EXAMPLES OF $\mathfrak{KC}$-MAPS AND $\mathfrak{KO}$-MAPS ON ABSTRACT CONVEX SPACES

Sehie Park

The National Academy of Sciences, Republic of Korea, and
School of Mathematical Sciences,
Seoul National University, Seoul 151–747, Korea
E-mail address: shpark@math.snu.ac.kr

Abstract. We give examples of the multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ in abstract convex spaces or connected ordered spaces. Some of our results are generalizations of those of Jeng, Huang, and Zhang [3]. Consequently, we have plenty of new examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps.

1. Introduction

In early 1990's, the author introduced the admissible class $\mathfrak{A}_c^\kappa$ of multimaps in the KKM theory. Since then there have appeared new classes of multimaps such as the KKM class, the S-KKM class, the 'better' admissible class $\mathfrak{B}$, and modifications of them. Those classes of multimaps were first applied to convex spaces and later to generalized convex spaces (simply, G-convex spaces) due to the author.

Recently, in [6], we introduced a new concept of abstract convex spaces and multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ having certain KKM property which are adequate to establish the KKM theory. With these new concepts, in [7], we generalized and simplified known results of the theory on convex spaces, H-spaces, G-convex

Received
2000 Mathematics Subject Classification. Primary 47H04, 47H10. Secondary 46A16, 46A55, 52A07, 54C60, 54H25, 55M20.
Key words and phrases. multimap (map), abstract convex space, multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$, connected ordered space.

spaces, convexity spaces in the classical sense, and others. Note that the class $\mathfrak{KC}$ contains the KKM class and the S-KKM class.

In the present paper, our aim is to give examples of multimaps in the classes $\mathfrak{KC}$ and $\mathfrak{KO}$ on abstract convex spaces. Some of our results are generalizations of those of Jeng, Huang, and Zhang [3], where some significant examples of multimaps in the class $\mathfrak{KC}$ not in the class $\mathfrak{A}_c^\kappa$ were given for convex spaces.

Section 2 is concerned with definitions of abstract convex spaces and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$. We also introduce the concept and examples of connected ordered spaces. In Sections 3 and 4, we give generalizations of results of Jeng, Huang, and Zhang [3] and other new results. In fact, we show that the main results in [3] and our comments to them in [8] can be generalized. Consequently, we have new examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps on abstract convex spaces or connected ordered spaces. Section 3 is concerned with results for abstract convex spaces, and Section 4 for connected ordered spaces.

2. Abstract convex spaces and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$

This section is concerned with definitions of abstract convex spaces and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ as in [6]. We also introduce the concept and examples of connected ordered spaces as in [5].

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definitions. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\}.$$

[co is reserved for the convex hull in topological vector spaces.] A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D'\rangle})$ itself is an abstract convex space called a *subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if, for any $A \in \langle X \cap D \rangle$, we have $\Gamma_A \subset X$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

If E is given a topology, then the abstract convex space $(E, D; \Gamma)$ is called *an abstract convex topological space.*

Typical examples of abstract convex spaces are convex subsets of topological vector spaces, convex spaces due to Lassonde, C-spaces (or H-spaces) due to Horvath, generalized convex (simply, G-convex) spaces due to Park, and convexity spaces in the classical sense; see [6-8] and references therein.

Definitions. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called *a* $\mathfrak{K}$-*map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. Then, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical.

Example. Every abstract convex space in our sense has a map $F \in \mathfrak{K}(E, Z)$ for any nonempty set Z. In fact, for each $x \in E$, choose $F(x) := Z$ or $F(x) := \{z_0\}$ for some $z_0 \in Z$.

We know several relations between multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ as in [8]. The following is a typical one:

Theorem 1. *Let* $(X, D; \Gamma)$ *be an abstract convex space,* Z *a topological space, and* $F : X \multimap Z$*. Suppose that for any* $A \in \langle D \rangle$ *with* $|A| = n+1$*, the set* $F(\Gamma_A)$ *in its induced topology is a normal space. If* $F \in \mathfrak{KC}(X, Z)$*, then* $F \in \mathfrak{KO}(X, Z)$*. The converse also holds.*

For the converse case, we may assume the set $\overline{F(\Gamma_A)}$ in its induced topology is a normal space. Therefore we have

