# COMMENTS ON SOME FIXED POINT THEOREMS ON GENERALIZED CONVEX SPACES

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Abstract. We show that a number of fixed point theorems or other results related to generalized convex spaces appeared in other author's works are simple consequences of known results.

## 1. Introduction

In 1993, the author introduced the concept of generalized convex spaces (or $G$-convex spaces) as a far-reaching generalization of various general convexities without linear structures due to a large number of other authors. We established within such a frame the foundations of the KKM theory initiated by Knaster, Kuratowski, and Mazurkiewicz, as well as fixed point theorems, and many other equilibrium results for multimaps. This direction of study has been followed by several other authors in a large number of publications.

In some cases certain results of these authors are meaningful generalizations of known theorems related to topological vector spaces. However, it is regretful to say that some of their results are mere imitations, sometimes incorrect, of our works. This is mainly because of that they misunderstood the abstract or topological nature of generalized convex spaces, and that, in order to claim their own results are more general than known ones, they introduce unnecessarily artificial concept or terminology and pretend to obtain new results. In certain cases, they quote the present author's works for only borrowing basic ideas and they do not check or

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Received

2000 *Mathematics Subject Classification.* Primary 47H04, 47H10; Secondary 52A07, 54H25.

*Key words and phrases.* Generalized convex ($G$-convex) space, u.s.c. multimap, $\Gamma$-convex (hull), $G$-convex subspace.

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

compare theirs with known results. Moreover, it is rather surprising that they could not supply any single proper example of generalized convex spaces other than the spaces listed by the present author.

In this paper, we show that a number of fixed point theorems or other results related a generalized convex spaces appeared in other author's works are simple consequences of our previous works. We confine ourselves only the papers appeared in the new millennium and give comments mainly on fixed point results.

## 2. Preliminaries

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set $X$ into the power set $2^Y$ of $Y$; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$. Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context.

For topological spaces $X$ and $Y$, a multimap $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) [resp. *lower semicontinuous* (l.s.c.)] if for each closed [resp. open] set $B \subset Y$, $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$ is closed [resp. open] in $X$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set $D$.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space $X$, a nonempty set $D$, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ having $n+1$ elements, there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J := \Gamma(J)$.

Note that $\Delta_n$ is an $n$-simplex with vertices $v_0, v_1, \cdots, v_n$, and $\Delta_J$ the face of $\Delta_n$ corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$.

It should be noted that $\phi_A$ depends on $A \in \langle D \rangle$. Hence, for example, for any $A, J$ in $\langle D \rangle$, if $A \supset J$ as above, $\phi_A|_{\Delta_J} : \Delta_J \to \Gamma(J)$ might be different from $\phi_J : \Delta_k \multimap \Gamma(J)$. In case when the family $\{\phi_A\}_{A \in \langle D \rangle}$ can be so chosen that for any $A \supset J$ as above, we have

$$\phi_A\left(\sum_{j=0}^{k} \lambda_j v_{i_j}\right) = \phi_J\left(\sum_{j=0}^{k} \lambda_j v_j\right) \text{ for any } \lambda_j \geq 0, \ \sum_{j=0}^{k} \lambda_j = 1;$$

then we can take $\Gamma(A) = \phi_A(\Delta_n)$ for all $A \in \langle D \rangle$. One of such cases is that if $A = D$ is finite, then we may put $\Gamma(J) = \phi_A(\Delta_J)$ for all $J \subset A$. Some authors try to rewrite our works in such a way.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$.

For a $G$-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be $\Gamma$-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$ (or $\phi_N(\Delta_{|N|-1}) \subset Y$, if you prefer); and for a nonempty set $Z \subset X$, its $\Gamma$-*convex hull* is defined by

$$\Gamma\text{-co}(Z) := \bigcap\{A \subset X \mid A \text{ is a } \Gamma\text{-convex subset of } X \text{ containing } Z\}.$$

For a $G$-convex space $(X; \Gamma)$ and a subset $Z \subset X$, we have

$$\Gamma\text{-co}(Z) = \bigcup\{\Gamma\text{-co}(A) \mid A \in \langle Z \rangle\}.$$

Examples of $G$-convex spaces can be found in [P2,4,7] and references therein.

For a $G$-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if, for each $N \in \langle D \rangle$, we have

$$\Gamma_N \subset F(N), \;\; (\text{or } \; \phi_N(\Delta_{|N|-1}) \subset F(N), \;\; \text{if you prefer}).$$

The following is our basic KKM theorem for $G$-convex spaces [P4,PL]:

**Theorem A.** *Let $(X, D; \Gamma)$ be a $G$-convex space and $F : D \multimap X$ a multimap such that*

(1) *$F$ has closed* [*resp. open*] *values*; *and*
(2) *$F$ is a KKM map.*

*Then $\{F(z)\}_{z \in D}$ has the finite intersection property. Furthermore, if*

(3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*

*then we have*

$$\bigcap_{z \in D} \overline{F(z)} \neq \emptyset.$$

From the closed version of Theorem A, we can deduce the following equivalent formulation:

