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VARIOUS SUBCLASSES OF ABSTRACT CONVEX SPACES FOR THE KKM THEORY

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Abstract. We introduce some subclasses of the class of abstract convex spaces, namely, abstract convex minimal spaces, minimal KKM spaces, generalized convex minimal spaces, and others. Each of these subclasses is convenient for the KKM theory and contains G-convex spaces properly. The class of G-convex spaces contains the classes of L-spaces, spaces having property (H), FC-spaces, and others. Some related matters are also discussed.

1. Introduction

The KKM theory, originally called by the author in [26], is nowadays the study of applications of various equivalent formulations of the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in 1929 [23]. In the last decade, the theory has been extensively studied for generalized convex (G-convex) spaces in a sequence of papers of the author; for the details, see [27-36,46-49] and references therein.

Since the concept of G-convex spaces first appeared in 1993 [46], a number of modifications or imitations of the concept have followed. Such examples are L-spaces due to Ben-El-Mechaiekh et al. [2], spaces having property (H) due to Huang [22], FC-spaces due to Ding [6-15], and others. It is known that all of such examples are particular forms of G-convex spaces; see [42].

In our recent work [37], we introduced a new concept of abstract convex spaces and the multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ having certain KKM property. These new spaces and multimap classes are known to be adequate to establish the KKM theory; see [37-41]. Especially, in [41], we generalized and simplified the known results of the theory on convex spaces, H-spaces, G-convex spaces, and others. It is noticed there that the class of abstract convex spaces $(E, D; \Gamma)$ satisfying the KKM principle plays the major role in the KKM theory. It seems to be quite natural to call such spaces the KKM spaces. In our work [43], we show that a large number of well-known results in the KKM theory on G-convex spaces also hold on the KKM spaces.

Moreover, apparently motivated by the author's earlier works, Alimohammady et al. [1] introduced the notion of minimal G-convex spaces and obtained the open and closed versions of the KKM principle in this new setting. Their method is

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just replacing the topological structure in the relevant results by the more general minimal structure.

In our previous work [44], we introduced a new concept of abstract convex minimal spaces which can be also useful to establish some results in the KKM theory. With this new concept, we obtained generalizations of the KKM principle. Furthermore, the KKM type maps were used to obtain coincidence theorems, the Fan-Browder type fixed point theorems, the Fan intersection theorem, and the Nash equilibrium theorem on abstract convex minimal spaces.

Now it is evident that the class of abstract convex spaces contains many subclasses which are adequate to establish the KKM theory. In the present paper, we introduce such subclasses, namely, abstract convex minimal spaces, minimal KKM spaces, generalized convex minimal spaces, and the corresponding ones for topological spaces. All of these contains G-convex spaces properly. The class of G-convex spaces contains the classes of L-spaces, spaces having property (H), FC-spaces, and others. In general, a *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ is shown to be a G-convex space, where $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Some related matters are also discussed.

2. Abstract convex spaces

In this section, we recall definition of abstract convex spaces given in [37-41]:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \to 2^E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\}.$$

[co is reserved for the convex hull in topological vector spaces (t.v.s.).] A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D'\rangle})$ itself is an abstract convex space called a *subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

If E is given a topology, then the abstract convex space $(E, D; \Gamma)$ is called *an abstract convex topological space.*

We already gave plenty of examples of abstract convex spaces in [37,41].

The following is given in [1]:

Definition 2.2. A family $\mathcal{M} \subset 2^X$ is called a *minimal structure* on a set X if $\emptyset, X \in \mathcal{M}$. In this case, $(X, \mathcal{M})$ is called a *minimal space.* Any element of $\mathcal{M}$ is called an *m-open set* of X and a complement of an m-open set is called an *m-closed set* of X. For minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a function $f : X \to Y$ is said to be *continuous* (more precisely, *m-continuous* or $(\mathcal{M}, \mathcal{N})$-*continuous*) if $f^{-1}(V) \in \mathcal{M}$ for each $V \in \mathcal{N}$.

Definition 2.3. If E is given a minimal structure, then the abstract convex space $(E, D; \Gamma)$ is called *an abstract convex minimal space.*

Example 2.4. (1) Any topological space is a minimal space and not conversely.

