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A Unified Fixed Point Theory in Generalized Convex Spaces

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Abstract Let $\mathfrak{B}$ be the class of 'better' admissible multimaps due to the author. We introduce new concepts of admissibility (in the sense of Klee) and of Klee approximability for subsets of G-convex uniform spaces and show that *any compact closed multimap in $\mathfrak{B}$ from a G-convex space into itself with the Klee approximable range has a fixed point.* This new theorem contains a large number of known results on topological vector spaces or on various subclasses of the class of admissible G-convex spaces. Such subclasses are those of Φ-spaces, sets of the Zima–Hadžić type, locally G-convex spaces, and LG-spaces. Mutual relations among those subclasses and some related results are added.

Keywords multimap classes $\mathfrak{B}$ and $\mathfrak{A}_c^\kappa$, Φ-map, Φ-set, Φ-space, admissible G-convex space, the Zima type, locally G-convex space, LG-space

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1 Introduction

Since we introduced the concept of generalized convex spaces (simply, G-convex spaces) in 1995, there have appeared a large number of works contributing mainly to the KKM theory and equilibria theory on these spaces. Many known results on fixed points, coincidence points, minimax theorems, saddle points, variational inequalities, monotone extensions, and others in topological vector spaces are extended to the corresponding ones in G-convex spaces. For the literature, see [1–19].

On the other hand, in the fixed point theory of multimaps on topological vector spaces, the author introduced the admissible class $\mathfrak{A}_c^\kappa$ and the 'better' admissible class $\mathfrak{B}$ of multimaps and gave new fixed point theorems on such classes. One of them is that *any compact closed multimap in $\mathfrak{B}$ from an admissible* (*in the sense of Klee*) *convex subset of a Hausdorff topological vector space into itself has a fixed point*; see [20–21]. It is known that this theorem unifies and generalizes the known results in more than sixty previous works.

Our principal aim in the present paper is to show that the above-mentioned theorem holds for G-convex uniform spaces. In fact, we introduce new concepts of admissibility (in the sense of Klee) and of Klee approximability for subsets of G-convex uniform spaces and show that *any compact closed multimap in $\mathfrak{B}$ from a G-convex space into itself with the Klee approximable range has a fixed point.* This new theorem contains a large number of known results on topological vector spaces or on various subclasses of the class of admissible G-convex spaces. Such subclasses are those of Φ-spaces, sets of the Zima–Hadžić type, locally G-convex spaces, and LG-spaces. Mutual relations among those subclasses and some related results on approximable maps, Kakutani maps, acyclic maps, Φ-maps, and others are investigated.

Sections 2 and 3 deal with preliminaries on G-convex spaces and the class $\mathfrak{B}$ of multimaps. In Section 4, we introduce new concepts of admissibility (in the sense of Klee) and of Klee

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approximability for subsets of the G-convex uniform space X and show that any compact closed map in $\mathfrak{B}$ from X into itself with Klee approximable range has a fixed point. Sections 5-8 deal with particular subclasses of the class of admissible G-convex spaces and fixed point results on spaces in those subclasses.

In Section 5, we are concerned with Φ-maps (or Fan–Browder maps) and Φ-spaces due to Horvath [22–23]. Section 6 deals with fixed point theorems on G-convex spaces of the Zima-Hadžić type, and many results due to Hadžić [24–30] and others are extended. In Section 7, we are concerned with locally G-convex spaces which are generalizations of the corresponding concepts due to Bielawski [31] and Ben-El-Mechaiekh et al. [32]. Section 8 deals with fixed point theorems on LG-spaces, which are generalizations of LC-spaces originated from Horvath [22–23].

Consequently, most of the known fixed point theorems for multimaps on topological vector spaces are extended to the corresponding ones on G-convex spaces. Finally, in Section 9, we give some historical remarks on the current state of the fixed point theory of multimaps in topological vector spaces or in generalized convex spaces. We note that the only important result which refuses to be generalized seems to be the well-known theorem of Idzik [33] on Kakutani maps having convexly totally bounded ranges.

2 Generalized Convex Spaces

In this paper, all topological spaces are assumed to be Hausdorff unless otherwise explicitly stated, "t.v.s." means a topological vector space, and "co" denotes the convex hull.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that, for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$. A G-convex space $(X, D; \Gamma)$ with $X \supset D$ is denoted by $(X \supset D; \Gamma)$ and $(X; \Gamma) := (X, X; \Gamma)$. For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if, for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$.

For the details on G-convex spaces, see [1–19], where basic theory was extensively developed. There are lots of examples of G-convex spaces.

Example 2.1 If X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology, then $(X \supset D; \Gamma)$ becomes a *convex space* generalizing the one due to Lassonde [34]. Note that any convex subset of a t.v.s. is a convex space, but not conversely.

Example 2.2 If Γ_A is assumed to be contractible or, more generally, ω-connected (that is, n-connected for all $n \geq 0$), and if for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then $(X, D; \Gamma)$ becomes an *H-space* [13]. For $X = D$, an H-space reduces to a C-space of Horvath [22–23]. There are many examples of C-spaces due to Horvath [22]. It is notable that a torus, the Möbius band, or the Klein bottle can be regarded as C-spaces, and are examples of compact G-convex spaces without having the fixed point property.