**Theorem A′.** *Let $(X, D; \Gamma)$ be a $G$-convex space and $F : D \multimap X$ a map such that*

(1) *$\bigcap_{z \in D} \overline{F(z)} = \bigcap_{z \in D} F(z)$* [*$F$ is transfer closed-valued*];
(2) *$\overline{F}$ is a KKM map*; *and*
(3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$.*

*Then we have $\bigcap_{z \in D} F(z) \neq \emptyset$.*

From Theorem A, we have the following Fan-Browder type fixed point theorem:

**Theorem B.** *Let $(X, D; \Gamma)$ be a $G$-convex space, and $S : D \multimap X$, $T : X \multimap X$ two maps satisfying*

(1) *for each $z \in D$, $S(z)$ is open [resp. closed];*
(2) *for each $y \in X$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$; and*
(3) *$X = S(N)$ for some $N \in \langle D \rangle$.*

*Then $T$ has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

Theorem B is obtained in [P5] and applied to give various forms of the Fan–Browder theorem, the Ky Fan intersection theorem, and the Nash equilibrium theorem for $G$-convex spaces.

From Theorem B, we deduced the following new result [P5]:

**Theorem C.** *Let $(X \supset D; \Gamma)$ be a $G$-convex space and $A : X \multimap X$ be a multimap such that $A(x)$ is $\Gamma$-convex for each $x \in X$. If there exist $z_1, z_2, \cdots, z_n \in D$ and nonempty open [resp. closed] subsets $G_i \subset A^-(z_i)$ for $i = 1, 2, \cdots, n$ such that $X = \bigcup_{i=1}^n G_i$, then $A$ has a fixed point.*

For a $G$-convex space $(X \supset D; \Gamma)$, a subset $Y$ of $X$ is called a *$G$-convex subspace* of $(X \supset D; \Gamma)$ if $(Y, Y \cap D; \Gamma')$ is a $G$-convex space where $\Gamma'_A := \Gamma_A \cap Y$ for $A \in \langle Y \cap D \rangle$. Note that for a $G$-convex space $(X; \Gamma)$, any nonempty $\Gamma$-convex subset of $X$ is a $G$-convex subspace.

The following in [P6] is a simple consequence of Theorem C:

**Theorem D.** *Let $(X \supset D; \Gamma)$ be a $G$-convex space, $K$ a nonempty subset of $X$, and $S : X \multimap D$, $T : X \multimap X$ multimaps. Suppose that*

(1) *for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$;*
(2) *$K \subset \bigcup \{ \mathrm{Int}\, S^-(z) \mid z \in N \}$ for some $N \in \langle D \rangle$; and*
(3) *there exists a $G$-convex subspace $L_N$ of $X$ containing $N$ such that*

$$L_N \backslash K \subset \bigcup \{ \mathrm{Int}\, S^-(z) \mid z \in M \}$$

*for some $M \in \langle L_N \cap D \rangle$.*

*Then $T$ has a fixed point in $L_N$.*

The following particular form of Theorem D is essentially known in 1996 [PK1]:

**Theorem E.** *Let $(X \supset D; \Gamma)$ be a $G$-convex space, $K$ a nonempty compact subset of $X$, and $S : X \multimap D$, $T : X \multimap X$ multimaps. Suppose that*

(1) *for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$;*
(2) *$K \subset \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in D\}$; and*
(3) *for each $N \in \langle D \rangle$, there exists a compact $G$-convex subspace $L_N$ of $X$ containing $N$ such that*

$$L_N \backslash K \subset \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in L_N \cap D\}.$$

*Then $T$ has a fixed point.*

The following was the basis of our theory in [PK1, 2]:

**Theorem F.** *Let $(X, D; \Gamma)$ be a $G$-convex space, $Y$ a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$ maps, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1) *for each $x \in D$, $S(x)$ is compactly open in $Y$;*
(2) *for each $y \in F(X)$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$;*
(3) *there exists a nonempty compact subset $K$ of $Y$ such that $\overline{F(X)} \cap K \subset S(D)$; and*
(4) *either*
(i) *$Y \backslash K \subset S(M)$ for some $M \in \langle D \rangle$; or*
(ii) *$X \supset D$ and for each $N \in \langle D \rangle$, there exists a compact $G$-convex subspace $L_N$ of $X$ containing $N$ such that $F(L_N) \backslash K \subset S(L_N \cap D)$.*

*Then there exists an $\overline{x} \in X$ such that $F(\overline{x}) \cap T(\overline{x}) \neq \emptyset$.*

Here $\mathfrak{A}_c^\kappa$ denotes the admissible class of multimaps due to Park [P1, PK1, 2], and if $F$ is single-valued, then $Y$ is not necessarily Hausdorff in Theorem 5.

For a topological space $(X, \mathcal{T})$, the compactly generated extension (or the $k$-extension) $\mathcal{T}_k$ of the original topology $\mathcal{T}$ is a topology of $X$ finer than $\mathcal{T}$ such that $\mathcal{T}_k$ consists of all compactly open [resp. compactly closed] subsets of $(X, \mathcal{T})$.

Note that 'compactly open' in Theorem F can be replaced simply by 'open' by adopting the compactly generated extension of the original topology on $Y$.