(2) Any t.v.s. is a minimal vector space. There is some linear minimal space which is not a t.v.s.; see [1].

3. The KKM spaces

Definition 3.1. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \to 2^Z$ with nonempty values, if a multimap $G : D \to 2^Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. When $E = Z$, a *KKM map* $G : D \to 2^E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \to 2^Z$ is said to have the *KKM property* and called *a* $\mathfrak{K}$*-map* if, for any KKM map $G : D \to 2^Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \to 2^Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical.

Furthermore, when $(Z, \mathcal{M})$ is a minimal space, an $m\mathfrak{KC}$-map is defined for m-closed-valued maps G, and an $m\mathfrak{KO}$-map for m-open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset m\mathfrak{KC}(E, Z) \cap m\mathfrak{KO}(E, Z).$$

Example 3.2. (1) Every abstract convex space in our sense has a map $F \in \mathfrak{K}(E, Z)$ for any nonempty set Z and for any class of KKM maps $G : D \to 2^Z$ with respect to F. In fact, for each $x \in E$, choose $F(x) := Z$ or $F(x)$ contains some $z_0 \in Z$.

(2) Further examples were given in Section 5 of [37].

Definition 3.3. For an abstract convex topological space $(E, D; \Gamma)$, the *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$. A *KKM space* is an abstract convex topological space satisfying the KKM principle [43].

For an abstract convex minimal space $(E, D; \Gamma)$, the *KKM principle* is the statement $1_E \in m\mathfrak{KC}(E, E) \cap m\mathfrak{KO}(E, E)$. A *minimal KKM space* (or simply, *mKKM space*) is an abstract convex minimal space satisfying the KKM principle.

Example 3.4. We give examples of KKM spaces:

(1) The triple $(\Delta_n, V; \mathrm{co})$, where V denotes the set of vertices and $\mathrm{co} : \langle V \rangle \to 2^{\Delta_n}$ the convex hull operation, is a KKM space. The original KKM principle [23] is $1_{\Delta_n} \in \mathfrak{KC}(\Delta_n, \Delta_n)$. Later it was known that $1_{\Delta_n} \in \mathfrak{KO}(\Delta_n, \Delta_n)$.

(2) Ky Fan's $(E, D; \mathrm{co})$, where D is a nonempty subset of a topological vector space E, is a KKM space. His celebrated lemma [19] is $1_E \in \mathfrak{KC}(E, E)$.

These are the origins of our G-convex space $(X, D; \Gamma)$. Note that any KKM type theorem on $(X; \Gamma)$ can not generalize the KKM principle and the Ky Fan lemma.

(3) Every generalized convex space is a KKM space; see [29,30,49].

(4) A connected ordered space $(X, \leq)$ can be made into an abstract convex topological space $(X \supset D; \Gamma)$ for any nonempty $D \subset X$ by defining

$$\Gamma_A := [\min A,\ \max A] = \{x \in X \mid \min A \leq x \leq \max A\} \ \text{ for each } A \in \langle D \rangle.$$

Moreover, it is a KKM space; see [40, Theorem 5(i)].

(5) The extended long line L^* can be made into a KKM space $(L^* \supset D; \Gamma)$; see [40]. In fact, L^* is constructed from the ordinal space $D := [0, \Omega]$ consisting of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology. Recall that L^* is a generalized arc obtained from $[0, \Omega]$ by placing a copy of the interval $(0, 1)$ between each ordinal α and its successor $\alpha + 1$ and we give L^* the order topology. Now let $\Gamma : \langle D \rangle \to 2^{L^*}$ be the one as in (2).

In our previous work [29], for G-convex spaces, there exist more than 15 equivalent formulations of the KKM principle such as Alexandroff-Pasynkoff theorem, Ky Fan type matching theorem, Tarafdar type intersection theorem, geometric or section properties, Fan-Browder type fixed point theorems, maximal element theorems, analytic alternatives, Ky Fan type minimax inequalities, variational inequalities, and others. This is also true for KKM spaces.

