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COMMENTS ON SOME ABSTRACT CONVEX SPACES AND THE KKM MAPS

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Abstract. We introduce basic results in the KKM theory on abstract convex spaces and the KKM maps. These are applied to various modifications of the concepts of generalized convex spaces and KKM type maps. We discuss the nature of those modifications and criticize recently appeared 'generalizations' of our previous works due to many other authors.

1. Introduction

Since 1993, the author has studied generalized convex spaces (or G-convex spaces) and the better admissible class $\mathfrak{B}$ of multimaps as common generalizations of various general convexities without linear structures and of multimaps due to a large number of other authors, respectively. We have established within such a frame the foundations of the KKM theory initiated by Knaster, Kuratowski, and Mazurkiewicz, as well as fixed point theorems and many other equilibrium results for multimaps. This direction of study has been followed by a number of other authors.

In order to polish up the G-convex space theory, in our previous work [36], we suggested to destroy many of artificial terminology adopted by other authors in the KKM theory on such spaces. Moreover, in [40,41], we showed that a number of fixed point theorems or other results related to G-convex spaces appeared in many works are simple consequences of known results. Furthermore, in a recent work [43], we found that, in the framework of certain abstract convex spaces including G-convex spaces properly, the elements or basic results in the KKM theory can be established without assuming any topology on those spaces.

Our principal aim in the present paper is to introduce basic results in the KKM theory on abstract convex spaces and the map class $\mathfrak{K}$ as in [43]. These are applied to simplify various modifications of the concept of generalized convex spaces. We discuss the nature of these modifications and criticize recently appeared so-called generalizations of our previous works due to other authors.

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In Section 2, we introduce our new abstract convex spaces, KKM maps, and the map class $\mathfrak{KC}$ [or $\mathfrak{KO}$] in [43], and, in Section 3, a few basic theorems in our KKM theory for those spaces given in [43]. Section 4 deals with KKM type theorems for G-convex spaces, which are shown to be easily deduced from our new results on abstract convex spaces.

Sections 5-8 are devoted to various modifications of G-convex spaces and KKM type maps appeared in the 21st century. We show that most of them are mere modifications without having any proper example or any applicability. Such modifications are, for examples, L-spaces, generalized R-KKM maps, pseudo H-spaces, and others.

2. Abstract convex spaces and the map classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Definitions [43]. An *abstract convex space* $(E, D; \Gamma)$ consists of nonempty sets E, D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

[co is reserved for the convex hull in vector spaces.]

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D' \rangle})$ is called a Γ-*convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

An abstract convex space with a topology on E is called *an abstract convex topological space.*

If the reader prefers, abstract convex spaces can be called A-convex spaces.

Examples. 1. A convexity space $(E, \mathcal{C})$ in the classical sense; see [49], where the bibliography lists 283 papers.

2. A *generalized convex space* due to Park or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. In certain cases, it is possible to assume $\Gamma(A) = \phi_A(\Delta_n)$.

For details, see [33-38,40-42,44-47].

3. A G-convex space $(X, D; \Gamma)$ is called a C-*space* (or an H-*space*) if each Γ_A is ω-connected (that is, n-connected for all $n \geq 0$) and $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D \rangle$; see Horvath [17,18] for particular forms for $X = D$.

4. A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde [23].

5. Let E be a topological vector space with a neighborhood system $\mathcal{V}$ of its origin. A subset X of E is said to be *almost convex* [16] if for any $V \in \mathcal{V}$ and for any finite subset $A := \{x_1, x_2, \dots, x_n\}$ of X, there exists a subset $B := \{y_1, y_2, \dots, y_n\}$ of X such that $y_i - x_i \in V$ for each $i = 1, 2, \dots, n$ and $\mathrm{co}\, B \subset X$. By choosing one of such B, let $\Gamma_A := \mathrm{co}\, B$ for each $A \in \langle X \rangle$. Then $(X; \Gamma)$ becomes an abstract convex space.

Definitions. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called *a* $\mathfrak{K}$*-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$*-map* is defined for closed-valued maps G, and a $\mathfrak{KO}$*-map* for open-valued maps G. Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation $\mathrm{KKM}(E, Z)$ instead of $\mathfrak{KC}(E, Z)$.

Examples. 1. Every abstract convex space in our sense has a map $F \in \mathfrak{K}(E, Z)$ for any nonempty set Z. In fact, for each $x \in E$, choose $F(x) := Z$ or $F(x) := \{z_0\}$ for some $z_0 \in Z$.

If $1_E \in \mathfrak{K}(E, E)$, then $f \in \mathfrak{K}(E, Z)$ for any function $f : E \to Z$. If E and Z are topological spaces, this holds for $\mathfrak{KC}$ or $\mathfrak{KO}$ for any continuous f.