In [PK2, p.556], the following was given:

**Corollary.** *Let $(X, D; \Gamma)$ be a $G$-convex space, $Y$ a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, and $H : D \multimap Y$ such that, for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset H(N)$. Then the family $\{\overline{H(X)} \mid x \in D\}$ has the finite intersection property.*

The above corollary is the origin of considering the so-called KKM class of multimaps.

Recently Theorem F is simplified and generalized as follows [P5, PK3]:

**Theorem G.** *Let $(X, D; \Gamma)$ be a $G$-convex space, $Y$ a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1) *$S$ has open [resp. closed] values;*
(2) *for each $y \in F(X)$, $M \in \langle S^-(y)\rangle$ implies $\Gamma_M \subset T^-(y)$;*
(3) *$Y = S(N)$ for some $N \in \langle D\rangle$.*

*Then there is a point $x_* \in X$ such that $F(x_*) \cap T(x_*) \neq \emptyset$.*

## 3. Comments on recent works

In this section, we review some of recent results of other author's works on generalized convex spaces. In fact, most of the results due to other authors in this section are simple consequences of the author's results given as early as in 1994 [P1] and so on.

The following is a particular form of [P1, Theorem 6] in 1994:

**Theorem 5.** [B] *Let $D$ be a nonempty subset of a convex space, $Y$ a Hausdorff space, $G : D \to Y$ a map and $F : \operatorname{co} D \to Y$ an admissible u.s.c. map. Suppose that*

(i) *for each $x \in D$, $G(x)$ is compactly open in $Y$;*
(ii) *there exists a nonempty compact subset $K$ of $Y$ such that $\overline{F(\operatorname{co}D)} \cap K \subset G(D)$; and*
(iii) *either*
(a) *$Y \backslash K \subset G(A_0)$ for some $A_0 \in \langle D\rangle$; or*
(b) *for each $A \in \langle D\rangle$, there exists a compact convex subset $L_A$ of $\operatorname{co}D$ containing $A$ such that $F(L_A)\backslash K \subset G(L_A \cap D)$.*

*Then there exists an $A \in \langle D\rangle$ such that $F(\operatorname{co}A) \cap \bigcap\{G(x) | \, x \in A\} \neq \emptyset$.*

From now on, other author's theorem number is the same as in its original source. In Theorem 5 [B], an admissible map is in the sense of Górniewicz, which belongs to a particular subclass of $\mathfrak{A}_c^\kappa$.

As an application of the preceding result, its author also obtained

**Theorem 9.** [B] *Let $D$ be a nonempty subset of a convex space, $Y$ a Hausdorff space, $G : D \to Y$, $T : \operatorname{co}D \to Y$ maps and $F : \operatorname{co}D \to Y$ an admissible u.s.c. map. Suppose that the conditions (i)-(iii) in Theorem 5 are satisfied and moreover assume that:*

(iv) *for each $y \in F(\operatorname{co}D)$, $\operatorname{co}(G^-(y)) \subset T^-(y)$.*

*Then there exists $x_0 \in \operatorname{co}D$ such that $F(x_0) \cap T(x_0) \neq \emptyset$.*

This is a particular form of [P1, Theorem 5] and Theorem F in this paper.

One of the worst forms of Theorem D or E is the following (where, $G$-co is our $\Gamma$-co):

**Theorem 2.3-A.** [CTT] *Let $X$ be a nonempty $G$-convex subset of a $G$-convex space $(E, \Gamma)$ such that for each $x \in E$, $\{x\}$ is $G$-convex, i.e., $\Gamma(\{x\}) = \{x\}$ and for each $A \in \langle E \rangle$, $G$-co$(A)$ is compact and $\Gamma(A) = G$-co$(A)$. Let $P, Q : X \multimap X$ be maps such that*

(a) *for each $x \in X$, $P(x) \subset Q(x)$;*
(b) *for each $x \in X$, $P^-(x)$ is compactly open in $X$;*
(c) *for each $y \in X$, $Q(y)$ is $G$-convex;*
(d) *there exist a nonempty closed and compact subset $K$ of $X$ and $x_0 \in X$ such that $X \backslash K \subset Q^-(x_0)$;*
(e) *for each $y \in K$, $P(y) \neq \emptyset$.*

*Then there exists a point $x \in X$ such that $x \in Q(x)$.*

This theorem can be reformulated to a particular form of our Theorems C and D as follows:

**Theorem H.** *Let $(X; \Gamma)$ be a $G$-convex space and $P, Q : X \multimap X$ multimaps such that*

(1) *for each $x \in X$, $P(x) \subset Q(x)$ and $Q(x)$ is $\Gamma$-convex;*
(2) *for each $x \in X$, $P^-(x)$ is open in $X$;*
(3) *there exist a compact subset $K$ of $X$ and $x_0 \in X$ such that $X \backslash K \subset P^-(x_0)$; and*
(4) *for each $y \in K$, $P(y) \neq \emptyset$.*

*Then $Q$ has a fixed point.*

*Proof.* Condition (4) implies $K \subset \bigcup_{x \in X} P^-(x)$. Since $K$ is compact, there exists an $N \in \langle X \rangle$ such that $K \subset \bigcup_{x \in N} P^-(x)$. Therefore, $X = (X \backslash K) \cup K = P^-(x_0) \cup \bigcup_{x \in N} P^-(x)$. Therefore, by putting $A := Q$, $D := \{x_0\} \cup N$ and $G_x := P^-(x)$ for $x \in D$, in Theorem C, we have the conclusion.