In our forthcoming paper [43], we show that some of well-known results in the KKM theory on G-convex spaces also hold on the KKM spaces. Examples of such results are theorems of Sperner and Alexandroff-Pasynkoff, the Horvath type fixed point theorem, the Fan-Browder type coincidence theorems, the Ky Fan type minimax inequalities, variational inequalities, the von Neumann type minimax theorem, and the Nash type equilibrium theorem.

4. Generalized convex spaces

Recall the following appeared in [27-36,49]:

Definition 4.1. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \to 2^X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$.

Definition 4.2. A *generalized convex minimal space* or a *G-convex minimal space* $(X, D; \Gamma)$ consists of a minimal space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \to 2^X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. See [1].

Remark. (1) A G-convex space is a G-convex minimal space, and the converse does not hold; for example, see [1].

(2) A G-convex space is a KKM space and the converse does not hold; for example, the extended long line L^* is a KKM space $(L^* \supset D; \Gamma)$, but not a G-convex space.

In fact, since $\Gamma\{0, \Omega\} = L^*$ is not path connected, for $A := \{0, \Omega\} \in \langle L^* \rangle$ and $\Delta_1 := [0, 1]$, there does not exist a continuous function $\phi_A : [0, 1] \to \Gamma_A$ such that $\phi_A\{0\} \subset \Gamma\{0\} = \{0\}$ and $\phi_A\{1\} \subset \Gamma\{\Omega\} = \{\Omega\}$. Therefore $(L^* \supset D; \Gamma)$ is not G-convex.

(3) A G-convex minimal space $(X, D; \Gamma)$ is a minimal KKM space in view of the following:

Theorem 4.3. *Let $(E, D; \Gamma)$ be a G-convex minimal space and $F : D \to 2^E$ a KKM map with m-closed values [resp., m-open values]. Then $\{F(z)\}_{z\in D}$ has the finite intersection property.*

Essentially, the proof of Theorem 4.3 [1, Theorems 3.2 and 3.5] is the one in [29,30,49] with slight modifications.

It is obvious that many facts on G-convex spaces can be extended to corresponding ones on G-convex minimal spaces.

In the category of topological vector spaces or C-spaces, the concepts of locally convex spaces, LC-spaces, Φ-spaces, subsets of the Zima-Hadžić type, admissible subsets, and sets of Klee approximability are quite well-known; see [36]. They were introduced to establish generalization of the known fixed point theorems.

In our previous work [36], we extended these concepts to G-convex uniform spaces and established the mutual relations among them as follows:

Theorem 4.4. *In the class of G-convex uniform spaces, the following hold:*

(1) *Any LG-space is of the Zima-Hadžić type.*

(2) *Every LG-space is locally G-convex whenever every singleton is Γ-convex.*

(3) *Any nonempty subset of a locally G-convex space is a Φ-set.*

(4) *Any Zima-Hadžić type subset of a G-convex uniform space such that every singleton is Γ-convex is a Φ-set.*

(5) *Every Φ-space is admissible. More generally, every nonempty compact Φ-subset is Klee approximable.*

Note that Theorem 4.4 can be extended to the KKM uniform spaces.

For details on G-convex spaces, see [27-36,46-49], where basic theory was extensively developed.

5. Examples of G-convex spaces

There are lots of examples of G-convex spaces:

Example 5.1. If X is a subset of a vector space, $D \subset X$ such that $\operatorname{co} D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology, then $(X \supset D; \Gamma)$ becomes a *convex space* generalizing the one due to Lassonde [24]. Note that any convex subset of a t.v.s. is a convex space, but not conversely.

Example 5.2. If Γ_A is assumed to be contractible or, more generally, ω-connected (that is, n-connected for all $n \geq 0$), and if for each $A, B \in \langle D\rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then $(X, D; \Gamma)$ becomes an *H-space* [45]. For $X = D$, an H-space reduces to a C-space of Horvath [21]. There are many examples of C-spaces due to Horvath [21]. It is notable that a torus, the Möbius band, or the Klein bottle can be regarded as C-spaces, and are examples of compact G-convex spaces without having the fixed point property.

