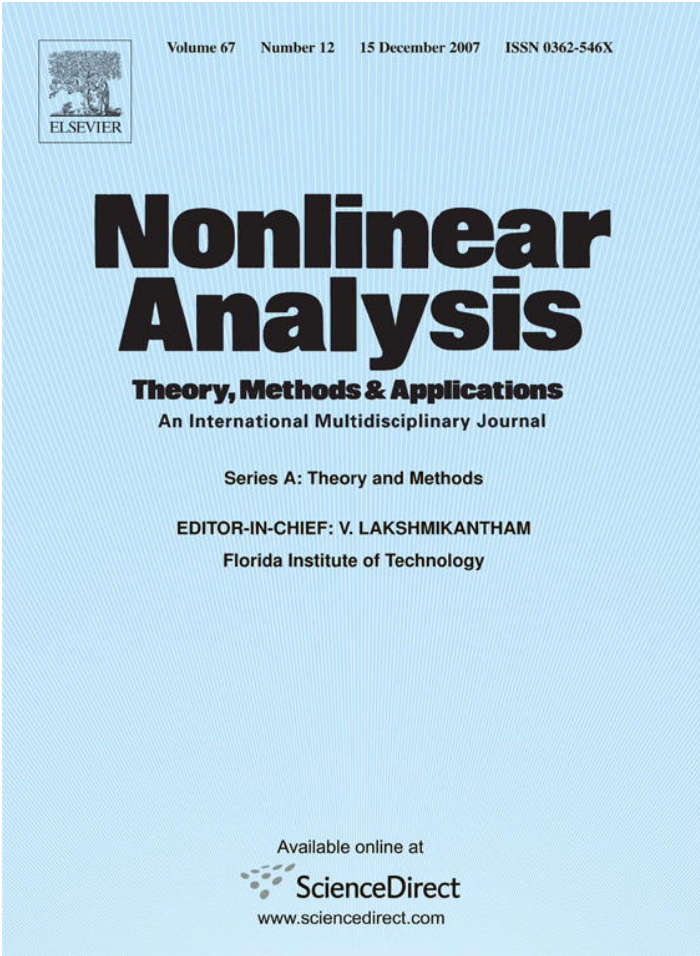
Volume 67 Number 12 15 December 2007 ISSN 0362-546X
ELSEVIER
Nonlinear Analysis
Theory, Methods & Applications
An International Multidisciplinary Journal
Series A: Theory and Methods
EDITOR-IN-CHIEF: V. LAKSHMIKANTHAM
Florida Institute of Technology
Available online at
ScienceDirect
www.sciencedirect.com

Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 67 (2007) 3401–3410
Nonlinear Analysis
www.elsevier.com/locate/na

Generalizations of the Krasnoselskii fixed point theorem

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Received 20 June 2006; accepted 23 October 2006

Abstract

We obtain new generalized forms of the Krasnoselskii theorem on fixed points for a sum of two operators and show that our new results encompass a number of previously known generalizations of the theorem.

MSC: 47H10; 54H25

Keywords: Compact fixed point property (f.p.p.); Klee approximable; Banach contraction; Boyd–Wong contraction; Meir–Keeler contraction; Hoa–Schmitt contraction; Sequentially complete locally convex t.v.s.

1. Introduction

In 1955, Krasnoselskii [13] proved a fixed point theorem motivated by an observation that the inversion of a perturbed differential operator may yield the sum of compact and contraction operators. His theorem actually combines both the Banach contraction principle and the Schauder fixed point theorem, and is useful in establishing existence theorems for perturbed operator equations. Since then there have appeared a large number of papers contributing generalizations or modifications of the Krasnoselskii fixed point theorem and their applications; see the references at the end of this paper. One of the main features of such generalizations is the adopting of generalized forms of the Banach principle or the Schauder theorem.

One of the most impressive generalizations of the Krasnoselskii theorem was given by Hoa and Schmitt [12] in 1994. More recently, in 2005, Barroso and Teixeira [3] established various extended forms of the Krasnoselskii theorem and several applications of such new theorems.

Our aim in this paper is to obtain new generalized forms of the Krasnoselskii theorem and to show that our new results encompass a number of previously known generalizations or modifications of the Krasnoselskii theorem.

Section 2 deals with several known generalizations of the Schauder fixed point theorem and the Banach contraction principle which can be applicable to certain subsets of locally convex Hausdorff topological vector spaces. In Section 3, we obtain a basic abstract theorem of the Krasnoselskii type and several direct consequences of it. In Section 4, we show that our results reduce to a number of previously known generalizations or modifications of the

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0362-546X/$ - see front matter
doi:10.1016/j.na.2006.10.024

Krasnoselskii theorem. We list them in chronological order with comments on proofs of them and mutual relations among them. Finally, Section 5 deals with certain variants of generalizations of the Krasnoselskii theorem.