Corollary 1.1. *Let* $(X, D; \Gamma)$ *be an abstract convex space and* Z *a normal space. Then* $\mathfrak{KO}(X, Z) \subset \mathfrak{KC}(X, Z)$*.*

We need the following:

Definitions. A linearly ordered set $(X, \leq)$ is called an *ordered space* if it has the order topology whose subbase consists of all sets of the form $\{x \in X \mid x < s\}$ and $\{x \in X \mid x > s\}$ for $s \in X$. Note that an ordered space X is connected iff it is Dedekind complete (that is, every subset of X having an upper bound has a supremum) and whenever $x < y$ in X, then $x < z < y$ for some z in X. For details, see Willard [11].

A connected ordered space $(X, \leq)$ can be made into an abstract convex topological space $(X \supset D; \Gamma)$ for any nonempty $D \subset X$ by defining $\Gamma_A := [\min A,\ \max A] = \{x \in X \mid \ \min A \leq x \leq \max A\}$ for each $A \in \langle D \rangle$.

Examples. We give some examples of connected ordered spaces.

(1) Any nonempty interval of the real line $\mathbb{R}$.

(2) Connected $[0, 1]$-spaces; that is, connected spaces admitting a continuous bijection onto the unit interval; see [4, 9].

(3) The following is a connected ordered space:

$$X = \{(0, 0)\} \cup \{(x, y) \mid x \in (0, 1] \ \text{ and } \ y = \sin 1/x\} \subset \mathbb{R}^2.$$

(4) A generalized arc; that is, a continuum which has exactly two non-cut points. For example, the extended long line L^* constructed from the ordinal space $[0, \Omega]$ consisting of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology. Recall that L^* is a generalized arc obtained from $[0, \Omega]$ by placing a copy of the interval $(0, 1)$ between each ordinal α and its successor $\alpha + 1$ and we give L^* the order topology; see [10].

Examples. For subsets of a topological vector spaces or other topological spaces, the following sequence of implications is clear:

$$\text{convex} \implies \text{star-shaped} \implies \text{contractible} \implies \text{acyclic} \implies \text{connected.}$$

Here, a space is *contractible* if the identity map is homotopic to a constant map; and a nonempty space is *acyclic* if it is connected and its Čech homology (with a fixed coefficient field) is zero in dimensions greater than zero.

3. On $\mathfrak{KC}$ and $\mathfrak{KO}$ on abstract convex spaces

In 2002, Jeng, Huang, and Zhang [3] obtained some interesting examples of $\mathfrak{KC}$-maps on convex spaces or nonempty intervals on the real line. In our previous work [8], we gave some comments and observations on results of [3]. In Sections 3 and 4 in the present paper, we show that some of the results in [3] and our comments to them in [8] can be generalized. Consequently, we give new examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps on abstract convex spaces or connected ordered spaces.

In this section, we are concerned with results for abstract convex spaces:

Theorem 2. *Let $(X;\Gamma)$ be an abstract convex space such that $x \in \Gamma_A$ for all $A \in \langle X \rangle$ and $x \in A$, $(Y;\Omega)$ an abstract convex topological space, and $F : X \multimap Y$ such that, for each $A \in \langle X \rangle$, $F(\Gamma_A)$ is Ω-convex in Y.*

(1) *If $1_Y \in \mathfrak{KC}(Y,Y)$, then $F \in \mathfrak{KC}(X,Y)$.*

(2) *If $1_Y \in \mathfrak{KO}(Y,Y)$, then $F \in \mathfrak{KO}(X,Y)$.*

Proof. We follow that of [3, Theorem 2.2]. Let $G : X \multimap Y$ be a closed-valued [resp. open-valued] KKM map with respect to F. Let $A := \{x_1, \cdots, x_n\} \in \langle X \rangle$, choose $y_i \in F(x_i)$, and put $A_i := \{x_j \mid 1 \leq j \leq n, y_i \in F(x_j)\}$ for each $i = 1, \cdots, n$. Define $H : Y \multimap Y$ by

$$H(y) := \begin{cases} Y, & \text{if } y \notin \{y_1, \cdots, y_n\}; \\ \bigcap_{x_j \in A_i} G(x_j), & \text{if } y = y_i,\ i = 1, \cdots, n. \end{cases}$$