Example 5.3. Let $X = D = [0, 1)$ and $Y = D' = \mathbb{S}^1 = \{e^{2\pi it} \mid t \in [0, 1)\}$ in the complex plane $\mathbb{C}$. Let $f : X \to Y$ be a continuous function defined by $f(t) = e^{2\pi it}$. Define $\Gamma : \langle D'\rangle \to 2^Y$ by

$$\Gamma_A = f(\operatorname{co}(f^{-1}(A))) \quad \text{for} \quad A \in \langle D'\rangle.$$

Then $(Y \supset D'; \Gamma)$ is a compact G-convex space. Recall that any continuous image of a G-convex space is a G-convex space. We note the following:

(1) $\mathbb{S}^1$ lacks the fixed point property. Moreover, $(Y \supset D'; \Gamma)$ is an example of a compact H-space since each Γ_A is contractible. Therefore, it shows that the Schauder conjecture (that is, any compact convex subset of a topological vector space has the fixed point property) does not hold for G-convex spaces.

(2) Note that, in $(Y \supset D'; \Gamma)$, singletons are Γ-convex; that is, $\Gamma_{\{y\}} = \{y\}$ for each $y \in D'$.

(3) $(Y, D; \Gamma)$ with $\Gamma : \langle D \rangle \to 2^Y$ defined by

$$\Gamma_A = f(\operatorname{co} A) \quad \text{for } A \in \langle D \rangle$$

is an example of an H-space satisfying $D \not\subset Y$.

Example 5.4. Let $X = D = [0,1]$ and $Y = D' = \mathbb{S}^1 = \{e^{2\pi i t} \mid t \in [0,1]\}$. Define f and Γ_A as in Example 5.3. Then $(Y \supset D'; \Gamma)$ is a compact G-convex space.

(1) Note that $1 \in \mathbb{S}^1$ and that $\Gamma_{\{1\}} = \mathbb{S}^1$ is not contractible. Hence, $(Y \supset D'; \Gamma)$ is not an H-space.

(2) Moreover $\Gamma_{\{1\}} \neq \{1\}$. Therefore, in general, $\Gamma_{\{y\}} \neq \{y\}$ in an H-space.

Example 5.5. Other major examples of G-convex spaces are metric spaces having the Michael convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, topological semilattices with path-connected intervals, and so on. For the literature, see [27-36,46]. Moreover, further examples of G-convex spaces were given by the author [32] as follows: L-spaces and B'-simplicial convexity of Ben-El-Mechaiekh et al. [2], continuous images of G-convex spaces, Verma's or Stachó's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, mc-spaces of Llinares (In [Y], it is incorrectly stated that G-convex spaces are mc-spaces), hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces; see [28-34]. Note also that Ding's FC-space [6-15] is a particular form of G-convex spaces.

Example 5.6. Any hyperbolic space X in the sense of Kirk and Reich-Shafrir is a G-convex space, since the closed convex hull of any $A \in \langle X \rangle$ is contractible. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic; see [50].

In the remainder of this paper, we deal with particular subclasses of the class of G-convex spaces. Such subclasses are incorrectly claimed to contain G-convex spaces by some authors even in the 21st century.

6. L-SPACES

For the definition of a G-convex space, at first in [46-48], we assumed an additional isotonicity condition as follows:

$(*)$ for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

From 1998, this isotonicity has been removed; see [27-36].

Example 6.1. Let $\Delta_3 = \operatorname{co} V$ where $V = \{e_0, e_1, e_2, e_3\}$.

(1) We have a G-convex space $(\Delta_3, V; \operatorname{co})$ where $\operatorname{co} : \langle V \rangle \to 2^{\Delta_3}$ is the convex hull operator.

(2) Let $(\Delta_3, V; \Gamma)$ be a G-convex space given by $\Gamma\{e_0, e_1\} := \mathrm{co}\{e_0, e_1, e_2\}$ for $\{e_0, e_1\} \in \langle V\rangle$ and $\Gamma(N) := \mathrm{co}\, N$ for all other $N \in \langle V\rangle$. Then Γ violates the isotonicity $(*)$.

In the same year, Ben-El-Mechaiekh et al. [2] defined an L-space (E, Γ), which is a particular form of our G-convex space $(X, D; \Gamma)$ [without assuming $(*)$] for the case $E = X = D$ as follows:

Definition 6.2. ([2], Def. 3.1) An L-structure on a topological space E is given by a nonempty set-valued map $\Gamma : \langle E\rangle \to E$ verifying

$(**)$ for each $A \in \langle E\rangle$, say $A = \{x_0, x_1, \ldots, x_n\}$, there exists a continuous function $f^A : \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0, 1, \ldots, n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_i, i \in J\})$.