2. Preliminaries

Throughout this paper, t.v.s. means Hausdorff topological vector spaces with fundamental systems $\mathcal{V}$ of neighborhoods of the origin 0.

The following was raised by Schauder in 1935:

Conjecture. *Let X be a convex subset of a t.v.s. and $f : X \to X$ a continuous map. If f is compact, then f has a fixed point $\hat{x} \in X$, that is, $\hat{x} = f(\hat{x})$.*

Let us say that a topological space X has the *(compact) fixed point property* (f.p.p.) if any (compact) continuous self-map $f : X \to X$ has a fixed point $x_0 \in X$; that is, $x_0 = f(x_0)$.

For a subset X of a t.v.s. E, a multimap $T : X \multimap X$ is called a *Φ-map* (or *a Fan–Browder map*) if there exists a multimap $S : X \multimap X$ such that (1) for each $x \in X$, co $Sx \subset Tx$ and (2) $X = \bigcup\{\mathrm{Int}\, S^{-}x \mid x \in X\}$.

A subset X of a t.v.s. E is called a *Φ-space* if for each $U \in \mathcal{V}$, there is a Φ-map $T : X \multimap X$ such that $Tx \subset x + U$ for each $x \in X$.

We list some examples of cases when the conjecture holds:

Lemma 1. *The Schauder conjecture is affirmed if one of the following conditions holds:*

(1) *(Schauder) E is a normed vector space.*
(2) *(Hukuhara) E is locally convex.*
(3) *(Klee) X is admissible (in the sense of Klee).*
(4) *(Idzik) $\overline{f(X)}$ is convexly totally bounded.*
(5) *(Horvath) X is a Φ-space.*
(6) *(Nhu and Arandelovič) X is compact and weakly admissible.*

For the literature on Lemma 1, see [20–22]. Note that (1) ⇒ (2) ⇒ (3) and (2) ⇒ (5).

A *polytope* P in a t.v.s. E is a nonempty compact convex subset of E contained in a finite dimensional subspace of E.

A nonempty subset K of E is said to be *Klee approximable* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in E. In particular, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

Examples. We give some examples of Klee approximable sets in [22,23]:

(1) A subset X of E is admissible (in the sense of Klee), if and only if every compact subset K of X is Klee approximable into E.
(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.
(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.
(4) Any compact subset K of a convex subset X of a t.v.s. is Klee approximable into X whenever X is locally convex.
(5) Any compact subset K of an admissible almost convex subset X of a t.v.s. is Klee approximable into X.
(6) Any compact subset of a Φ-space in a t.v.s. is Klee approximable.
(7) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.

The following has recently become known [22]:

Lemma 2. *Let X be a subset of a t.v.s. E and $f : X \to X$ a compact continuous map. If $f(X)$ is Klee approximable into X, then f has a fixed point.*

Lemma 2 implies Lemma 1(1)–(3), and (5).

Let E be a locally convex t.v.s., $\mathcal{P}$ a family of seminorms which generates the topology of E, and D a nonempty subset of E. Then a map $U : D \to E$ is called

(i) a *Banach contraction* if, for each $p \in \mathcal{P}$, there is a γ_p, $0 \le \gamma_p < 1$, such that for all $x, y \in D$, $p(Ux - Uy) \le \gamma_p p(x - y)$.

Let $\mathbb{R}_+$ denote the nonnegative reals and Ψ a family of real functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which are nondecreasing, continuous on the right, and satisfy $\phi(t) < t$ for $t > 0$.

A map $U : D \to E$ is called

(ii) a *Boyd–Wong contraction* if, for each $p \in \mathcal{P}$, there exists a $\phi_p \in \Psi$ such that for each pair $x, y \in D$, $p(Ux - Uy) \le \phi_p(p(x - y))$;

(iii) a *Meir–Keeler contraction* if, for each $p \in \mathcal{P}$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in D$,

$$\varepsilon \le p(x - y) < \varepsilon + \delta \quad \text{implies } p(Ux - Uy) < \epsilon;$$

(iv) a *Hoa–Schmitt contraction* [12] if it satisfies the following conditions on a subset Ω of E:

(A.1) For any $a \in \Omega$, $U_a(D) \subset D$ where $U_a : D \to E$ is defined by $U_a(x) := U(x) + a$.