Then each $H(y)$ is nonempty and closed [resp. open]. We show that H is a KKM map on Y, that is, $\Gamma_B \subset H(B)$ for each $B \in \langle \{y_1, \cdots, y_n\} \rangle$. Let $B := \{y_{i_1}, \cdots, y_{i_l}\}$. For any $x_{j(k)} \in A_{i_k}$, $k = 1, \cdots, l$, we have

$$y_{i_k} \in F(x_{j(k)}) \subset F(\Gamma(\{x_{j(1)}, \cdots, x_{j(l)}\}),$$

and hence

$$\Omega_B \subset F(\Gamma(\{x_{j(1)}, \cdots, x_{j(l)}\}) \subset \bigcup_{k=1}^{l} G(x_{j(k)}).$$

Since $x_{j(k)} \in A_{i_k}$ is arbitrary, we have

$$\Omega_B \subset \bigcup_{k=1}^{l} (\bigcap_{x_{j(k)} \in A_{i_k}} G(x_{j(k)})) = \bigcup_{k=1}^{l} H(y_{i(k)}).$$

Therefore H is a KKM map. Since $1_Y \in \mathfrak{KC}(Y, Y)$ [resp. $1_Y \in \mathfrak{KO}(Y, Y)$], $\{H(y) \mid y \in Y\}$ has the finite intersection property. In particular, $\bigcap_{i=1}^{n} H(y_i) \neq \emptyset$ and hence $\bigcap_{i=1}^{n} G(x_i) \neq \emptyset$. Therefore the conclusion follows.

Corollary 2.1. *Let $(X; \Gamma)$ be an abstract convex space such that $x \in \Gamma_A$ for all $A \in \langle X \rangle$ and $x \in A$, $(Y; \Omega)$ a G-convex space, and $F : X \multimap Y$ such that, for each $A \in \langle X \rangle$, $F(\Gamma_A)$ is Ω-convex in Y. Then $F \in \mathfrak{KC}(X, Y) \cap \mathfrak{KO}(X, Y)$.*

Proof. Since $(Y; \Omega)$ is a G-convex space, we have $1_Y \in \mathfrak{KC}(Y, Y) \cap \mathfrak{KO}(Y, Y)$.

Corollary 2.2. *Let X and Y be two convex spaces and $F : X \multimap Y$ satisfy that $F(C)$ is convex for any convex subset C of X. Then $F \in \mathfrak{KC}(X, Y) \cap \mathfrak{KO}(X, Y)$.*

Remark. For the case (1) of Theorem 2, we may assume that $\overline{F(\Gamma_A)}$ is Ω-convex in Y instead of $F(\Gamma_A)$. From this, we have the following:

Corollary 2.3. [3, Theorem 2.2] *Let X and Y be two convex spaces and $F : X \multimap Y$ satisfy that $\overline{F(C)}$ is convex for any convex subset C of X. Then $F \in \mathfrak{KC}(X, Y)$.*

Theorem 3. *Let $(X; \Gamma)$ be an abstract convex space, Y a topological space, and $F \in \mathfrak{KC}(X, Y) \cup \mathfrak{KO}(X, Y)$ such that $F(x)$ is connected for each $x \in X$. Then $F(\Gamma_A)$ is connected for each $A \in \langle X \rangle$.*

Proof. Suppose that $F(\Gamma_A)$ is not connected for some $A \in \langle X \rangle$. Then there exist two nonempty disjoint closed [resp. open] subsets U and V of Y such that $F(\Gamma_A) \subset U \cup V$. Moreover, there exist two points $p, q \in \Gamma_A$ such that $F(p) \cap U \neq \emptyset$ and $F(q) \cap V \neq \emptyset$. Since $F(p)$ and $F(q)$ are connected, $F(p) \subset U$ and $F(q) \subset V$. Define a map $G : X \multimap Y$ by $G(p) := U$, $G(q) := V$, and $G(x) := Y$ for $x \notin \{p, q\}$. Then G is closed-valued [resp. open-valued] and

$$F(\Gamma_A) \subset G(A) \qquad \text{for all } A \in \langle X \rangle.$$

Therefore G is a KKM map with respect to F. But $G(p) \cap G(q) = \emptyset$ contradicts $F \in \mathfrak{KC}(X,Y) \cup \mathfrak{KO}(X,Y)$. This completes our proof.