The pair (E, Γ) is then called an *L-space*, and $X \subset E$ is said to be *L-convex* if $\forall A \in \langle X\rangle$, $\Gamma(A) \subset X$.

Then the authors of [2] stated that, in particular, if Γ, as in Definition 3.1 [2], verifies the additional condition $(*)$, then the pair (E, Γ) is what is called by Park and Kim [46], a *G-convex space.*

This does not mean that the class of L-spaces contains G-convex spaces. In fact, the authors of [2] imitated our definition of G-convex spaces and implicitly stated that, under the condition $(*)$, their L-spaces reduce to our original G-convex spaces. From the beginning, any L-space is a G-convex space and not conversely.

With the misconception that the class of L-spaces contains our G-convex spaces, some authors [4,5,17,25,51] restated a number of particular results on L-spaces which were previously known for G-convex spaces. All of these authors failed to give any proper example justifying their misconception.

Note that the concept of minimal L-spaces is possible.

7. Spaces having property (H)

In [22], a topological space Y is said to have property (H) if, for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. Then the following is introduced:

Definition 7.1. ([22], Def. 2.1) Let X be a nonempty set and Y be a topological space with property (H). $T : X \to 2^Y$ is said to be a generalized R-KKM mapping if for each $\{x_0, \ldots, x_n\} \in \langle X\rangle$, there exists $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ such that

$$\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} T x_{i_j},$$

for all $\{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$.

Adopting this concept, in [22], its author obtained modifications of some known results in the G-convex space theory in which we had already supplied a large number of examples of such spaces. It is noteworthy that the authors of [3,16,18,22] adopted R-KKM maps and claimed to obtain generalizations of known results without giving any justifications or any proper examples.

For a G-convex space $(X, D; \Gamma)$, a multimap $G : D \to 2^X$ is called a *KKM map* if $\Gamma_A \subset G(A)$ for each $A \in \langle D\rangle$.

We should recognize that, in the KKM theory on G-convex spaces, every argument is related to the finite intersection property of functional values of KKM maps having closed [resp. open] values, in other words, related to some $N \in \langle D \rangle$ in $(X, D; \Gamma)$. Therefore, the works in [3,16,18,22] can be reduced to the ones in our G-convex space theory as follows:

Theorem 7.2. *Every argument on KKM maps on a space Y having property (H) can be switched to the one for the G-convex space $(Y, N; \Gamma')$ for some $N \in \langle Y \rangle$ where*

$$\Gamma'_J = \Gamma'(J) := \varphi_N(\Delta_J) \quad \textit{for all} \quad J \in \langle N \rangle.$$

Moreover, we have

Theorem 7.3. *A generalized R-KKM map $T : X \to 2^Y$ is simply a KKM map for a G-convex space $(Y, X; \Gamma)$.*

Proof. Let $A \in \langle X \rangle$ with $|A| = n + 1$. Then there corresponds an $N \in \langle Y \rangle$ with $|N| = n + 1$. Define $\Gamma : \langle X \rangle \to 2^Y$ by $\Gamma_A := T(A)$ for each $A \in \langle X \rangle$. Then $(Y, X; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n + 1$, we have a continuous function $\phi_A := \varphi_N : \Delta_n \to T(A) =: \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle X \rangle$ and hence $T : X \to 2^Y$ is a KKM map on a G-convex space $(Y, X; \Gamma)$. □

Note that the concepts of the property (H) and the generalized R-KKM map can be extended with respect to minimal spaces instead of topological spaces.

8. FC-spaces

In 2005, Ding [6] introduced the following notion of "a finitely continuous" topological space (in short, FC-space):

Definition 8.1. ([6], Def. 1.1) $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. A subset D of $(Y, \{\varphi_N\})$ is said to be a FC-subspace of Y if for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subset N \cap D$, $\varphi_N(\Delta_k) \subset D$ where $\Delta_k = \mathrm{co}\{e_{i_j} : j = 0, \ldots, k\}$.