(A.2) For any $a \in \Omega$ and $p \in \mathcal{P}$ there exists $k_a \in \mathbb{Z}_+$ with the following property: for any $\varepsilon > 0$, there exist $r \in \mathbb{N}$ and $\delta > 0$ such that for $x, y \in D$, $\alpha_a^p(x, y) < \varepsilon + \delta$ implies

$$\alpha_a^p(U_a^r(x), U_a^r(y)) < \varepsilon,$$

where $\alpha_a^p(x, y) := \max\{p(U_a^i(x) - U_a^j(y)) \mid i, j = 0, 1, 2, \ldots, k_a\}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Remarks. 1. Note that (i) ⇒ (ii) ⇒ (iii).

2. It is stated, in [12], that a Meir–Keeler contraction $U : D \to E$ is a Hoa–Schmitt contraction on E itself with $k_a = 0$ and $r = 1$. However, in order to satisfy condition (A.1), we have to assume an extra condition, for example,

$$U(x) + a \in D \quad \text{for all } a \in \Omega \text{ and } x \in D.$$

3. Moreover, if U satisfies (A.1) and if U^r for some $r \in \mathbb{N}$ is (i) a Banach contraction, (ii) a Boyd–Wong contraction, or (iii) a Meir–Keeler contraction, then it is a Hoa–Schmitt contraction.

We need the following:

Lemma 3 (*[12] Theorem 1*). *Let D be a sequentially complete subset of a locally convex t.v.s. E and $U : D \to E$ be such that*

(a) *U is uniformly continuous (that is, for $p \in \mathcal{P}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $p(x - y) < \delta$ implies $p(Ux - Uy) < \varepsilon$); and*
(b) *U is a Hoa–Schmitt contraction on a subset Ω of E.*

Then $(I - U)^{-1}$ is well defined and uniformly continuous on Ω.

Note that Lemma 3 is closely related to [27, p. 32].

The following is step 1 in the proof of [12, Theorem 1]:

Lemma 4. *Under the hypothesis of Lemma 3, for any $a \in \Omega$, the operator U_a has a unique fixed point in D, say $\phi(a)$, and the iterated sequence $\{U_a^n(x)\}_n$ converges to $\phi(a)$, for all $x \in D$. Furthermore, the map $a \mapsto \phi(a)$ is injective.*

Note that Lemma 4 is a far-reaching generalization of the Banach contraction principle; for other generalizations, see [18,19]. For example, we have the following:

Corollary. *Let D be a sequentially complete subset of a locally convex t.v.s. E and $U : D \to D$ a Meir–Keeler contraction. Then U has a unique fixed point $x_0 \in D$ and the iterated sequence $\{U^n(x)\}_n$ for any $x \in D$ converges to x_0.*

Since any complete metric space can be isometrically embedded in a closed subset of a Banach space, the Corollary can be stated for complete metric spaces and actually is a generalization of the Meir–Keeler fixed point theorem [14].

For a normed vector space $(E, \|\cdot\|)$ and a subset D of E, a map $F : D \to E$ is called (i)′ *a (Banach) contraction* if it satisfies the condition that for all $x, y \in D$,

$$\|Fx - Fy\| \leq q\|x - y\|$$

for some $q, 0 \leq q < 1$;

(ii)′ *a Boyd–Wong contraction* [4] if it satisfies the condition that for all $x, y \in S$,

$$\|Fx - Fy\| \leq \phi(\|x - y\|),$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a real function in Ψ; and

(iii)′ *a Meir–Keeler contraction* [14] if for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x, y \in D$,

$$\varepsilon \leq \|x - y\| < \varepsilon + \delta \quad \text{implies } \|Fx - Fy\| < \varepsilon.$$

Note that (i)′, (ii)′ and (iii)′ are particular cases of (i), (ii) and (iii), respectively.

3. New Krasnoselskii type theorems

We begin with the following new version of the Krasnoselskii fixed point theorem:

Theorem 1. *Let D be a subset of a t.v.s. E, $I : D \to D$ the identity map, and $A, B : D \to E$ continuous maps such that*

(a) *A is compact (that is, $A(D)$ is contained in a compact subset of E); and*

(b) *$I - B$ is continuously invertible on $\overline{A(D)}$.*

If D has the compact f.p.p. or if $(I - B)^{-1}A(D)$ is Klee approximable into D, then $A + B$ has a fixed point.