2. For a G-convex space $(X, D; \Gamma)$ and a topological space Z, we defined the classes $\mathfrak{K}, \mathfrak{KC}, \mathfrak{KO}$ of multimaps $F : X \multimap Z$ [38]. It is known that for a G-convex space $(X, D; \Gamma)$, we have the identity map $1_X \in \mathfrak{KC}(X, X) \cap \mathfrak{KO}(X, X)$; see [35,38,47]. Moreover, for any topological space Y, if $F : X \to Y$ is a continuous single-valued map or if $F : X \multimap Y$ has a continuous selection, then $F \in \mathfrak{KC}(X, Y) \cap \mathfrak{KO}(X, Y)$.

3. The admissible class $\mathfrak{A}_c^\kappa(X, Y)$ [29] of multimaps $X \multimap Y$ is a subclass of $\mathfrak{KC}(X, Y)$ when X is a convex space and Y is a Hausdorff space [30]. Motivated by this, Chang and Yen [6] defined the KKM class of maps on convex subsets of topological vector spaces, and further, Chang et al. [5] extended the KKM-class to S-KKM class. On the other hand, the author showed that, in the class of compact closed multimaps from convex spaces to Hausdorff spaces, two subclasses $\mathfrak{B}$ and $\mathfrak{KC}$ coincide [31]. Moreover, recently H. Kim [21] showed that two classes KKM and s-KKM of multimaps from a convex space into a topological space are identical whenever s is surjective [this is the only case S-KKM is meaningful]. For a G-convex space $(X, D; \Gamma)$ and a Hausdorff space Y, it is known that $\mathfrak{A}_c^\kappa(X, Y) \subset \mathfrak{KC}(X, Y) \cap \mathfrak{KO}(X, Y)$ by H. Kim and the author [22].

4. Modifying the original definition of S-KKM maps of Chang et al. [5], Amini et al. [1] defined the S-KKM class for a classical convexity space $(X, \mathcal{C})$ and a topological space Y. Their $\text{S-KKM}_{\mathcal{C}}(Z, X, Y)$ becomes simply $\mathfrak{KC}(X, Y)$.

3. Basic theorems in the KKM theory

In our KKM theory on abstract convex spaces given in [39,43], there exist some basic theorems from which we can deduce several equivalent formulations that can be used for applications. In this section, we introduce some of such basic theorems.

We begin with the following prototype of KKM type theorems:

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F \in \mathfrak{K}(E, Z)$. Let $G : D \multimap Z$ be a map such that*

(A.1) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$.*

Then for each $N \in \langle D \rangle$, $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$.

Remarks. 1. If Z is a topological space and G is open-valued [resp., closed-valued], then we can assume $F \in \mathfrak{KO}(E, Z)$ [resp., $F \in \mathfrak{KC}(E, Z)$].

2. If $E = Z$ and if the identity map $1_E = F \in \mathfrak{K}(E, E)$, then Condition (A.1) says that G is a KKM map. If $E = Z = \Delta_n$ is an n-simplex, D is the set of its vertices, and $\Gamma = \mathrm{co}$ is the convex hull operation, then the celebrated KKM principle says that $1_E \in \mathfrak{KC}(E, E)$.

3. If D is a nonempty subset of a topological vector space $E = Z$ (not necessarily Hausdorff), Fan's KKM lemma says that $1_E \in \mathfrak{KC}(E, E)$ [15].

4. For another forms of the KKM theorem for convex spaces, H-spaces, or G-convex spaces and their applications, there are a large number of works; see [34,35,43] and references therein.

From Theorem A, we have another finite intersection property as follows:

Theorem B. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F \in \mathfrak{K}(E, Z)$. Let $G : D \multimap Z$ and $H : E \multimap Z$ be maps satisfying*

(B.1) *for each $x \in E$, $F(x) \subset H(x)$; and*

(B.2) *for each $z \in F(E)$, $M \in \langle D \setminus G^-(z) \rangle$ implies $\Gamma_M \subset E \setminus H^-(z)$.*

Then $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$ for all $N \in \langle D \rangle$.

The following coincidence theorem follows from Theorem B.

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, $S : D \multimap Z$, $T : E \multimap Z$ maps, and $F \in \mathfrak{K}(E, Z)$. Suppose that*

(C.1) *for each $z \in F(E)$, $\mathrm{co}_\Gamma S^-(z) \subset T^-(z)$; and*

(C.2) *$F(E) \subset S(N)$ for some $N \in \langle D \rangle$.*

Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Remark. If Z is a topological space and S has open [resp., closed] values, then we can assume $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KO}(E, Z)$].