In 2006, Ding added the following to [6, Def. 1.1]:

Definition 8.2. ([7], Def. 2.1) If A and B are two subsets of Y, B is said to be a FC-subspace of Y relative to A if for each $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ and for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset A \cap N$, $\varphi_N(\Delta_k) \subset B$ where $\Delta_k = \mathrm{co}\{e_{i_j} : j = 0, \ldots, k\}$. If $A = B$, then B is called a FC-subspace of Y.

Then Ding [7] wrote: "It is easy to see that the class of FC-spaces includes the classes of convex sets in topological vector spaces, C-spaces (or H-spaces) [21], G-convex spaces [48], L-convex spaces [2], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces." Here he failed to give any justification or any proper example of his space which is not G-convex.

The above definition and Ding's claim have appeared also in [8-15,20,52,54], and possibly more because one dozen of such papers on FC-spaces have appeared already within two years. In these papers, known results in KKM theory on G-convex spaces are restated or modified for the so-called FC-spaces. In order to prevent such unnecessary efforts, something has to be done.

Note that L-spaces, spaces having property (H), and FC-spaces have a family $\{\phi_N\}$ of continuous functions; see Section 9. Recall that our G-convex space $(X, D; \Gamma)$ is originated from the KKM principle [23] and the celebrated Ky Fan lemma [19] from the beginning. The case $X = D$ is not applicable to them and this is the most serious defect of L-spaces or FC-spaces. Hence, they are inadequate for the KKM theory.

Now Ding's FC-subspace relative to some subset A in Definition 8.2 ([7], Def. 2.1) can be extended as follows:

Definition 8.3. Let $(E, D; \Gamma)$ be a G-convex space and $X \subset E,\ D' \subset D$. Then X is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$.

Recall that, for a G-convex space $(E \supset D; \Gamma)$, we used to say that a subset X of E is Γ-*convex* if, for any $N \in \langle X \cap D \rangle$, we have $\Gamma_N \subset X$. This is now saying that X is Γ-convex relative to $D' := X \cap D$.

Therefore, instead of using the concept of an FC-subspace of $(Y, \{\varphi_N\})$ relative to A as in Definition 8.2 [7, Def. 2.1], we may use a Γ-convex subset of the G-convex space $(Y, D; \Gamma)$ relative to $A \subset D$. Any interested reader can check this matter in all of [6-15,20,54].

For a topological space $(X, \mathcal{T})$, the compactly generated extension (or the k-extension) $\mathcal{T}_k$ of the original topology $\mathcal{T}$ is a new topology of X finer than $\mathcal{T}$ such that $\mathcal{T}_k$ is the collection of all compactly open [resp., compactly closed] subsets of $(X, \mathcal{T})$. Note that the artificial terminology of compact interior, compact closure, etc., are not practical and can be eliminated by switching the original topology of the underlying space to its compactly generated extension; see [30].

Such inadequate artificial terminology was used in [4-8,10,15,20,54], but disappeared or withdrawed in [9,11-14].

Moreover, the topological structure in FC-space can be replaced by the minimal structure with minor modification.

9. ϕ_A-SPACES

Recently, we are concerned with particular subclasses or variants of G-convex spaces as follows:

Definition 9.1. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Similarly, a *minimal* ϕ_A-*space* can be defined whenever X is a minimal space.

Note that, by putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$, a ϕ_A-space becomes an abstract convex space.

It is clear that any G-convex space is a ϕ_A-space. The converse also holds:

Theorem 9.2. *Any* ϕ_A*-space* $(X, D; \{\phi_N\})$ *can be made into a* G*-convex space* $(X, D; \Gamma)$.

Proof. In fact, this can be done at least in three ways.

(1) For each $A \in \langle D\rangle$, by putting $\Gamma_A := X$, we obtain a trivial G-convex space $(X, D; \Gamma)$.

