A form of Theorem D′ was obtained for $F \in \mathfrak{K}(X, Y)$ and $Y = \overline{F(X)}$ when S has open values: see [P8].

Now the readers are sure to replace Γ_J by $\phi_N(\Delta_J)$ for each $J \in \langle N\rangle$, and that such replacements do not mean any generalizations. So our theory on generalized convex spaces is still firmly standing.

4. On the Michael convex structure

In a sequence of the author's works on generalized convex spaces, as an example of them, we listed a metric space with the Michael convex structure [M].

However, in a recent work, Saveliev [S] argued against that the Michael convex structures are generalized convex spaces. But we show that his counter-example is just a trivial Michael convex structure and so a generalized convex space.

The Michael convex structure [M] is given as follows:

If Y is any set, and $i \leq n$, then $\partial_i : Y^{n+1} \to Y^n$ is defined by

$$\partial_i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

A *convex structure* on a metric space Y with metric ρ assigns each non-negative integer n to a subset M_n of Y^{n+1} and a function $k_n : M_n \times \Delta_n \to Y$ such that

(1) if $x \in M_0$, then $k_0(x, 1) = x$;
(2) if $x \in M_n$ $(n \geq 1)$ and $i \leq n$, then $\partial_i x \in M_{n-1}$ and, for any $t \in \Delta_n$ with $t_i = 0$, $k_n(x, t) = k_{n-1}(\partial_i x, \partial_i t)$;
(3) if $x \in M_n$ $(n \geq 1)$ with $x_i = x_{i+1}$ for some $i < n$, and if $t \in \Delta_n$, then $k_n(x, t) = k_{n-1}(\partial_i x, t^*)$, where $t^* = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_n)$;
(4) if $x \in M_n$, then the function $t \mapsto k_n(x, t)$, from Δ_n to Y, is continuous; and
(5) for all $\epsilon > 0$ there exists a neighborhood V_ϵ of the diagonal in $Y \times Y$ such that, for all n and all $x, y \in M_n$, $(x_i, y_i) \in V_\epsilon$ for $i = 0, \ldots, n$ implies $\rho(k_n(x, t), k_n(y, t)) < \epsilon$ for all $t \in \Delta_n$.

For a subset S of a space Y with convex structure is *admissible* if $S^{n+1} \subset M_n$ for all nonnegative integer n. If S is admissible, then the *convex hull* of S, denoted by $\mathrm{conv}(S)$, is

$$\{k_n(x,t) \,|\, x \in S^{n+1},\ t \in \Delta_n,\ n = 0,1,2,\dots\}.$$

Saveliev claimed that the following example has a nontrivial Michael convex structure:

$Y = M_0 = \{0,1\} \subset \mathbf{R}$, $M_n = \emptyset$ for $n \geq 1$, $k_0(x,t) = x$ for all $x \in Y$, $t \in [0,1]$.
It is obvious that $k_n(x,t) = \emptyset$ for all $x \in Y$, $t \in [0,1]$ and $n \geq 1$.

But in this example, there is no admissible set in Y except $\emptyset$. Hence it only has a trivial Michael convex structure.

By putting $\Gamma_A = \mathrm{co}\, A$ for all $A \in \langle Y \rangle$ satisfying $A^{n+1} \subset M_n$ for all $n \in \mathbf{N} \cup \{0\}$ (or $n+1 \leq |A|$), a metric space Y with the convex structure becomes a G-convex space $(Y;\Gamma)$.

Recall that the Michael convex structure was shown by Bielawski to be his simplicial convexity, which is an example of a G-convex space; see [PK1].

5. On the L-spaces

At first, we assumed an additional condition that

$(*)$ for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$

in the definition of our generalized convex spaces. This monotonicity was removed since 1998. However, until now, no one showed any example of a G-convex space which does not satisfy $(*)$.