From Theorem C, we obtain the following Ky Fan type matching theorem:

Theorem D. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, $S : D \multimap Z$, and $F \in \mathfrak{K}(E, Z)$ satisfying* (C.2). *Then there exists an $M \in \langle D \rangle$ such that $F(\Gamma_M) \cap \bigcap\{S(x) \mid x \in M\} \neq \emptyset$.*

Theorem D can be stated in its contrapositive form and in terms of the complement $G(x)$ of $S(x)$ in Z. Then we obtain Theorem A. Therefore, Theorems A–D are equivalent.

From Theorem C, we have the following prototype of the Fan-Browder fixed point theorem:

Theorem E. *Let $(E, D; \Gamma)$ be an abstract convex topological space, and $G : E \multimap D$, $F : E \multimap E$ maps satisfying*

(E.1) *for each $x \in E$, $\mathrm{co}_\Gamma G(x) \subset F(x)$;*

(E.2) *$E = G^-(N)$ for some $N \in \langle D \rangle$; and*

(E.3) *G^- has open [resp., closed] values.*

If the identity map $1_E \in \mathfrak{KC}(E, E)$ [resp. $1_E \in \mathfrak{KO}(E, E)$], then F has a fixed point $\bar{x} \in E$, that is, $\bar{x} \in F(\bar{x})$.

Definition. For a given abstract convex space $(E, D; \Gamma)$ and a topological space X, a map $H : X \multimap E$ is called a Φ-*map* (or a *Fan-Browder map*) if there exists a map $G : X \multimap D$ such that

(i) for each $x \in X$, $\mathrm{co}_\Gamma G(x) \subset H(x)$; and

(ii) $X = \bigcup\{\mathrm{Int}\, G^-(y) \mid y \in D\}$.

Definitions. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is an abstract convex space with a basis $\mathcal{U}$ of a uniform structure of E.

In $(E, D; \Gamma; \mathcal{U})$, a subset Z of E is called a Φ-*set* if, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : Z \multimap E$ such that $\mathrm{Gr}(H) \subset U$. If E itself is a Φ-set, then it is called a Φ-*space*.

A point $x \in E$ is called a *U-fixed point* of a map $F : E \multimap E$ if $F(x) \cap U[x] \neq \emptyset$. The map F is said to have the *almost fixed point property* whenever it has a U-fixed point for each $U \in \mathcal{U}$.

Theorem F. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space and $F \in \mathfrak{KC}(E, E)$ a compact map. If $\overline{F(E)}$ is a Φ-set, then F has the almost fixed point property.*

Corollary F.1. *Under the hypothesis of Theorem F, further if $(E, \mathcal{U})$ is separated and if F is closed, then it has a fixed point.*

From now on, numbers attached to Theorems and Definitions are the ones in their original sources.

4. Generalized KKM maps

Our aim in this section is to give examples of our KKM maps with respect to a certain map F.

One of the most general KKM theorems for G-convex spaces is due to the present author and H. Kim [45,46]. Later in 2001, it is refined by Kalmoun and Rihai [20] as follows:

Theorem 2.1 [20]. *Let $(X, D; \Gamma)$ be a G-convex space, Z a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Z)$, and $G : D \multimap Z$ such that*

(1.1) *G is transfer closed-valued [that is, $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$];*

(1.2) *G is weakly FG-KKM [that is, $F(\Gamma_A) \subset \bigcup_{y\in A} \overline{G(y)}$ for all $A \in \langle D \rangle$];*

(1.3) *there exists a nonempty compact subset K of Z such that either*

(i) *$\bigcap_{y\in B} G(y) \subset K$ for some $B \in \langle D \rangle$, or*

(ii) *$X \supset D$ and for each $N \in \langle D \rangle$, there exists a compact G-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap_{y\in L_N\cap D} \overline{G(y)} \subset K$.*

Then $\bigcap_{y\in D} G(y) \cap \overline{F(X)} \cap K \neq \emptyset$.

We replaced T and F in Theorem 2.1 [20] by F and G, resp., as above. Here, if F is single-valued, then Z is not necessarily Hausdorff. The original proof is based on a result of the present author and H. Kim [44]. But we give a simple proof based on Theorem A. Note that $F \in \mathfrak{KC}(X, Z)$.

Proof. From Theorem A with $E := X$, we immediately have $\overline{F(X)} \cap \bigcap_{y\in N} \overline{G(y)} \neq \emptyset$ for each $N \in \langle D\rangle$.

(i) Let $K' := \overline{F(X)} \cap \bigcap_{y\in B} \overline{G(y)} \cap K \neq \emptyset$. Then the family $\{\overline{G(y)} \cap K'\}_{y\in D}$ is a family of closed sets in the compact set K and hence has the whole intersection property, that is, $\bigcap_{y\in D} \overline{G(y)} \cap \overline{F(X)} \cap K \neq \emptyset$. By (1.1) the conclusion follows.

(ii) Let $K' := \overline{F(X)} \cap \bigcap_{y\in N} \overline{G(y)} \cap K \supset \overline{F(L_N)} \cap \bigcap_{y\in L_N\cap D} \overline{G(y)} \cap K \neq \emptyset$. Then as in above, we have the conclusion.