Condition (d) of Theorem 2.3-A seems to be an incorrect form of condition (3) of Theorem H. Switching the topology of $(X; \Gamma)$ to the compactly generated one, condition (b) of Theorem 2.3-A reduces to condition (2) of the above.

The following is equivalent to the preceding [CTT, Theorem 2.3-A]:

**Theorem 2.5.** [CTT] *Let $X$ be a nonempty $G$-convex subset of a $G$-convex space $(E, \Gamma)$ such that for each $x \in E$, $\{x\}$ is $G$-convex, i.e., $\Gamma(\{x\}) = \{x\}$ and for each* *compact and $G$-convex subset $A$ of $X$ and for each $B \in \langle E \rangle$, $G$-co$(A \cup B)$ is compact and $\Gamma(B) = G$-co$B$. Let $Q : X \multimap X$ be such that*

(1) *for each $y \in X$, $G^{-}(y)$ contains a compactly open subset $O_y$ of $X$ (which may be empty);*
(2) *there exists a nonempty compact and $G$-convex subset $X_0$ of $X$ and a nonempty compact subset $K$ of $X$ such that for each $y \in X \backslash K$, there exists an $x \in G$-co$(X_0 \cup \{y\})$ with $y \notin cl_C((X \backslash (G\text{-co}\, Q)^{-}(x) \cap C)$ for any nonempty compact subset $C$ of $X$, and $K \subset \bigcup_{y \in X} O_y$.*

*Then there is a point $\hat{y} \in G$-co$\,(Q(\hat{y}))$.*

This result can be simplified as in our Theorem E. Similarly, most of other results in [CTT] can be reformulated to much more clear forms as in our previous works.

The following appears in [LY, L]:

**Theorem 1.** [LY, L] *Let $(X, \Gamma)$ be a $G$-convex space and $G : X \multimap X$ a nonempty map such that*

(i) $X = \bigcup \{\text{int}(G^{-}(y)) \mid y \in X\}$; *and*
(ii) *there exists a nonempty compact subset $K$ of $X$ such that for each $N \in \langle X \rangle$, there exists a compact $G$-convex subset $L_N$ of $X$ containing $N$ such that*

$$L_N \cap \bigcap \{X \backslash \text{int}(G\text{-co}\, G)^{-}(x) \mid x \in L_N\} \subset K.$$

*Then there exists a $y \in X$ such that $y \in G$-co$\,G(y)$.*

Perhaps 'nonempty' map might be redundant. This is a form of Theorem E with $T := \Gamma$-co$\,G$ noting that

$$L_N \backslash K \subset \bigcup \{\text{int}(\Gamma\text{-co}\, G)^{-}(x) \mid x \in L_N\}.$$

This is also a form of Theorem F(ii) for the case $X = D = Y$, $F = 1_X$, $T = \Gamma$-co$\,G$, and $S = \text{int}\,(\Gamma\text{-co}\, G)^{-}$.

Let $(X \supset D; \Gamma)$ be a generalized convex space, $Y$ a Hausdorff space, and $T \in \mathfrak{A}_c^{\kappa}(X, Y)$ (the class of admissible maps due to Park).

Under the above assumption, the authors of [KR] obtained the following:

**Theorem 4.2.** [KR] *Let $R : D \rightrightarrows Y$ satisfy* :

(1) *for each $y \in T(X)$, $R^-(y)$ is $G$-convex*;
(2) *there exists nonempty compact subset $C$ in $Y$ such that either*
(i) *there is a finite subset $B$ in $\mathcal{F}(D)$ such that $Y \backslash C \subset \bigcup_{x \in B}$ int $R(x)$, or*
(ii) *for every $A \in \mathcal{F}(D)$ there is a compact $G$-convex $C_A \subset X$ containing $A$ such that $Y \backslash C \subset \bigcup_{x \in C_A \cap D} \mathrm{int} R(x)$.*

*Suppose in addition that there is a transfer compactly open-valued selection $S$ of $R$ satisfying $S^-(y) \neq \emptyset$ for all $y \in C \cap \overline{T(X)}$. Then there exists $\overline{x} \in D$ such that $T(\overline{x}) \cap R(\overline{x}) \neq \emptyset$.*

Note that $S$ is transfer open-valued means $\bigcup_{z \in D} S(z) = \bigcup_{z \in D} \mathrm{Int}_Y S(z)$ and that we may regard $Y$ has the compactly generated extension of its original topology without loss of generality.