Proof. Let us define a map $T : D \to D$ by

$$T(u) := (I - B)^{-1}A(u) \quad \text{for each } u \in D.$$

Then T is well defined and continuous by (b). Since condition (b) means that $I - B : D \to E$ gives a homeomorphism between $(I - B)^{-1}\overline{A(D)}$ and $\overline{A(D)}$, we have

$$T(D) = (I - B)^{-1}A(D) \subset (I - B)^{-1}\overline{A(D)} \subset D$$

and hence T is compact. In the case D where has the compact f.p.p., T has a fixed point, which is actually a fixed point of $A + B$. For the other case, $T(D)$ is Klee approximable into D, and T has a fixed point by Lemma 2. This completes our proof. □

Remarks. 1. Condition (b) is properly implied by the following:

(1) $I - B$ is a homeomorphism on D; and

(2) $\overline{A(D)} \subset (I - B)(D)$.

2. As we shall see later, a simple way of checking the continuous invertibility of $I - B$ on $\overline{A(D)}$ in Theorem 1 is to ask that B is a certain type of generalized contraction as in Section 2.
3. In view of Lemmas 1 and 2, Theorem 1 contains several particular subcases.

Corollary 1.1. *Let D be a compact subset of a t.v.s. E and $A, B : D \to E$ continuous maps such that*

(d) *there exists a sequence $\lambda_n \to 1$ such that $I - \lambda_n B$ is injective and $A(D) \subset (I - \lambda_n B)(D)$ for all n.*

If D has the f.p.p., then $A + B$ has a fixed point.

Proof. Since E is Hausdorff, D is compact and B is continuous, for any closed subset C of D, we have that $(I - \lambda_n B)(C) = ((I - \lambda_n B)^{-1})^{-1}(C)$ is compact and hence closed. Therefore $(I - \lambda_n B)^{-1}$ is continuous. Moreover, $A(D)$ is compact. Hence, by Theorem 1, there exists a fixed point $u_n \in D$ for $A + \lambda_n B$, that is,

$$u_n = \lambda_n B(u_n) + A(u_n).$$

Since D is compact, up to a subnet we may suppose $u_n \to u$ in D. Passing to the limit in the above equality, $u \in D$ is a fixed point of $A + B$. □

Note that Theorem 1 and Corollary 1.1 improve Theorem 2.2 and Corollary 2.3 of [3], respectively, which are for locally convex t.v.s.

Theorem 2. *Let D be a subset of a t.v.s. E, $I : E \to E$ the identity map, and $A : D \to E$, $B : E \to E$ continuous maps such that*

(a) *A is compact;*
(b) *$I - B$ is a homeomorphism on D; and*
(c) *for every $z \in \overline{A(D)}$, the map $B_z : E \to E$ defined by $B_z(y) := z + By$ for $y \in E$ has a fixed point in D.*

If D has the compact f.p.p. or if $(I - B)^{-1}A(D)$ is Klee approximable into D, then $A + B$ has a fixed point.

Proof. For each $z \in \overline{A(D)}$, by (c), there exists a $y \in D$ such that $y = z + By$ or $z = (I - B)y$. Hence we have $\overline{A(D)} \subset (I - B)(D)$ and, by (b), $I - B$ is continuously invertible on $\overline{A(D)}$. Now, by applying Theorem 1, we have the conclusion. □

Remarks. 1. The maps I and B can be replaced by the ones defined on D.
2. For a map $B : E \to E$, in order to show that B_{Ax} has a fixed point in D as in (c) or that $A(D) \subset (I - B)(D)$, Burton [6] first assumed the condition

$$[y = Ax + By, x \in D] \Rightarrow y \in D,$$

which is more general than

$$Ax + By \in D \quad \text{for all } x, y \in D.$$

For a sequentially complete convex subset of a locally convex t.v.s. E, we have some more concrete forms of the Krasnoselskii theorem adopting a certain type of generalized contraction.

From Lemmas 1 and 3 and Theorem 2, we have the following:

Theorem 3. *Let D be a sequentially complete convex subset of a locally convex t.v.s. E and $A, B : D \to E$ be such that*

(a) *A is continuous and compact; and*
(b) *B is uniformly continuous and a Hoa–Schmitt contraction on $\overline{A(D)}$.*

Then $A + B$ has a fixed point in D.

Proof. By Lemma 3, (b) implies that $I - B$ is continuously invertible on $\overline{A(D)}$. Hence all of the requirements of Theorem 1 are satisfied. □

Remark. Note that, for the constant map $A := 0$, Theorem 3 reduces to Lemma 4 for the case $a = 0$.