Using our new terminology, (1.2) can be stated as follows:

(1.2)$'$ $\overline{G}$ is a KKM map with respect F.

Motivated by a large number of recent works on generalized KKM maps, we introduced the following definition in 2001 [47]:

Let $(X, D; \Gamma)$ be a G-convex space and I a nonempty set. A map $F : I \multimap X$ is called a *generalized KKM map* provided that for each $N \in \langle I\rangle$, there exists a function $\sigma : N \to D$ such that $\Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle N\rangle$.

In [47], a unified account on results for such maps was given; for example, the KKM type theorem, characterizations of such maps, an equilibrium theorem implying minimax inequalities, variational inequalities, and so on.

In our new theory, a generalized KKM map can be regarded simply as a KKM map for the G-convex space $(X, I; \Gamma')$, where $\Gamma'_M := \Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle N\rangle$ and each $N \in \langle I\rangle$.

In 2002, similar results appeared in [10], which have trivial defects in certain aspects.

In 2004, the authors of [13] defined the following (where $\mathfrak{F}(X) = \langle X\rangle$):

Definition 2.1 [13]. Let X be a nonempty subset of a G-convex space $(E \supset D; \Gamma)$ and let $T : X \to 2^E$ be a set-valued mapping. T is said to be a GKKM mapping if $N \in \mathfrak{F}(X \cap D)$ implies $\Gamma(N) \subseteq \bigcup_{x\in N} T(x)$.

Definition 2.2 [13]. Let Y be a nonempty set and let X be a nonempty subset of a G-convex space $(E \supset D; \Gamma)$. The set-valued mapping $T : Y \to 2^X$ is said to be a generalized GKKM mapping if, for each $B = \{y_1, \ldots, y_n\} \in \mathfrak{F}(Y)$, there exists $A = \{x_1, \ldots, x_n\} \in \mathfrak{F}(X \cap D)$ such that, for any subset $\{x_{i_1}, \ldots, x_{i_k}\}$ of A, $\Gamma(\{x_{i_1}, \ldots, x_{i_k}\}) \subseteq \bigcup_{j=1}^{k} T(y_{i_j})$.

In order to match our new definitions, in Definition 2.1 [13], let $D' := X \cap D$ and $\Gamma' := \Gamma|_{\langle D'\rangle}$. Then a GKKM map $T : D' \multimap E$ becomes a KKM map if $N \in \mathfrak{F}(X \cap D)$ implies $\Gamma(N) \subseteq \bigcup_{x\in N} T(x)$. (Notice that T is not necessarily defined on $X \setminus D'$.)

In Definition 2.2 [13], for the G-convex space $(E, Y; \Gamma')$, where $\Gamma'(B) := \Gamma(A)$ for each $B \in \langle Y \rangle$, a map $T : Y \multimap E$ is a KKM map if, for any subset $\{x_{i_1}, \ldots, x_{i_k}\}$ of A, $\Gamma(\{x_{i_1}, \ldots, x_{i_k}\}) \subseteq \bigcup_{j=1}^{k} T(y_{i_j})$.

Consequently, the concepts of generalized KKM-maps and GKKM maps in [10,13, 20,46] are disguised forms of KKM maps in our new sense.

5. On L-spaces

The original KKM principle is for the triple $(\Delta_n, V; \mathrm{co})$, where V denotes the set of vertices and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation, and Ky Fan's celebrated lemma [15] is for $(E, D; \mathrm{co})$, where D is a nonempty subset of a topological vector space E. These are the origins of our G-convex space $(X, D; \Gamma)$. Note that any KKM type theorem on $(X; \Gamma)$ can not contain the KKM principle and the Ky Fan lemma.

For the definition of a G-convex space, at first, we assumed an additional condition that

$(*)$ for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

In 1998, this isotonicity was removed from the definition of G-convex spaces; see [33-38].

Examples. Let $\Delta_3 = \mathrm{co}\, V$ where $V = \{e_0, e_1, e_2, e_3\}$.

1. We have a G-convex space $(\Delta_3, V; \mathrm{co})$ where $\mathrm{co} : \langle V \rangle \multimap \Delta_3$ is the convex hull operator.

2. Let $(\Delta_3, V; \Gamma)$ be a G-convex space given by $\Gamma\{e_0, e_1\} := \mathrm{co}\{e_0, e_1, e_2\}$ and $\Gamma(N) := \mathrm{co}\, N$ for all other $N \in \langle V \rangle$. Then Γ violates the isotonicity $(*)$.