We show that this is a particular case of our Theorem F, where $\langle D \rangle = \mathcal{F}(D)$. First of all we have to correct one thing; that is, condition (1) is incorrect. Hence we have to consider another map as follows:

*Proof using Theorem F.* In our Theorem F, let $F := T$, $S := \mathrm{Int}\, R : D \multimap Y$, $T^-(y) := \Gamma\text{-co}R^-(y)$ [We have to define another map $T : Y \multimap X$ instead of (1).], $K := C$, $N := A$, and $L_N := C_A$. Then "suppose" part simply tells that

$$K \cap \overline{F(X)} \subset \bigcup \{S(x) | \, x \in D\}.$$

Hence the result follows from our Theorem F.

Recall that $T : X \to Y$ is transfer closed-valued means $\bigcap_{x \in X} T(x) = \bigcap_{x \in X} \overline{T(x)}$, and that ccl and cint denote closure and interior, respectively, with respect to the compactly generated topology.

**Theorem 4.6.** [D1] *Let $X$ be a nonempty subset of a $G$-convex space $(E, D; \Gamma)$, $K$ be a nonempty compact subset of $E$, and $Z$ be a nonempty set. Let $F, G : X \multimap Z$ be two set-valued mappings such that*

(i) *for each $x \in X$, the set $\{y \in X \mid F(y) \cap G(x) \neq \emptyset\}$ is $G$-convex,*
(ii) *the mapping $T : X \multimap E$ defined by*

$$T(x) = \{y \in X \mid F(x) \cap G(y) = \emptyset\}$$

*is transfer compactly closed-valued,*
(iii) *for each $N \in \mathcal{F}(X \cap D)$, there exists a compact $G$-convex subset $L_N$ of $E$ containing $N$ such that for each $y \in L_N \backslash K$, there is a $x \in L_N \cap X$ satisfying $y \notin \mathrm{ccl}\, T(x)$,*
(iv) *for each $x \in K$, $F(x) \cap G(x) \neq \emptyset$.*

*Then there exists* $\hat{x} \in K$ *such that* $F(\hat{x}) \cap G(\hat{x}) \neq \emptyset$.

Note that condition (iv) readily implies the conclusion, and hence the above theorem needs correction.

**Corollary 4.2.** [D1] *Let* $X$ *be a nonempty subset of a* $G$*-convex space* $(E, D; \Gamma)$, $K$ *be a nonempty compact subset of* $E$, *and* $G : X \multimap E$ *be map such that*

- (i) *for each* $x \in K$, *the set* $G(x) \cap X \neq \emptyset$ *and each* $G(x)$ *is* $G$*-convex,*
- (ii) *for each* $y \in X$, $G^-(y)$ *is transfer compactly open-valued,*
- (iii) *for each* $N \in \mathcal{F}(X \cap D)$, *there exists a compact* $G$*-convex subset* $L_N$ *of* $E$ *containing* $N$ *such that for each* $y \in L_N \backslash K$, *there is a* $x \in L_N \cap X$ *satisfying* $y \in \operatorname{cint} G^-(x)$.

*Then there exists* $\hat{x} \in K$ *such that* $\hat{x} \in G(\hat{x})$.

It seems to be easy to correct the preceding results in order to match our theorems. In fact, Theorem 4.6 [D1] is essentially the following:

**Theorem I.** *Let* $(X \supset D; \Gamma)$ *be a* $G$*-convex space,* $K$ *a nonempty compact subset of* $X$, *and* $Z$ *a nonempty set. Let* $F$, $G : X \multimap Z$ *be two multimaps such that*

- (i) *for each* $x \in X$, $T(x) := \{y \in X \mid F(x) \cap G(y) \neq \emptyset\}$ *is* $\Gamma$*-convex;*
- (ii) $X = \bigcup\{\text{Int } T^-(z) \mid z \in D\}$;
- (iii) *for each* $N \in \langle D \rangle$, *there exists a compact* $G$*-convex subspace* $L_N$ *of* $X$ *containing* $N$ *such that* $L_N \setminus K \subset \bigcup\{\text{Int } T^-(z) \mid z \in L_N \cap D\}$

*Then there exists* $\hat{x} \in X$ *such that* $F\hat{x} \cap G(\hat{x}) \neq \emptyset$.

*Proof.* We use Theorem D with $S^-|_D = T|_D$. Since $K$ is compact and $K \subset X \subset \bigcup\{\text{Int } T^-(z) \mid z \in D\}$ by (ii), $K \subset \bigcup\{\text{Int } T^-(z) \mid z \in N\}$ for some $N \in \langle D \rangle$. By (iii), $L_N$ is compact and

$$L_N = (L_N \setminus K) \cap (L_N \cup K) \subset \bigcup\{\text{Int } T^-(z) \mid z \in L_N \cap D\}$$

since $L_N \cap K \subset K$ and $N \subset L_N \cap D$. Therefore $L_N \subset \bigcup\{\text{Int}\, T^-(z) \mid z \in M\}$ for some $M \in \langle L_N \cap D \rangle$. By Theorem D, we have an $\hat{x} \in L_N$ such that $\hat{x} \in T(\hat{x})$. This completes our proof.