(2) Let $\{\Gamma^\alpha\}_\alpha$ be the family of maps $\Gamma^\alpha : \langle D\rangle \to 2^X$ giving a G-convex space $(X, D; \Gamma^\alpha)$. Note that, by (1), this family is not empty. Then, for each α and each $A \in \langle D\rangle$ with $|A| = n + 1$, we have

$$\phi_A(\Delta_n) \subset \Gamma^\alpha_A \ \text{ and } \ \phi_A(\Delta_J) \subset \Gamma^\alpha_J \quad \text{for } \ J \subset A.$$

Let $\Gamma := \bigcap_\alpha \Gamma^\alpha$, that is, $\Gamma_A = \bigcap_\alpha \Gamma^\alpha_A$. Then

$$\phi_A(\Delta_n) \subset \Gamma_A \ \text{ and } \ \phi_A(\Delta_J) \subset \Gamma_J \quad \text{for } \ J \subset A.$$

Therefore, $(X, D; \Gamma)$ is a G-convex space.

(3) Let $N \in \langle D\rangle$ with $|N| = n + 1$. For each $M \in \langle D\rangle$ with $N \subset M$, $M = \{a_0, \dots, a_m\}$ and $N = \{a_{i_0}, \dots, a_{i_n}\}$, there exists a subset $\phi_M(\Delta^M_n)$ of X such that $\Delta^M_n := \mathrm{co}\{e_{i_j} \mid j = 0, \dots, n\} \subset \Delta_m$. Now let

$$\Gamma_N = \Gamma(N) := \bigcup_{M\supset N} \phi_M(\Delta^M_n).$$

Then $\Gamma : \langle D\rangle \to 2^X$ is well-defined and $(X, D; \Gamma)$ becomes a G-convex space: In fact, for each $A \in \langle D\rangle$ with $|A| = n + 1$, there exists a continuous map $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. □

Therefore, G-convex spaces and ϕ_A-spaces are essentially same. Similarly, minimal G-convex spaces and minimal ϕ_A-spaces are also same.

Example 9.3. Let $(\Delta_3, V; \{\phi_N\})$ be a triple where $\phi_N(\Delta_n) = \Gamma(N)$ as in Example 5.1(2). Then

$$\phi_{\{e_0,e_1\}}(\Delta_1) = \phi_{\{e_0,e_1\}}(\mathrm{co}\,\{e_0, e_1\}) = \Gamma\{e_0, e_1\} = \mathrm{co}\,\{e_0, e_1, e_2\},$$

where we may assume $\phi_{\{e_0,e_1\}}$ is a surjective space-filling curve such that $\phi_{\{e_0,e_1\}}(e_0) = e_0$ and $\phi_{\{e_0,e_1\}}(e_1) = e_1$. Then it is easily checked that Γ itself is the one in the proof (3) of Theorem 9.2 corresponding to $\{\phi_N\}$.

From Theorem 9.1, contrary to Ding's claim, we have the following:

Theorem 9.4. *An FC-space $(Y, \{\varphi_N\})$ can be made into an L-space $(Y; \Gamma)$, a particular type of G-convex spaces $(Y, D; \Gamma)$.*

Proof. In Definition 8.1 ([6], Def. 1.1), we can eliminate the case where some elements in N may be same. Then we can define a map $\Gamma : \langle Y\rangle \to Y$ as in the proof of Theorem 9.1. Therefore, the so-called FC-spaces are L-spaces and hence very particular forms of our G-convex spaces. □

Theorem 7.3 can be extended as follows:

Theorem 9.5. *For a ϕ_A-space $(X, D; \{\phi_A\})$, any map $T : D \to 2^X$ satisfying*

$$\phi_A(\Delta_J) \subset T(J) \ \text{ for each } \ A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is a KKM map on a G-convex space $(X, D; \Gamma)$.

Proof. Define $\Gamma : \langle D\rangle \to 2^X$ by $\Gamma_A := T(A)$ for each $A \in \langle D\rangle$. Then $(X, D; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n+1$, we have a continuous function $\phi_A : \Delta_n \to T(A) =: \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle D\rangle$ and hence $T : D \to 2^X$ is a KKM map on a G-convex space $(X, D; \Gamma)$. □

Note that every ϕ_A-space is a KKM space since it is a G-convex space. Note also that, in the recent study on abstract convex spaces in [37-41], many basic theorems on G-convex spaces are further generalized; and that all facts in this section can be extended to minimal spaces.

Remark. This is a review article delivered at the NIMS Workshop. Parts of this may appear somewhere else.

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