Corollary 3.1. *Let X be any convex space and $F \in \mathfrak{KC}(X,Y)$ or $F \in \mathfrak{KO}(X,Y)$ such that $F(x)$ is connected for any $x \in X$. Then $F(C)$ is connected for any convex subset C of X.*

Remark. In Theorem 3, if $F \in \mathfrak{KC}(X,Y)$ such that $\overline{F(x)}$ is connected for each $x \in X$, then $\overline{F(\Gamma_A)}$ is connected for each $A \in \langle X \rangle$.

Corollary 3.2. [3, Theorem 2.4] *Let X be any convex space and $F \in \mathfrak{KC}(X,Y)$ such that $\overline{F(x)}$ is connected for any $x \in X$. Then $\overline{F(C)}$ is connected for any convex subset C of X.*

Remark. In [3], an example showing that the converse of Corollary 3.2 does not hold is given.

4. On $\mathfrak{KC}$ and $\mathfrak{KO}$ in connected ordered spaces

In this section, we are concerned with results for connected ordered spaces:

Theorem 4. *Let $(X;\leq)$ be a connected ordered space, Y a topological space, and $F : X \multimap Y$ such that $F([a,b])$ is connected in Y for each $a,b \in X$ with $a < b$. Then $F \in \mathfrak{KC}(X,Y) \cap \mathfrak{KO}(X,Y)$.*

Proof. Just follow that of [3, Theorem 2.3] with necessary modifications.

Corollary 4.1. *Let X be a nonempty interval of $\mathbb{R}$ and Y a topological space. If $F : X \multimap Y$ satisfies that $F([a,b])$ is connected for any $a,b \in X$ with $a < b$, then $F \in \mathfrak{KC}(X,Y) \cap \mathfrak{KO}(X,Y)$.*

Remarks. 1. In Theorem 4, if we assume that $\overline{F([a,b])}$ is connected in Y instead of $F([a,b])$, then $F \in \mathfrak{KC}(X,Y)$.

2. When X is an interval of $\mathbb{R}$, then the above remark reduces to the following:

Corollary 4.2. [3, Theorem 2.3] *Let X be a nonempty interval of $\mathbb{R}$ and Y a topological space. If $F : X \multimap Y$ satisfies that $\overline{F([a,b])}$ is connected for any $a,b \in X$ with $a < b$, then $F \in \mathfrak{KC}(X,Y)$.*

Example. Let $X = Y = [0,1]$ with the usual topology, and let $F : X \multimap Y$ be defined as

$$F(x) := \begin{cases} \{|\sin 1/x|\}, & \text{for } x \in (0,1]; \\ \{0\}, & \text{for } x = 0. \end{cases}$$

This example was originally given to show that $\mathfrak{KC}(X,Y) \supsetneq \mathfrak{A}_c^\kappa$. Now it is also an example of Corollaries 4.1 and 4.2.

Note that The converses of Corollaries 4.1 and 4.2 are not true:

Example. [3] Let $X := [0,1]$ and $F : X \multimap X$ be defined by $F(0) := \{0,1\}$ and $F(x) := \{1\}$ for $x \in (0,1]$. Since F has a continuous selection f such that $f(x) = 1$ for all $x \in [0,1]$, we have $F \in \mathfrak{KC}(X,X) \cap \mathfrak{KO}(X,X)$. But $\overline{F([0,1])}$ is not connected.

The class of maps having connected graph is quite large:

Lemma. (Hiriart-Urruty [2, Theorem 3.2]) *Let X, Y be a topological spaces, $C \subset X$ a connected subset, and $F : X \multimap Y$ be a multimap with connected values on C. Either of the next assumptions ensure that the graph of $F|_C$ is connected:*

(a) *F is l.s.c.*

(b) *F is u.s.c. and compact-valued.*

A map $F : X \multimap Y$ is called a *connectivity map* if the graph over each connected subset of X is a connected set. This concept was introduced by Nash for single-valued case; see Girolo [1].

From Theorem 4 and Lemma we have the following:

Theorem 5. *Let $(X; \leq)$ be a connected ordered space, Y a topological space, and $F : X \multimap Y$. Then $F \in \mathfrak{KC}(X,Y) \cap \mathfrak{KO}(X,Y)$ if it satisfies one of the following conditions:*

(i) *F is a connectivity map.*

(ii) *F is l.s.c. with connected values.*

(iii) *F is u.s.c. with compact connected values.*

(iv) *F has connected values and open fibers.*

(v) *F is a closed compact map with connected values.*

Proof. (i) Since F is a connectivity map and $[a,b]$ is connected for each $a < b$ in X, $F|_{[a,b]}$ has connected graph. Therefore, $F([a,b])$ is connected. Hence, by Theorem 4, the conclusion follows.