We note the following defects of Corollary 4.2 [D1]:

(1) By (i), for each $x \in K$, $G(x)$ is defined and hence $K$ should be a subset of $X$.
(2) By (iii), for each $y \in L_N \setminus K$, we have $y \in \operatorname{cint} G^-(x) \subset G^-(x)$ for some $x \in L_N \cap X$. Then $x \in G(y)$ and this means $y \in X$. Hence $L_N \subset X$.
(3) Instead of $G : X \multimap E$ and $G(x) \cap X \neq \emptyset$, we can simply assume $G : X \multimap X$.

Then Corollary 4.2 [D1] becomes a simple consequence of Theorems D and E.

**Theorem 1.1.** [D2] *Let $(X, \Gamma)$ be a $G$-convex space, $K$ be a nonempty compact subset of $X$, and $G, T : X \multimap X$ be two set-valued mappings such that we have the following.*

(i) *$G^- : X \multimap X$ is transfer compactly open-valued on $X$.*
(ii) *for each $x \in X$, $N \in \langle (\operatorname{cint} G^-)^-(x) \rangle$ implies $\Gamma(N) \subset T(x)$.*
(iii) *for each $x \in X$, $G(x) \neq \emptyset$.*
(iv) *foe each $N \in \langle X \rangle$, there exists a compact $G$-convex subset $L_N$ of $X$ containing $N$ such that*

$$L_N \backslash K \subset \bigcup_{y \in L_N} \operatorname{cint}(G^-(y)).$$

*Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.*

*Proof using Theorem E.* In Theorem E, put $X = D$. Define $S(x) := (\operatorname{cint} G^-)^-(x) \subset G(x)$ for $x \in X$. Then

(1) for each $x \in X$, $N \in \langle S(x) \rangle$ implies $\Gamma_N \subset T(x)$, by (ii).
(2) by (i) and (iii), $X = \bigcup\{\operatorname{cint} G^-(y) : y \in X\}$. Therefore $K \subset \bigcup\{\operatorname{cint} G^-(y) : y \in X\}$.
(3) immediate from (iv).

Therefore, all the requirements of Theorem E are satisfied.

In [D3], its author followed the present author's way and claimed that a number of his results generalize or improve corresponding ones appeared in Park et al. [PK2] and other papers.

In [D3], its author begins with the following:

**Theorem 3.1.** [D3] *Let $(X, D, \Gamma)$ be a $G$-convex space, $Y$ be a topological space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G : Y \to 2^D$ be such that*

(i) *for each $N \in \mathcal{F}(D)$ and $x \in N$, $F(\Gamma(N)) \cap G^{-1}(x)$ is relatively open in $F(\Gamma(N))$,*
(ii) *for each $N \in \mathcal{F}(N)$, $F(\Gamma(N)) \subset \bigcup_{x \in N}(Y \backslash G^{-1}(x))$.*

*Then we have*

(1) *for each $N \in \mathcal{F}(D)$, $F(\Gamma(N)) \cap (\bigcap_{x \in N}(Y \backslash G^{-1}(x))) \neq \emptyset$,*
(2) *for each $N \in \mathcal{F}(D)$, there exists a $y \in F(\Gamma(N))$ such that $G(y) \cap N = \emptyset$.*

This is proved by the partition of unity argument on a finite open cover of a compact subset of $F(\Gamma(N))$, and hence its author should assume that $F(\Gamma(N)) \subset Y$ is Hausdorff.

We show that Theorem 3.1 [D3], under the assumption that $Y$ is Hausdorff, follows from our Corollary to Theorem F.

*Proof.* For any $N \in \langle D \rangle$, we define $H : N \multimap F(\Gamma_N)$ by $H(z) = F(\Gamma_N) \cap (Y \backslash G^-(z))$ for $z \in N$. Then each $H(z)$ is closed by (i). Moreover, for each $N' \in \langle N \rangle$, $F(\Gamma_{N'}) \subset F(\Gamma_{N'}) \cap \bigcup_{z \in N'} (Y \backslash G^-(z)) = \bigcup_{z \in N'} H(z) = H(N')$ by (ii). Then, by our Corollary for the $G$-convex space $(X, N; \Gamma)$, the family $\{H(z) \mid z \in N\}$ has the finite intersection property. This proves (1).

Note that conclusion (2) is equivalent to (1).

From Theorem 3.1 [D3], its author obtains a number of results essentially due to the present author et al. and claims that some of his results generalize or improve our results. However, by replacing the topology of $X$ by its compactly generated extension, those results in [D3] become simple consequences of corresponding known ones of the present author.

The following is an example:

**Theorem 3.5.** [D3] *Let $(X, D; \Gamma)$ be a $G$-convex space and $K$ be nonempty compact subset of a topological space $Y$. Let $F \in \mathfrak{A}_c^\kappa(X, Y)$, $G : Y \to 2^D$ and $T : Y \to 2^X$ be such that*

(i) *$G$ satisfies one of the conditions (I)-(V) in Lemma 2.1,*
(ii) *for $y \in \overline{F(X)}$, $N \in \mathcal{F}((\operatorname{cint} G^{-1})^{-1}(y))$ implies $\Gamma(N) \subset T(y)$,*
(iii) *$\overline{F(X)} \cap K \subset \bigcup_{x \in D} G^{-1}(x)$,*
(iv) *one of the following conditions hold:*
(a) *for some $M \in \mathcal{F}(D)$, $Y \backslash K \subset \bigcup_{x \in M} \operatorname{cint}(G^{-1}(x))$;*
(b) *for each $N \in \mathcal{F}(D)$, there exists a compact $G$-convex subset $L_N$ of $X$ containing $N$ such that*

$$F(L_N) \backslash K \subset \bigcup_{x \in L_N \cap D} \operatorname{cint}(G^{-1}(x)).$$

*Then there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in T(y_0)$ and $y_0 \in F(x_0)$.*

Here, it is assumed that $X \supset D$ and (i) means $K \subset \bigcup_{x \in D} \mathrm{Int}_X G^{-1}(z)$.