In the same year, Ben-El-Mechaiekh et al. [4] defined an L-space (E, Γ), which is a particular form of our G-convex space $(X, D; \Gamma)$ [without assuming $(*)$] for the case $E = X = D$. With the misconception that the class of L-spaces contains our class of G-convex spaces, a number of careless authors of [9,10,14,28,48] restated a number of particular results (with certain defects) of known ones. All of these authors failed to give any proper example justifying their misconception.

In 2003, the authors of [48] considered the same particular form of our G-convex spaces as follows (where $\mathfrak{F}(X) = \langle X \rangle$):

Definition 1 [48]. Let X be a topological space, X has an *L-structure* if there exists a non-empty valued correspondence $\Psi : \mathfrak{F}(X) \to X$ and for all $B \in \mathfrak{F}(X)$, namely $B = \{b_0, b_1, \cdots, b_n\}$, there exists a continuous function $f^B : \Delta_n \to \Psi(B)$ such that for all $J \subseteq \{0, 1, \cdots, n\}$, $f^B(\Delta_J) \subseteq \Psi(\{b_i : i \in J\})$. The pair (X, Ψ) is called an L-space.

In [48], its authors defined a generalized KKM-correspondence and obtained a necessary and sufficient condition for a multimap to be a such correspondence as Theorem 1 [48]. But, in our previous work [41], we noted that this theorem has already several generalizations in [47].

In [9], Ding obtained the following results similar to the Himmelberg fixed point theorem [16]:

Theorem 3.1 [9]. *Let $(X, \Gamma, \mathcal{U})$ be a locally L-convex space, D be an L-compact subset of X. If $T \in KKM(X, D)$, then for each open entourage $U \in \mathcal{U}$ there exists $x_U \in E$ such that $Tx_U \cap U(x_U) \neq \emptyset$, where E is the compact L-convex subset of X containing D.*

Theorem 3.2 [9]. *Let $(X, \Gamma, \mathcal{U})$ be a locally L-convex space, D be an L-compact subset of X, and $T \in KKM(X, D)$ be upper semicontinuous with closed values. Then T has a fixed point in X.*

Since any subset of a locally G-convex space is a Φ-set [42], these are particular forms of Theorem F and Corollary F.1, resp. Ding's theorems are actually for maps in $KKM(D, D)$ and hence can not generalize the Himmelberg theorem.

In [10], Ding defined a GLKKM mapping equivalent to the one defined as above in [48] and obtained KKM type theorems and some of its consequences. All results in [10] are already known in more general forms in [47] and others.

In 2002, the authors of [14] incorrectedly repeated that a G-convex space is essentially an L-convex space introduced by Ben-El-Mechaiekh et al. [4] and gave the following particular form of an earlier work of Ding:

Theorem 1.1 [14]. *Let (X, Γ) be a G-convex space, K be a nonempty compact subset of X, and $G, T : X \to 2^X$ be two set-valued mappings such that we have the following.*

(i) *For each $x \in K$, there exists $y \in X$ such that $x \in \text{cint } G^{-1}(y) \cap K$ and*

$$K = \bigcup_{y \in X} (\text{cint } G^{-1}(y) \cap K) = \bigcup_{y \in X} (G^{-1}(y) \cap K).$$

(ii) *For each $x \in X$, $N \in \langle (\text{cint } G^{-1})^{-1}(x) \rangle$ implies $\Gamma(N) \subset T(x)$.*

(iii) *For each $x \in K$, $G(x) \neq \emptyset$.*

(iv) *For each $N \in \langle X \rangle$, there exists a nonempty compact G-convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup_{y \in L_N} \text{cint } (G^{-1}(y)).$$

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

The authors of [13] incorrectly claimed that this improves and generalizes one of the earlier results of the present author. Notice that (iii) is redundant. By assuming $S^- := \text{cint } G^{-1}$ has (compactly) open values, the situation would become more simple and the conclusion follows from our Theorem E with $E = D = L_N$.

In 2005, Lu and Tang [28] obtained the following main theorem on L-spaces:

Theorem 1 [28]. *Let X be a Hausdorff space, $(Y;\Gamma)$ be an L-convex space, $C : Y \to 2^X$ be a map, and let M, N be two subsets of $X \times Y$. Suppose the following conditions are fulfilled:*

(i) *C is transfer compactly closed-valued;*

(ii) *for each $y \in Y$, the set $\{x \in X : (x, y) \in N\} \subset C(y)$.*

Suppose also that there exists a subset P of M and a compact subset K of X such that P is closed in $X \times Y$, and

(iii) *for each $x \in K$ and $A \in \langle\{y \in Y : (x, y) \notin N\}\rangle$, $\Gamma(A) \subset \{y \in Y : (x, y) \notin M\}$;*

(iv) *for each $y \in Y$, the set $\{x \in K : (x, y) \in P\}$ is nonempty acyclic.*

Then

$$\bigcap_{y\in Y} C(y) \cap K \neq \emptyset.$$

We can simplify the situation by using our Theorem B. We will not distinguish maps with their graphs.