In the same year, Ben-El-Mechaiekh et al. [BC] defined an L-space (E,Γ), which is a particular form of our G-convex space $(X,D;\Gamma)$ [without assuming $(*)$] for the case $E = X = D$. The authors of [BC] incorrectly claimed that their class of L-spaces contains our class of G-convex spaces. This is followed by [D1,2], which contain a number of particular results (with certain defects) of known ones.

In 2003, the authors of [SL] considered the same particular form of our G-convex spaces as follows (where $\mathfrak{F}(X) = \langle X \rangle$):

Definition 1. [SL] *Let X be a topological space, X has an L-*structure *if there exists a non-empty valued correspondence $\Psi : \mathfrak{F}(X) \to X$ and for all $B \in \mathfrak{F}(X)$, namely $B =$*

$\{b_0, b_1, \cdots, b_n\}$, *there exists a continuous function* $f^B : \Delta_n \to \Psi(B)$ *such that for all* $J \subseteq \{0, 1, \cdots, n\}$, $f^B(\Delta_J) \subseteq \Psi(\{b_i : i \in J\})$*. The pair* (X, Ψ) *is called an* L-space.

From now on, in this section, we follow [SL]:

Definition 2. [SL] *Let* X *and* Y *be topological spaces such that* X *has an* L*-structure defined by* $\Psi : \mathfrak{F}(X) \to X$ *and by* $f^B : \Delta_n \to \Psi(B)$ *for each* $B \in \mathfrak{F}(X)$*. A correspondence* $\Gamma : Y \to X$ *is said to be a* generalized KKM-correspondence*, if for all* $\{y_0, y_1, \cdots, y_n\} \in \mathfrak{F}(Y)$*, there exists a subset* $B = \{x_0, x_1, \cdots, x_n\} \in \mathfrak{F}(X)$*, such that for all* $J \subseteq \{0, 1, \cdots, n\}$*, it is satisfied that* $f^B(\Delta_J) \subseteq \bigcup_{j \in J} \Gamma(y_j)$*.*

Note that a generalized KKM-correspondence becomes simply our KKM by putting $D := Y$ and, for any $A \in \langle D \rangle$, by defining $\phi_A(\Delta_{|A|-1}) := f^B(\Delta_{|B|-1})$ and $\Gamma_A := \Psi(B)$ for $B \in \langle X \rangle$ corresponding to A.

Definition 3. [SL] (Transfer closedness) *Let* X *and* Y *be topological spaces. A correspondence* $\Phi : Y \to X$ *is said to be* transfer closed-valued *on* Y *if for every* $y \in Y$ *and* $x \in X$*, if* $x \notin \Phi(y)$ *then there exists* $y' \in Y$ *such that* $x \notin cl[\Phi(y')]$*.*

Theorem 1. [SL] *Let* X *and* Y *be topological spaces and* $\Gamma : Y \to X$ *a transfer closed-valued correspondence on* Y *such that there exists* $y^* \in Y$ *with* $cl[\Gamma(y^*)]$ *compact. Then, there exists an* L*-structure on* X *such that* Γ *is a generalized KKM-correspondence if and only if* $\bigcap_{y \in Y} \Gamma(y) \neq \emptyset$*.*

Note that $\Phi : Y \multimap X$ is transfer-closed means simply

$$\bigcap_{y \in Y} \overline{\Phi(y)} = \bigcap_{y \in Y} \Phi(y)$$

Note also that the necessity part follows simply from Theorem A′; see [P7, PL]. The sufficiency part is rather trivial and given in [SL] as follows:

"Conversely, if we assume that $\bigcap_{y \in Y} \Gamma(y) \neq \emptyset$, then we can take $x^* \in \bigcap_{y \in Y} \Gamma(y)$ and define an L-structure on X as follows: $\Psi : \mathfrak{F}(X) \to X$ is given by the constant function $\Psi(B) = x^*$ for all $B = \{b_0, b_1, \cdots, b_n\} \in \mathfrak{F}(X)$, and function $f^B : \Delta_n \to \Psi(B)$ is defined as $f^B(\lambda) = x^*$ for all $\lambda \in \Delta_n$. Then it is easy to verify that, with this L-structure, Γ is a generalized KKM-correspondence."