Note that, by adopting compactly generated extension of the original topology on $X$ and replacing $(S, T)$ by $(\operatorname{Int} G^-, T^-)$, Theorem 3.5 [D3] follows from our

Theorem F. Moreover, note that $Y$ should be Hausdorff, since its author forgot the Hausdorffness when he was using the partition of unity argument on a compact space in the proof of Theorem 3.1 [D3], which is the basis of the whole of [D3]. Therefore, by our Theorem F or G, there exists an $x_0 \in X$ such that $F(x_0) \cap T^-(x_0) \neq \emptyset$.

Similarly, numerous other papers of the author of [D1-3] contain results having certain defects as is shown for [D1-3].

In [CL], its authors claimed two generalizations of Nikaidô's theorem. However, all of their main results are already known in more general forms. For example, they gave the following:

**Theorem 2.6.** [CL] *Let $C$ be a nonempty compact convex subset of a locally convex Hausdorff t.v.s. $X$, and $D$ be a nonempty convex subset of a Hausdorff t.v.s. $Y$. If $S : C \multimap D$ is map with nonempty convex values such that $C = \bigcup_{y \in D} \mathrm{Int}\ S^-(y)$, and $T : D \multimap C$ is an acyclic map, then there exists an $(\bar{x}, \bar{y}) \in C \times D$ such that $\bar{y} \in S(\bar{x})$ and $\bar{x} \in T(\bar{y})$.*

This is a simple consequence of Theorems F or G with more generous requirements that (1) $C$ is a compact Hausdorff space, and (2) $D$ is a convex subset of a t.v.s.

The following in [A] is a simple consequence of our previous results:

**Theorem 2.1.** [A] *Let $X$ be a convex subset of a Hausdorff topological vector space $E$ and $S, T : X \multimap X$ two multimaps. Assume that*

(a) *for each $x \in X$, $\mathrm{co}\, S(x) \subset T(x)$;*
(b) *$X = \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in X\}$; and*
(c) *there exists a nonempty subset $Y_0$ of $X$ such that $Y_0$ is contained in a compact convex subset $Y_1$ of $X$ and the set $Z := \bigcap\{X \backslash \mathrm{Int}_X S^-(y) \mid y \in Y_0\}$ is either empty or compact.*

*Then $T$ has a fixed point.*

*Proof using Theorem C.* Since $Z$ is empty or compact and $Z \subset \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in Y_0\}$, we have an $N \in \langle X \rangle$ such that $Z \subset \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in N\}$. Let $Y :=\mathrm{co}(Y_1 \cup N) \subset X$. Since $Y_1$ is a compact convex subset of a Hausdorff t.v.s., so is $Y$. Note that $X \backslash Z \subset X = \bigcup\{\mathrm{Int}_X S^-(y) : y \in Y_0\}$ and hence

$$\begin{aligned} Y =& ((X \backslash Z) \cap Y) \cup (Z \cap Y) \\ \subset& \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in Y_0\} \cup \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in N\}. \end{aligned}$$

Since $Y$ is compact, there exists an $M \in \langle Y_0 \rangle$ such that

$$Y \subset \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in M \cup N\},$$

where $M \subset Y_0 \subset Y_1 \subset Y$ and $N \subset Y$.

Let $A : Y \multimap Y$ be defined by $A(x) := (\mathrm{co}S(x)) \cap Y$ for $x \in Y$. Let $M \cup N := \{z_1, z_2, \cdots, z_n\} \subset Y$ and $G_i := \mathrm{Int}_Y S^-(z_i) = \mathrm{Int}_X S^-(z_i) \cap Y$ for each $i$, $1 \leq i \leq n$. Therefore, by Theorem C, $A$ has a fixed point $x_0 \in Y$. Hence $x_0 \in A(x_0) \subset \mathrm{co}S(x_0) \subset T(x_0)$. This completes our proof.

*Proof using Theorems C and E.*

Case I. Suppose $Z = \emptyset$. Then $X = X \backslash Z = \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in Y_0\}$ and, since $Y_1$ is compact, we have

$$Y_1 \subset \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in N\}$$

for some $N = \{z_1, z_2, \cdots, z_n\} \subset Y_0$. Let $A : Y_1 \multimap Y_1$ be defined by

$$A(x) := (\mathrm{co}S(x)) \cap Y_1 \text{ for } x \in Y_1$$

and let $G_i := \mathrm{Int}_Y S^-(z_i)$ for $1 \leq i \leq n$. Then $A(x)$ is convex for each $x \in Y_1$ and $Y_1 = \bigcup_{i=1}^n G_i$. Hence, by Theorem C, $A$ has a fixed point $x_0 \in Y_1 \subset X$; that is, $x_0 \in A(x_0) \subset \mathrm{co}S(x_0) \subset T(x_0)$.