Proof using Theorem B. Let $(E, D; \Gamma) := (Y; \Gamma)$, $Z := X$, and $F : Y \multimap K \subset X$ defined by $F(y) := P^-(y) \cap K = \{x \in K \mid (x, y) \in P\}$ for $y \in Y$. Since P is closed, $P^- \cap (Y \times K)$ is closed in $Y \times X$ and hence F is a compact closed map with acyclic values. Therefore, $F \in \mathbb{V}(E, Z) \subset \mathfrak{KC}(E, Z)$, where $\mathbb{V}$ denotes the class of acyclic maps.

Let $G := N^- : D \multimap Z$ and $H := M^- : E \multimap Z$. Then

(1) for each $y \in E = Y$, $F(y) = P^-(y) \cap K \subset P^-(y) \subset M^-(y) = H(y)$ since $P \subset M$; and

(2) for each $z \in F(E) \subset K$, $A \in \langle D \setminus G^-(z)\rangle = \langle\{y \in Y \mid (x, y) \notin N\}\rangle$ implies $\Gamma_A \subset E \setminus H^-(z) = \{y \in Y \mid (x, y) \notin M\}$ by (iii).

Therefore, all of the requirements of Theorem B are satisfied. Hence, $\emptyset \neq F(E) \cap \bigcap\{G(y) \mid y \in A\} \subset K \cap \bigcap\{G(y) \mid y \in A\} = K \cap \bigcap_{y\in A}\{x \in X \mid (x, y) \in N\} \subset K \cap \bigcap_{y\in A} C(y)$ for all $A \in \langle Y\rangle$. From this finite intersection property, by (i), we have the conclusion.

Note that our proof is more simple than the original one. From Theorem 1 [28], its authors deduce a fixed point theorem, a maximal element theorem, a coincidence theorem, and some minimax theorems in G-convex space theory. As we have shown above, their results are for a particular type of G-convex spaces, can be simplified, and could be given by transparent proofs.

6. Generalized R-KKM maps

Recently, there have appeared authors in [7,8,12,19] and others who tried to rewrite our works on G-convex spaces by replacing $\Gamma(A)$ by $\phi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples. Those authors obtained KKM type theorems or equivalents which can not be applicable even to the KKM principle for $(\Delta_n, V; \mathrm{co})$ or to the Ky Fan lemma for $(E \supset D; \mathrm{co})$, where E is a topological vector space.

In 2003, in [7], the following definition appeared:

Definition 2.1 [7]. Let X be a nonempty set and Y be a topological space. $T : X \to 2^Y$ is said to be generalized relatively KKM (R-KKM) mapping if for any $N = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, there exists a continuous mapping $\phi_N : \Delta_n \to Y$ such that, for each $e_{i_0}, e_{i_1}, \cdots, e_{i_k}$,

$$\phi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

where Δ_k is a standard k-simplex of Δ_n with vertices $e_{i_0}, e_{i_1}, \cdots, e_{i_k}$.

The authors claim that their definition unifies and extends a lot of similar definitions due to other authors.

In our previous work [41], we mentioned that the key result in [7] is a simple consequence of known results. Moreover, in [7], its authors claimed that, applying their key result, they obtained new theorems which unify and extend many known results in recent literature. However, theirs are all disguised forms of known results and their applicability is doubtful.

Note that, by choosing $\Gamma_N := \phi_N(\Delta_n)$ for each $N \in \langle X \rangle$, $(Y, X; \Gamma)$ becomes an abstract convex space and their generalized relatively KKM (R-KKM) mapping becomes simply a KKM map. Recall that the key result of [7] states a necessary and sufficient condition for $1_Y \in \mathfrak{KC}(Y, Y)$.

In 2005, the authors of [12] obtained a necessary and sufficient condition for $1_Y \in \mathfrak{KO}(Y, Y)$ and some of its routine consequences parallel to the corresponding ones in [7].

In the same year, in [19], its author follows the method in [7]. First of all, we note that, in these three papers [7,12,19], their authors concern with maps having compactly closed [open] values and peculiar closures ccl and interior cint. These are not practical, not general, and can be immediately replaced by ordinary ones by switching the relevant topology to its compactly generated extension; see [36].

In [19], a topological space Y is said to have property (H) if, for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. Then the following is introduced:

Definition 2.1 [19]. Let X be a nonempty set and Y be a topological space with property (H). $T : X \to 2^Y$ is said to be a generalized R-KKM mapping if for each $\{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ such that

$$\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

for all $\{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$.

Adopting those concepts, in [19], the author obtained some known results or their modifications in the G-convex space theory in which we supplied a large number of examples of such spaces. It is noteworthy that the authors of [7,8,12,19] claimed to obtain generalizations of known results without giving any justifications or any proper examples.