Corollary 3.1. *In Theorem 3, compactness of A can be replaced by the following without affecting its conclusion:*
(a)′ *$K := \{v \in D : v = Bv + Au$ for some $u \in D\}$ is relatively compact.*

Proof. By Lemma 4, for any $x \in D$, the map $B_{Ax} : D \to D$ defined by $B_{Ax}(y) := Ax + By$ for $y \in D$ has a unique fixed point $Tx \in D$. Then we have a map $T : D \to D$ such that $Tx := B(Tx) + Ax \in D$ for each $x \in D$. In view of Theorem 3, it remains to show that A is compact. Note that $Tx \in K$ by (a)′ and hence $T(D) \subset K$. Since $Ax = (I - B)Tx$ for all $x \in D$, we have $A(D) = (I - B)T(D) \subset (I - B)(K) \subset (I - B)(D)$. Note that A is compact since K is relatively compact by (a)′ and $I - B$ is continuous. Therefore, by Theorem 3, the conclusion follows. □

Remark. For any $x \in D$, that the map $B_{Ax} : D \to E$ satisfies $B_{Ax}(D) \subset D$ in the proof is assumed in (A.1) in the definition of a Hoa–Schmitt contraction. Instead, if B is (i) a Banach contraction, (ii) a Boyd–Wong contraction, or (iii) a Meir–Keeler contraction, then it is sufficient to assume

$$Ax + By \in D \quad \text{for all } x, y \in D.$$

Corollary 3.1 is motivated by [3, Theorem 2.9 and Corollary 2.10].

Corollary 3.2. *Let D be a sequentially complete convex subset of a locally convex t.v.s. E and $A, B : D \to E$ be such that*

(a) *A is continuous and compact;*
(b) *B is uniformly continuous and is a Meir–Keeler contraction; and*
(c) *$Ax + By \in D$ for each $x, y \in D$.*

Then $A + B$ has a fixed point in D.

Proof. For each $z \in \overline{A(D)}$, the map $B_z : D \to E$ is defined by $B_z(y) = z + By$ for $y \in D$. Since D is closed, condition (c) implies $A(D) + B(D) \subset D$ and hence $\overline{A(D)} + B(D) \subset D$. Then $B_z(D) \subset D$. Therefore B is a Hoa–Schmitt contraction on $\overline{A(D)}$. Now, by Theorem 3, we have the conclusion. □

Similarly we have

Corollary 3.3. *In Corollary 3.2, for $B : D \to D$, condition* (b) *can be replaced by the following:*
(b)′ *B is uniformly continuous and, for some $r \in \mathbb{N}$, B^r is a Meir–Keeler contraction.*

4. Previously known results

In this section, we show that our new results in Section 3 reduce to a number of previously known generalizations of the Krasnoselskii theorem. We list them in chronological order with comments on their proofs and mutual relations among them.

(I) Krasnoselskii [13]: *Let E be a Banach space, D a bounded closed convex subset of D, and $A, B : D \to E$ be such that*

(a) *A is completely continuous (that is, continuous and maps bounded sets into compact sets)*;
(b) *B is a* (*Banach*) *contraction*; *and*
(c) *$Ax + By \in D$ for every $x, y \in D$.*

Then the equation $Ax + Bx = x$ has a solution in D.
Note that if A is compact, then boundedness of D is redundant, and condition (c) is needed to satisfy (A.1). The conclusion follows from Corollary 3.2.

(II) Nashed and Wong [16]: In (I), condition (b) can be replaced by one of the following:

(b)′ *B is a Boyd–Wong contraction.*
(b)″ *$B : D \to D$ is a bounded linear operator such that some iterate B^r is a Boyd–Wong contraction.*

For the case (b)′, the conclusion follows from Corollary 3.2 as above.
Note that (II) for (b′) generalizes (I).
For the case (b)″, B^r is a Meir–Keeler contraction. Hence Corollary 3.3 works.

(III) Cain and Nashed [18]: *Let E be a locally convex t.v.s., D a complete convex subset of E, and $A, B : D \to E$ be such that*

(a) *A is continuous and compact*;
(b) *B is a Banach contraction*; *and*
(c) *$Ax + By \in D$ for every pair $x, y \in D$.*

Then there is a point $\overline{x} \in D$ such that $A\overline{x} + B\overline{x} = \overline{x}$.
Note that (b) shows that B is a Meir–Keeler contraction. Hence Corollary 3.2 works.
(III) generalizes (I).