Case II. Suppose $Z \neq \emptyset$. Since $Z \subset X = \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in X\}$ and $Z$ is compact, then there exists an $N \in \langle X \rangle$ such that $Z \subset \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in N\}$. Since $E$ is Hausdorff and $Y_1$ is a compact convex subset of $E$, $L_N := \mathrm{co}(Y_1 \cup N)$ is a compact convex subset of $E$. Since

$$L_N \backslash Z \subset X \backslash Z = \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in Y_0\}$$

and $L_N \backslash Z$ is compact, we have

$$L_N \backslash Z \subset \bigcup\{\mathrm{Int}_X S^-(y) \mid y \in M\}$$

for some $M \in \langle Y_0 \rangle$ and hence $M \in \langle Y_1 \rangle \subset \langle L_N \rangle$. Therefore, by Theorem E, the conclusion follows.

In [A], its author gave a two-page proof of Theorem 2.1 [A]. Moreover it was assumed that each $S(x)$ is nonempty, which is redundant. Note also that other results in [A] are all simple consequences of Theorem 2.1 [A].

The following continuous selection theorem for multimaps with noncompact domain is given:

**Theorem 1.** [YL] *Let $X$ be a paracompact topological space, $(Y, D, \Gamma)$ be a $G$-convex space, $K$ a nonempty compact subset of $X$, and $S : X \multimap D$, $T : X \multimap Y$ be two maps satisfying the following conditions:*

(i) *for each $x \in X$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$;*
(ii) *$X = \bigcup\{\mathrm{Int}\, S^{-}(y) \,|\, y \in D\}$; and*
(iii) *$X \backslash K \subset \bigcup\{\mathrm{Int}\, S^{-}(y) \,|\, y \in M\}$ for some $M \in \langle D\rangle$.*

*Then there exist $A \in \langle D\rangle$, with $|A| = n+1$ for some $n \in N$ and continuous mappings $g : \Delta_n \to \Gamma_A$, $\phi : X \to \Delta_n$ such that $f = g\phi$ is a continuous selection of $T$.*

In [YL], all spaces are assumed to be Hausdorff. For each locally finite open cover of a normal space, there is a partition of unity subordinate to it. Therefore, the paracompactness in Theorem 1 [YL] can be replaced by the normality. In case $X$ has an infinite open cover, a number of selection results were given in Park [P3].

The above theorem can be improved, in view of [P3, Lemma 1], as follwos:

**Theorem J.** *Let $X$ be a normal space, $(Y, D; \Gamma)$ a $G$-convex space, and $S : X \multimap D$ a map such that $X = \bigcup\{\mathrm{Int}\ S^{-}(y) : y \in A\}$ for some $A \in \langle D\rangle$. Then there exists a continuous map $s : X \to \Gamma_A$ such that $s(x) \in \Gamma(A \cap S(x))$ for all $x \in X$. In fact, if $|A| = n+1$, then $s = \phi_A \circ p$, where $\phi_A : \Delta_n \to \Gamma_A$ and $p : X \to \Delta_n$ are continuous functions.*

The following particular form appears in [DP]:

**Lemma 2.1.** [DP] *Let $X$ be a normal space, $(Y, \Gamma)$ be a $G$-convex space and $G : X \to 2^Y$ be a set-valued mapping such that*

(i) *$G$ has a nonempty $G$-convex values,*
(ii) *$X = \bigcup_{y\in Y} int\, G^{-1}(y)$,*
(iii) *there exists a nonempty compact subset $K$ of $X$ and a finite subset $M$ of $Y$ such that $X\backslash K \subset \bigcup_{y\in M} int\, G^{-1}(y)$.*

*Then there exists a continuous selection $f : X \to Y$ of $G$ such that $f = \phi \circ \psi$ where $\phi : \Delta_n \to Y$ and $\psi : X \to \Delta_n$ are both continuous and $n$ is some positive integer.*

From the above lemma, its authors could deduce the following:

**Lemma 2.2.** [DP] *Let $(X, \Gamma)$ be a normal $G$-convex space and $G : X \to 2^X$ be a set-valued mapping such that*

(i) *for each $x \in X$, $G(x)$ is nonempty $G$-convex,*
(ii) *$X = \bigcup_{y\in Y} int\, G^{-1}(y)$,*

(iii) *there exists a nonempty compact subset $K$ of $X$ and a finite subset $M$ of $X$ such that*

$$X \backslash K \subset \bigcup_{y \in M} int\, G^{-1}(y).$$

*Then $G$ has a fixed point in $X$.*

Note that Lemma 2.2 [DP] is a simple consequence of our Theorems B - E and that normality is redundant.

Similarly, all of other results in [DP] are simple consequences of known results and have defects contrary to its authors' claim. This remark might work for results in several other papers of the authors of [DP] on collectively fixed points and other topics.

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