The works in [7,8,12,19] can be reduced to the ones in our new abstract convex space theory or G-convex space theory as follows:

Proposition 1. *The spaces having property* (H) *can be made into an abstract convex space* $(Y, X; \Gamma)$ *with* $\Gamma(A) = \Gamma_A := \varphi_N(\Delta_n)$ *for* $A := \{x_0, \ldots, x_n\} \in \langle X \rangle$, *where* $N \in \langle Y \rangle$ *is preassigned fixed one to* A.

We should recognize that, in the KKM theory on G-convex spaces, every argument is related to the finite intersection property of functional values of KKM maps, in other words, related to some $N \in \langle D \rangle$ in $(X, D; \Gamma)$. Therefore, the works in [7,8,12,19] can be reduced to the ones in our G-convex space theory as follows:

Proposition 2. *Every argument on KKM maps on a space having property* (H) *can be switched to the one for the* G*-convex space* $(Y, N; \Gamma')$ *for some* $N \in \langle Y \rangle$ *where*

$$\Gamma'_J = \Gamma'(J) := \varphi_N(\Delta_J) \quad \textit{for all} \quad J \in \langle N \rangle.$$

Moreover, we have

Proposition 3. *A generalized R-KKM map* $T : X \to 2^Y$ *is simply a KKM map for a* G*-convex space* $(Y, X; \Gamma)$.

In fact, let $A \in \langle X \rangle$ with $|A| = n + 1$. Then there corresponds an $N \in \langle Y \rangle$ with $|N| = n + 1$. Define $\Gamma : \langle X \rangle \multimap Y$ by $\Gamma_A := T(A)$ for each $A \in \langle X \rangle$. Then $(Y, X; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n + 1$, we have a continuous function $\phi_A := \varphi_N : \Delta_n \to T(A) =: \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle X \rangle$ and hence $T : X \multimap Y$ is a KKM map on a G-convex space $(Y, X; \Gamma)$.

Contrary to Proposition 3, Ding in [11] claimed as follows: "The above class of generalized *R-KKM* mappings include those classes of *KKM* mappings, *H-KKM* mappings, *G-KKM* mappings, generalized *G-KKM* mappings, generalized *S-KKM* mappings, *GLKKM* mappings and *GMKKM* mappings defined in topological vector spaces, H-spaces, G-convex spaces, G-H-spaces, L-convex spaces and hyperconvex metric spaces, respectively, as true subclasses."

Therefore, all of the KKM type theorems on such variants are simple consequences of our G-convex space theory. Consequently, all results in [11] are artificial disguised forms of known ones having no proper examples.

In 2006, the authors of [8] obtained the following coincidence theorem and its routine consequences:

Theorem 3.1 [8]. *Let* X, Y *be Hausdorff spaces. Let* $F \in \mathfrak{A}_c^\kappa(Y, X)$ *be a compact mapping,* $W : Y \to 2^Y$ *a generalized R-KKM mapping and* $G : X \to 2^Y$ *with nonempty values such that*

(i) $X = \bigcup_{y \in Y} \operatorname{Int} G^-(y)$;

(ii) *for each* $x \in X$, $M \in \langle G(x) \rangle$ *implies that* $W(M) \subset G(x)$.

Then there exist $(x_0, y_0) \in X \times Y$ *such that* $x_0 \in F(y_0)$ *and* $y_0 \in G(x_0)$.

This is a variant of a known result in [44,45]. We give a simple proof.

Proof using Theorem C. Recall that Y can be made into a G-convex space $(Y; \Gamma)$ with $\Gamma_M \subset W(M)$ as in Proposition 3. Note that $\mathfrak{A}_c^\kappa(Y, X) \subset \mathfrak{KC}(Y, X)$ [45] and that $F(Y) \subset \overline{F(Y)}$ is covered by a finite number of $G^-(y)$'s. The conclusion follows from Theorem C with $E = D := Y$, $Z := X$ and $S = T := G^-$.

Similarly all of other results in [8] can be simplified.

7. On pseudo H-spaces

In 2003, the authors of [27] introduced the following:

Definition 1 [27]. Let X be a topological space, D be a nonempty set. The triple (X, D, q) is said to be a pseudo H-space if for each nonempty finite subset A of D, the restricted mapping $q : \Delta^{|A|-1} \to 2^X$ is upper semi-continuous with nonempty compact values, where $\Delta^{|A|-1}$ is an $(|A|-1)$-simplex with vertices $\{e_1, e_2, \ldots, e_{|A|}\}$. If $D = X$, the triple (X, D, q) is written by (X, q).

The authors observed that a G-convex space $(X, D; \Gamma)$ with $|D| < \infty$ is an example of pseudo H-space and gave no other proper example. Therefore, they might obtain some results, but it seems to be not practical.