Recall that several generalizations of Theorem 1 [SL] have already appeared in [PL].

6. Generalized R-KKM maps

Recently, in [DX], the following definition appears:

Definition 2.1 [DX] Let X be a nonempty set and Y be a topological space. $T : X \to 2^Y$ is said to be generalized relatively KKM (R-KKM) mapping if for any $N = \{x_0, x_1, \cdots, x_n\} \in \langle X \rangle$, there exists a continuous mapping $\phi_N : \Delta_n \to Y$ such that, for each $e_{i_0}, e_{i_1}, \cdots, e_{i_k}$,

$$\phi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

where Δ_k is a standard k-simplex of Δ_n with vertices $e_{i_0}, e_{i_1}, \cdots, e_{i_k}$.

The authors claim that their definition unifies and extends a lot of similar definitions due to other authors. However, note that $(Y, N; \Gamma)$ is a G-convex space where $\Gamma(J) = \phi_N(\Delta_J)$ for $J \subset N$ and that $T : N \multimap Y$ becomes simply a KKM map.

The following is the key result in [DX] with almost one page proof:

Theorem 3.1. [DX] *Let X be a nonempty set and Y be a topological space. Let $T : X \to 2^Y$ be a set-valued mapping such that $T(x)$ is nonempty and compactly closed in Y for each $x \in X$.*

(i) *If T is a generalized R-KKM mapping, then for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle X \rangle$,*

$$\phi_N(\Delta_n) \cap \left(\bigcap_{x \in N} Tx \right) \neq \emptyset,$$

where ϕ_N is the continuous mapping in touch with N in definition of a generalized R-KKM mapping.

(ii) *If the family $\{T(x) : x \in X\}$ has finite intersection property, then T is a generalized R-KKM mapping.*

Note that (i) is a simple consequence of our Theorem A and (ii) is a simple observation as we remarked for Theorem 1 [SL].

We give a simple transparent proof as follows:

Proof. Switch the topology of Y to its compactly generated extension. Then we can eliminate 'compactly'.

(i) For each vertex e_i of Δ_n, we have $\phi_N(e_i) \in Tx_i$ for $0 \leq i \leq n$. Then $e_i \mapsto \phi_N^{-1} Tx_i$ is a closed-valued map such that $\Delta_k = \text{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1} Tx_{i_j}$ for each face

Δ_k of Δ_n. Therefore, by the original KKM theorem, $\Delta_n \supset \bigcap_{i=0}^n \phi_N^{-1} Tx_i \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \left(\bigcap_{x \in N} Tx\right) \neq \emptyset$.

(ii) Just follow the sufficiency part of Theorem 1 [SL] (the " " part in Section 5).

Remark. In [DX], its authors used the partition of unity subordinated to a cover of $\phi_{N_0}(\Delta_n)$ which should be assumed Hausdorff. They claim that, applying their Theorem 3.1, they obtained new theorems which unify and extend many known results in recent literature. However, theirs are all disguised forms of known results and their applicability is doubtful.

7. Generalized KKM maps

Motivated by a large number of recent works on generalized KKM maps, we introduced the following definition in [PL]:

Let (X, D, Γ) be a G-convex space and I a nonempty set. A map $F : I \multimap X$ is called a *generalized KKM map* provided that for each $N \in \langle I \rangle$, there exists a function $\sigma : N \to D$ such that $\Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle N \rangle$.

In [PL], a unified account on results for such maps was given; for example, the KKM type theorem, characterizations of such maps, an equilibrium theorem implying minimax inequalities, variational inequalities, and so on.

A little later than [PL], similar results appeared in [D], which has trivial defects in certain aspects.