(IV) Calvert [9]: *Let E be a Banach space, D a bounded nonempty closed convex subset of E, $A : D \to E$ a continuous map, and $B : E \to E$ a Lipschitz map such that*

(a) *A is completely continuous*;
(b) *for all $f \in E$, the map $B_f : E \to E$ defined by $B_f x = Bx + f$ for $x \in E$ satisfies*: *there exist an $r \in \mathbb{N}$ and a real number $\alpha < 1$ such that $(B_f)^r$ has Lipschitz norm $\le \alpha$*; *and*
(c) *$(I - B)^{-1}A(D) \subset D$.*

Then $A + B$ has a fixed point in D.

Since $B : E \to E$ is uniformly continuous and $(B_f)^r$ is a contraction by (b), B is a Hoa–Schmitt contraction on E. Then, by Lemma 3, $(I - B)^{-1}$ is well defined and uniformly continuous on E. For any $x \in D$ with $f = Ax$, the map B_f has a unique fixed point $y \in E$ by Lemma 4, that is, $y = Ax + By$ or $y = (I - B)^{-1}A(x) \in D$ by (c). Then by Theorem 2, the conclusion follows.

In fact, Calvert [9, Lemma 1] showed that the map B in (b) satisfies the conditions (i) $I - B$ is injective and (ii) $(I - B)^{-1}$ is Lipschitz (and hence, continuous), where I is the identity map on E. Therefore, (IV) simply follows from Theorem 1.

(IV) generalizes (I).

(V) Calvert [9]: *Instead of* (c) *in* (IV), *assume* $Ax + By \in D$ *for all* $x, y \in D$. *Then we have the same conclusion.*

Take $B := B|_D : D \to E$ instead of B. Since $Ax + By \in D$ for all $x, y \in D$, we have $\overline{A(D)} + B(D) \subset D$ and $B_f : D \to D$ for any $f \in \overline{A(D)}$. Hence, $B : D \to E$ is a Hoa–Schmitt contraction on $\Omega = \overline{A(D)}$. Therefore, by Lemma 3, $I - B$ is continuously invertible on $\overline{A(D)}$. Now Theorem 1 works.

(VI) Sehgal and Singh [26]: *Let* D *be a sequentially complete convex subset of a locally convex t.v.s.* E *and* $A, B : D \to E$ *be such that*

(a) A *is continuous and compact*;
(b) B *is a Boyd–Wong contraction*; *and*
(c) $Ax + By \in D$ *for each pair* $x, y \in D$.

Then there exists a $u \in D$ *with* $Au + Bu = u$.

Note that (b) implies that B is uniformly continuous and a Meir–Keeler contraction. Hence Corollary 3.2 works.

(VI) generalizes (I), (II) with (b)′, and (III).

(VII) Tan [28]: *In* (I) *and* (II), (b) *and* (b)′ *is replaced by the following*:
(b)″ B *is a Meir–Keeler contraction.*

The conclusion follows from Corollary 3.2.

(VII) generalizes (I) and (II)(b)′.

(VIII) Hoa and Schmitt [12]: *Let* E *be a sequentially complete locally convex t.v.s. and* D *a bounded closed convex subset of* E.

(a) *Let* $C : D \to E$ *be completely continuous.*
(b) *Let* $U : D \to E$ *be uniformly continuous and satisfy* (A.2) *on* $\overline{C(D)}$.
(c) *Suppose that* $Ux + Cy \in D$ *for all* $x, y \in D$.

Then $U + C$ *has a fixed point in* D.

In fact, by (c), U satisfies condition (A.1) on $\overline{C(D)}$ and hence it is a Hoa–Schmitt contraction on $C(D)$. Now the conclusion follows from Theorem 3.

Note that if C is compact, then the boundedness of D is redundant.

(VIII) generalizes (I), (II)(b)′, (III), and (V).

(IX) Hoa and Schmitt [12]: *Let* E *be a sequentially complete locally convex t.v.s.*, D *a bounded closed convex subset of* E, *and* $C, U : D \to E$ *be such that*

(a) $C : D \to E$ *is completely continuous*;
(b) $U : D \to E$ *is a Meir–Keeler contraction*; *and*
(c) $Ux + Cy \in D$ *for all* $x, y \in D$.

Then $U + C$ *has a fixed point in* D.

It is clear that (IX) $\Leftarrow$ (VIII).

(IX) generalizes (I), (II)(b)′, and (VII).

(X) Burton [5,6]: *Let* $(E, \|\cdot\|)$ *be a Banach space and* D *a nonempty closed convex subset of* E. *Suppose that* $A, B : D \to D$ *are such that*

(a) A *is continuous and compact*;
(b) B *is a large contraction*; *and*

(c) $x, y \in D \Rightarrow Ax + By \in D$.