(ii), (iii) By Lemma, F is a connectivity map. Therefore, (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i).

(iv) Since $F^-(y)$ is open for each $y \in X$, F is l.s.c. Indeed, for each open set $A \subset X$, we have

$$F^-(A) = \{x \in X \mid F(x) \cap A \neq \emptyset\} = \bigcup_{y \in A} F^-(y)$$

is open. Therefore, (iv) implies (ii).

(v) It is well-known that a closed compact map is u.s.c. with compact values. Therefore, (v) implies (iii).

The following characterizes maps in $\mathfrak{KC}$ and $\mathfrak{KD}$:

Theorem 6. *Let $(X;\leq)$ be a connected ordered space, Y a topological space, and $F : X \multimap Y$ such that $F(x)$ is connected for each $x \in X$. Then $F \in \mathfrak{KC}(X,Y) \cap \mathfrak{KD}(X,Y)$ if and only if $F([a,b])$ is connected for each $a,\ b \in X$ with $a < b$.*

Proof. This follows from Theorems 3 and 5.

Remarks. 1. In Theorem 6, if F is such that $\overline{F(x)}$ is connected for each $x \in X$, then $F \in \mathfrak{KC}(X,Y)$ if and only if $\overline{F([a,b])}$ is connected for each $a,\ b \in X$ with $a < b$.

2. For single-valued maps, Theorem 6 reduces to the following:

Corollary 6.1. *Let $(X;\leq)$ be a connected ordered space, Y a topological space, and $f : X \to Y$. Then $f \in \mathfrak{KC}(X,Y) \cap \mathfrak{KD}(X,Y)$ if and only if $f([a,b])$ is connected for each $a,\ b \in X$ with $a < b$.*

Remarks. 1. In Corollary 6.1, $f \in \mathfrak{KC}(X,Y)$ if and only if $\overline{f([a,b])}$ is connected for each $a,\ b \in X$ with $a < b$.

2. When X is an interval of $\mathbb{R}$, then the above remark reduces to the following:

Corollary 6.2. [3, Theorem 2.5] *Let X be a nonempty interval of $\mathbb{R}$, Y a topological space and $f : X \to Y$. Then $f \in \mathfrak{KC}(X,Y)$ if and only if $\overline{f([a,b])}$ is connected for any $a, b \in X$ with $a < b$.*

References

[1] J. Girolo, *The Schauder fixed-point theorem for connectivity maps*, Colloq. Math. **44** (1981), 59–64.

[2] J.-B. Hiriart-Urruty, *Images of connected sets by semicontinuous multifunctions*, J. Math. Anal. Appl. **111** (1985), 407–422.

[3] J.-C. Jeng, Y.-Y. Huang, and H.-L. Zhang, *Characterization of maps having the KKM property,* Soochow J. Math. **28** (2002), 329–338.

[4] S. Park, *Remarks on fixed point theorems of Ricceri*, Proc. '97 Workshop on Math. Anal. Appl. pp.1–9, Pusan Nat. Univ., 1997.

[5] S. Park, *Fixed point theorems on connected ordered spaces*, Nonlinear Anal. Forum **6** (2001), 185–192.

[6] S. Park, *On generalizations of the KKM principle on abstract convex spaces*, Nonlinear Anal. Forum **11** (2006), 67–77.

[7] S. Park, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc., to appear.

[8] S. Park, *Remarks on* $\mathfrak{KC}$*-maps and* $\mathfrak{KO}$*-maps on abstract convex spaces*, Nonlinear Anal. Forum, to appear.

[9] B. Ricceri, *Some topological mini-max theorems via an alternative principle for multifunctions*, Arch. Math. **60** (1993), 367–377.

[10] L. A. Steen and J. A. Seebach, *Counterexamples in Topology*, 2nd ed., Springer-Verlag, New York-Heidelberg-Berlin, 1978.

[11] S. Willard, *General Topology*, Addison-Wesley, Reading, MA, 1970.