More recently, in [F], its author introduced variants of several concepts in the KKM theory on generalized convex spaces as follows:

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \to 2^X$ is called a Φ-*KKM map* if, for each $A \in \langle D \rangle$, we have

$$\phi_A(\Delta_{|A|-1}) \subset F(A).$$

Similarly, for a nonempty set Y, a *generalized* Φ-*KKM map* $F : Y \to 2^X$ is defined.

Moreover, F is called *finitely G-closed* (*resp. open*) if for every finite subset B of D and $y \in Y$, $F(y) \cap G\text{-co}\, B$ is closed (resp. open) in

$$G\text{-co}\, B := \bigcap \{A \subset X : A \text{ is a } \Gamma\text{-convex subset of } X \text{ containing } B\}.$$

Further, F is called *finitely* Φ-*closed* (*resp, open*) if for every finite subset B of D with $|B| = n + 1$ and $y \in Y$, $F(y) \cap \phi_B(\Delta_n)$ is closed (resp, open) in $\phi_B(\Delta_n)$.

The author of [F] claims that these notions generalize that F has closed (resp, open) values.

Let a G-convex space $(X, D; \Gamma)$, a Hausdorff space Y, and a multimap $F : X \to 2^Y$ be given. A multimap $G : X \to 2^Y$ is called a *$F\Phi$-KKM map* if for any finite subset A of D, we have $F\phi_A(\Delta_{|A|-1}) \subseteq G(A)$. A multimap $G : D \to 2^Y$ is called a *generalized $F\Phi$-KKM map* if for any subset $A = \{x_0, \ldots, x_n\}$ of D, there exists a finite subset $B = \{y_0, \ldots, y_n\}$ of D, not necessarily all different, such that: for each $\{i_0, \ldots, i_j\}$ we have $F\phi_B(\Delta_J) \subseteq \bigcup_{k=0}^{j} G(x_{i_k})$. G is called *finitely $F\phi$-closed* (*resp. open*) *valued* if for any finite subset $A = \{x_0, \ldots, x_n\} \subseteq D$ and each $x \in D$, the set $G(x) \cap F\phi_A(\Delta_n)$ is closed (resp. open) in $F\phi_A(\Delta_n)$

Let Y be a topological space. A map $F : X \to 2^Y$ is said to have *the generalized Φ-KKM property* if, for any map $G : D \to 2^Y$ with compactly closed values which is a generalized $F\Phi$-KKM map the class $\{G(x) : x \in D\}$ has the finite intersection property. We denote

$$\mathfrak{K}(X, Y) := \{F : X \to 2^Y |\ F \text{ has the generalized } \Phi\text{-KKM property}\}.$$

The following are typical examples of results in [F]:

Theorem 4.5.1. [F] *Let $(X, D; \Gamma)$ be a G-convex space, $S : D \to 2^Y$, $T : X \to 2^Y$, and $F \in \mathfrak{K}(X, Y)$. Suppose that.*

1. *S has compactly open values,*
2. *for each $y \in F(X)$, any finite subset $A = \{x_0, \ldots, x_n\} \subseteq S^-(y)$ and any subset $J \subseteq A \cap S^-(y)$, we have $\phi_A(\Delta_J) \subseteq T^-(y)$,*
3. *$clF(X) \subseteq S(M)$ for some finite subset M of D.*

Then F and T have a coincidence point $x_0 \in X$; that is, $F(x_0) \cap T(x_0) \neq \emptyset$.

Theorem 4.5.5. [F] *Let $(X, D; \Gamma)$ be a G-convex space, $S : D \to 2^X$, $T : X \to 2^X$ two multimaps satisfying*

1. *S is finitely Φ-closed (resp. open) values,*
2. *$X = S(M)$ for some finite subset M of D,*
3. *for each $x \in X$, any finite subset $A \subseteq D$ and any subset $J \subseteq A \cap S^-(x)$, we have $\phi_A(\Delta_J) \subseteq T^-(x)$.*

Then T has a fixed point.

These two results are comparable to (a form of) Theorem D′ and Theorem B, respectively. We reserve any comments on [F], since every reader would have some comments to the new notions and the above results in [F].

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