Then there is a $y \in D$ *with* $Ay + By = y$.

Recall that B is a *large contraction* if for $x, y \in D$ with $x \neq y$ then $\|Fx - Fy\| < \|x - y\|$ and if $\forall \varepsilon > 0\, \exists \delta < 1$ such that $[x, y \in D, \|x - y\| \geq \varepsilon] \Rightarrow \|Fx - Fy\| \leq \delta \|x - y\|$; see Burton [5,6].

Lemma 5 ([5, Theorem 1]). *Let* (E, ρ) *be a complete metric space and* B *a large contraction. Suppose there exist an* $x \in E$ *and an* $L > 0$ *such that* $\rho(x, B^n x) \leq L$ *for all* $n \geq 1$. *Then* B *has a unique fixed point in* E.

Moreover, Burton [5, Lemma] showed that if $(E, \|\cdot\|)$ is a normed space, $D \subset E$, and $B : D \to E$ is a large contraction, then $I - B$ is a homeomorphism of D onto $(I - B)(D)$. Note that, for each $f \in \overline{A(D)}$, the map $B_f : D \to D$ defined by $B_f(y) := f + By$ for $y \in D$ has a unique fixed point by Lemma 5 (since D is bounded, the L in Lemma 5 is assured). Now (X) follows from Theorem 2.

(X) generalizes (I).

(XI) Burton [6]: *Let* $(E, \|\cdot\|)$ *be a Banach space and* D *a nonempty closed convex subset of* E. *Suppose that* $A : D \to E$ *and* $B : E \to E$ *are such that*

(a) A *is continuous and compact*;
(b) B *is a Banach contraction*; *and*
(c) $[y = Ax + By, x \in D] \Rightarrow y \in D$.

Then there is a $y \in D$ *with* $Ay + By = y$.

(XI) is applied in studying stability by Burton and Furumochi [7].

(XII) Dhage [10]: *In (XI), (c) is replaced by the following*:
(c′) $y = Ax + By \Rightarrow y \in D, \forall x \in D$.
Note that (c) or (c)′ implies

$$[y = f + By,\ f \in \overline{A(D)}] \Rightarrow y \in D.$$

(XI) and (XII) follows from Theorem 2. In fact, $I - B$ is a homeomorphism on D, and for any $f \in \overline{A(D)}$, the map $B_f : E \to E$ defined by $B_f(y) := f + By$ for $y \in E$ is a Banach contraction and, hence, has a unique fixed point $y = f + By$ and hence $y \in D$ by (c) or (c)′. Therefore all of the requirements of Theorem 2 are satisfied.

(XIII) Dhage [10]: *Let* D *be a closed bounded convex subset of a Banach space* E *and* $A : D \to E, B : E \to E$ *be such that*

(a) A *is completely continuous*;
(b) B *is linear and bounded, and* B^r *is a Boyd–Wong contraction for some* $r \in \mathbb{N}$; *and*
(c) $y = Ax + By \Rightarrow y \in D, \forall x \in D$.

Then $x = Ax + Bx$ *has a solution*.

Note that $B : E \to E$ is linear and bounded and hence uniformly continuous. So, B is a Hoa–Schmitt contraction on E, and hence, on $\overline{A(D)}$ by (b). Therefore, by Lemma 3, $I - B$ is continuously invertible on $\overline{A(D)}$. Then Theorem 1 works.

(XIV) Avramescu and Vladimirescu [1]: *Let* $(E, \|\cdot\|)$ *be a Banach space*, D *a closed convex subset of* E, *and* $A : D \to E,\ B : E \to E$ *be such that*

(a) A *is continuous and compact*;
(b) $I - B : E \to E$ *is injective and* $(I - B)^{-1}$ *is continuous such that*

$$A(D) \subset (I - B)(E) \quad \textit{and} \quad (I - B)^{-1} A(D) \subset D.$$

Then $x = Ax + Bx$ *has solutions in* D.

In view of Theorem 1, if we assume that $I - B : D \to E$ is continuously invertible on $\overline{A(D)}$, then E can be any t.v.s. and the closedness of D and the condition $A(D) \subset (I - B)(E)$ are redundant.

(XV) Avramescu and Vladimirescu [1]: *Let* E *and* D *be the same as above such that*

(a) $I - A : D \to E$ *is continuous and compact*;

(b) *$B : E \to E$ is injective, B^{-1} is continuous, and*

$$(I - A)(D) \subset B(E) \quad \textit{and} \quad B^{-1}(I - A)(D) \subset D.$$

Then the same conclusion holds.