Now, by defining $\Gamma_A := q(\Delta^{|A|-1})$ for each nonempty finite subset A of D, then (X, D, q) can be an abstract convex space $(X, D; \Gamma)$. Therefore the basic theorems in [43] can be applied.

Moreover, the following is given in [27]:

Definition 2 [27]. Let (X, D, q) be a pseudo H-space. A mapping $F : D \to 2^X$ is a q-map if for each nonempty finite subset A of D, $q(\Delta^{|A|-1}) \subset \bigcup_{x \in A} F(x)$ and $q(\Delta^{|J|-1}) \subset \bigcup_{x \in J} F(x)$ for all nonempty finite subset J of A, where $\Delta^{|J|-1}$ is the convex hull of $\{e_{i_1}, e_{i_2}, \ldots, e_{i_{k+1}}\}$ if $A = \{a_1, a_2, \ldots, a_{n+1}\}$, $J = \{a_{i_1}, a_{i_2}, \ldots, a_{i_{k+1}}\}$.

The authors of [27] then showed that, under a strong restriction, a q-map can be a KKM map; that is, $1_X \in \mathfrak{KC}(X, X)$. From this KKM theorem, they deduced routine intersection result, Fan-Browder type fixed point theorems, a selection theorem, a Ky Fan type minimax inequality, and an application to abstract economies.

8. Comments on other works

In 2001, Balaj [2] obtained the following:

Theorem 1 [2]. *Let D be a nonempty subset of a convex space, Y a topological space and $G : D \to Y$ a map such that*

(i) *for each $x \in D$, Gx is compactly open in Y;*

(ii) $G(D) = Y$.

Then for each admissible compact map $F : \operatorname{co} D \to Y$ there exists $A \in \langle D \rangle$ such that $F(\operatorname{co} A) \cap \bigcap \{Gx : x \in A\} \neq \emptyset$.

Here an admissible map is in the sense of Gorniewicz and hence belongs to $\mathfrak{B}$ and $\mathfrak{KC}$.

Proof by using our Theorem D. Let $E := \operatorname{co} D$, $Z := Y$, $S := G$ in our Theorem D. Since F is compact, by (i) and (ii), $F(\operatorname{co} D) \subset G(N)$ for some $N \in \langle D \rangle$. Therefore all of the requirements of our Theorem D are satisfied and so the conclusion follows.

In [2], its author showed some equivalent formulations of his Theorem 1 and some applications. All of them are particular forms of results already appeared in [30] and can be stated in more general forms as in our work [40].

Moreover, in [3], its author repeated to state particular results of the ones in [45].

In 2003, Lin, Ansari, and Wu [24] and, in 2005, Lin and Wan [26], Lin and Chen [25] obtained the following:

Theorem 2.6 [24]. *Let X be a convex space, let Y be a topological space, and let $F \in KKM(X,Y)$. Let $P,Q : Y \to 2^X$ be multivalued maps. Assume that*

(i) *for each $y \in Y$, $\mathrm{co}(P(y)) \subset Q(y)$;*

(ii) *$P^{-1} : X \to 2^Y$ is transfer open, $P(y)$ is nonempty;*

(iii) *for each compact subset A of X, $\overline{F(A)}$ is compact;*

(iv) *there exists a nonempty compact subset K of Y such that, for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that*

$$F(L_N) \setminus K \subset \bigcup \{\mathrm{int}\, P^{-1}(x) \ : \ x \in L_N\}.$$

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in F(\bar{x})$ and $\bar{x} \in Q(\bar{y})$.

Proof by using Theorem C. Note that $F \in \mathfrak{KC}(X,Y)$ and

(ii) simply tells that $Y = \bigcup_{x \in X} \mathrm{int}\, P^{-1}(x)$.

Since K is a compact subset of Y, there is an $N \in \langle X \rangle$ such that $K = \bigcup_{x \in N} \mathrm{int}\, P^{-1}(x)$. Here we correct (iii) as follows:

(iii)$'$ $F(L_N)$ is compact.

Now, let $E = D := L_N$, $Z := Y$, $S := P^{-1}$, and $T := Q^{-1}$. Then $F|_{L_N} \in \mathfrak{KC}(L_N, Y)$ can be easily checked. Condition (i) implies (C.1). From (iv), we have

$$F(L_N) \subset (F(L_N) \setminus K) \cup K \subset \bigcup_{x \in L_N} \mathrm{int}\, P^{-1}(x).$$

This implies (C.2) by (iii)$'$. Hence there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{y} \in F(\bar{x}) \cap T(\bar{x})$. This completes our proof.

Note that, in [26], Theorem 3.2 and Corollary 3.1 follow from our Theorem C and, similarly, Theorems 3.3, 3.4 and Corollary 3.2 also follow from our Theorem C. The authors applied those results to the existence of solutions of certain generalized vector equilibrium problems.

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