In the previous result (XIV), replace $I - A$ and B by A and $I - B$, respectively.

(XVI) Barroso [2]: *Let D be a closed convex subset of a Banach space $(E, \|\cdot\|)$. Assume that $A : D \to E$ and $B \in \mathcal{L}(E)$ satisfies*

(a) *A is weakly continuous and $A(D)$ is weakly precompact*;
(b) *$\|B^r\| < 1$ for some $r \geq 1$; and*
(c) *$[y = Ax + By, x \in D] \Rightarrow y \in D$.*

Then there is $y \in D$ such that $Ay + By = y$.

Recall that $\mathcal{L}(E)$ denotes the class of all linear and bounded self-maps of E.

Note that (XVI) is a particular form of (XI).

(XVII) Barroso and Teixeira [3]: *Let D be a closed convex subset of a locally convex t.v.s. E and $A, B : D \to E$ be continuous operators such that*

(a) *$A(D)$ is relatively compact*;
(b) *$I - B$ is continuously invertible and $A(D) \subset (I - B)(D)$*;
(c) *if $y = Ax + By$ for some $x \in D$ then $y \in D$.*

Then $A + B$ has a fixed point in D.

In view of Theorem 1, if we assume that $I - B$ is continuously invertible on $\overline{A(D)}$, then the closedness of D, the local convexity of E, and condition (c) are redundant. In fact, (c) follows from (b): For any $x \in D$, $Ax = (I - B)y$ for some $y \in D$ by (b). Suppose $y' = Ax + By'$ for some $y' \in E$. Then $(I - B)y' = Ax = (I - B)y$. Since $I - B$ is injective by (b), we have $y' = y \in D$.

In [3], (XVII) is applied to formulate various variants of the Krasnoselskii theorem and to study nonlinear elliptic problems and integral equations. Following our method, other Krasnoselskii type results in [3, Corollaries 2.3 and 2.4, Theorem 2.9, Corollaries 2.10 and 2.11], might have variants or improvements.

5. Previously known variants

According to Nashed and Wong [16], fixed point theorems of the Krasnoselskii type may be formulated in a slightly more general setting of a single map $F : D \times D \to D$ instead of the special case $F(x, y) + Ax + By$.

In particular, the same argument as in the proof of (I) in Section 4 yields the following extension:

(I)$'$ Nashed and Wong [16]: *Let D be a bounded closed convex subset of a given Banach space E, and $F : D \times D \to D$ a map such that for all $x \in E$,*

$$\|F(x, y_1) - F(x, y_2)\| \leq \gamma \|y_1 - y_2\|, \quad 0 \leq \gamma < 1,$$

and for all $y \in E$,

$$\|F(x_1, y) - F(x_2, y)\| \leq \|Ax_1 - Ax_2\|,$$

where $A : D \to D$ is completely continuous. Then there exists an element $u \in D$ such that

$$F(u, u) = u.$$

Similarly, as the proofs of (III) and (IV), we can deduce the following:

(III)$'$ Cain and Nashed [8]: *Let E be a locally convex t.v.s. and D a complete convex subset of E. Suppose $F : D \times D \to D$ is such that for each $p \in \mathcal{P}$, there is a constant γ_p, $0 \leq \gamma_p < 1$, so that*

$$\|F(x, y) - F(x, z)\| \leq \gamma_p p(y - z)$$

for all $y, z \in D$. *Suppose further that* $A : D \to D$ *is continuous and compact, and*

$$p(F(x, y) - F(z, y)) \le p(Ax - Az).$$

Then there is a point $\overline{x} \in D$ *for which* $F(\overline{x}, \overline{x}) = \overline{x}$.

(IV)′ Sehgal and Singh [26]: *Let* D *be a sequentially complete subset of a locally convex t.v.s.* E *and suppose* $F : D \times D \to D$ *is such that for each* $p \in \mathcal{P}$, *there exists a* $\phi_p \in \Psi$ *satisfying*

$$p(F(x, y) - F(x, z)) \le \gamma \phi_p(p(y - z))$$

for all $x, y, z \in D$. *If* $A : D \to E$ *is continuous and compact, and for each* $p \in \mathcal{P}$,

$$p(F(x, y) - F(z, y)) \le p(Ax - Az),$$

then there is a point $u \in D$ *for which* $F(u, u) = u$.

Final remark. There are another generalizations of the Krasnoselskii theorem, which are not of the type covered in this paper; see for example, Reinermann [24] and O'Regan [17], where compact maps are replaced by condensing maps. See also [11,15,25] for other types of generalization.

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