Example 2.3 Other major examples of G-convex spaces are metric spaces having the Michael convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, topological semi-lattices with path-connected intervals, and so on; for the literature, see [1, 4–10, 14–16]. Moreover, further examples of G-convex spaces were given by the author [7] as follows: L-spaces and B'-simplicial convexity of Ben-El-Mechaiekh et al. [32], continuous images of G-convex spaces, Verma's or Stachó's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, *mc*-spaces of Llinares, hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces; see [7–10]. Note also that Ding's FC-space [35–36] is a particular form of G-convex spaces.

Example 2.4 Any hyperbolic space X in the sense of Kirk and Reich–Shafrir is a G-convex space, since the closed convex hull of any $A \in \langle X \rangle$ is contractible. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic; see [37].

Example 2.5 Let $X = D = [0,1)$ and $Y = D' = \mathbb{S}^1 = \{e^{2\pi it} : t \in [0,1)\}$ in the complex plane $\mathbb{C}$. Let $f : X \to Y$ be a continuous function defined by $f(t) = e^{2\pi it}$. Define $\Gamma : \langle D' \rangle \multimap Y$ by $\Gamma_A = f(\mathrm{co}(f^{-1}(A)))$ for $A \in \langle D' \rangle$. Then $(Y \supset D'; \Gamma)$ is a compact G-convex space. Recall that any continuous image of a G-convex space is a G-convex space. We note the following:

(1) $\mathbb{S}^1$ lacks the fixed point property. Moreover, $\mathbb{S}^1$ is an example of a compact C-space since each Γ_A is contractible. Therefore, it shows that the Schauder conjecture (that is, any compact convex subset of a topological vector space has the fixed point property) does not hold for G-convex spaces.

(2) Note that, in $(Y \supset D'; \Gamma)$, singletons are Γ-convex; that is, $\Gamma_{\{y\}} = \{y\}$ for each $y \in D'$.

(3) $(Y, D; \Gamma)$ with $\Gamma : \langle D \rangle \multimap Y$ defined by $\Gamma_A = f(\mathrm{co}\, A)$ for $A \in \langle D \rangle$ is an example of an H-space satisfying $D \not\subset Y$.

Example 2.6 Let $X = D = [0,1]$ and $Y = D' = \mathbb{S}^1 = \{e^{2\pi it} : t \in [0,1]\}$. Define f and Γ_A as in Example 2.5. Then $(Y \supset D'; \Gamma)$ is a compact G-convex space:

(1) Note that $1 \in \mathbb{S}^1$ and that $\Gamma_{\{1\}} = \mathbb{S}^1$ is not contractible. Hence, $(Y \supset D'; \Gamma)$ is not an H-space.

(2) Moreover $\Gamma_{\{1\}} \neq \{1\}$. Therefore, in general, $\Gamma_{\{y\}} \neq \{y\}$ in an H-space.

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from X into the power set 2^Y of Y. $T(x)$ is called the *value* of T at $x \in X$ and $T^-(y) := \{x \in X : y \in T(x)\}$ the *fiber* of T at $y \in Y$. Let $T(A) := \bigcup\{T(x) : x \in A\}$ for $A \subset X$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its *graph* $\mathrm{Gr}(T) := \{(x,y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if its range $T(X)$ is contained in a compact subset of Y.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, the set $T^-(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, the set $T^-(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that a compact closed map is u.s.c. and compact-valued, and that every u.s.c. map with closed values is closed whenever its range is regular.

For a G-convex space $(X, D; \Gamma)$, a map $F : D \multimap X$ is called a *KKM map* if

$$\Gamma_N \subset F(N) \quad \text{for each } N \in \langle D \rangle.$$

The following is a KKM theorem for G-convex spaces [6]:

Theorem 2.1 *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ be a map such that*

(1) *F has closed [resp., open] values; and,*

(2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property [more precisely, for each $N \in \langle D \rangle$, we have $\Gamma_N \cap \bigcap_{z \in N} F(z) \neq \emptyset$].

Furthermore, if :

(3) *$\cap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have

$$\bigcap_{z \in D} \overline{F(z)} \neq \emptyset.$$

Recall that, in Theorem 2.1, X is not necessarily Hausdorff.

3 The Class $\mathfrak{B}$ of Multimaps

Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of multimaps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$ and any continuous function $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n)$.

We give some subclasses of $\mathfrak{B}$ as follows:

Example 3.1 For topological spaces X and Y, an *admissible* class $\mathfrak{A}^\kappa_c(X, Y)$ of maps $F : X \multimap Y$ is one such that, for each nonempty compact subset K of X, there exists a map $G \in \mathfrak{A}_c(K, Y)$ satisfying $G(x) \subset F(x)$ for all $x \in K$, where $\mathfrak{A}_c$ consists of finite compositions of maps in a class $\mathfrak{A}$ of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) Each $T \in \mathfrak{A}_c$ is u.s.c. with nonempty compact values; and,

(iii) For any polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Here, a polytope P is a homeomorphic image of a standard simplex. There are lots of examples of $\mathfrak{A}$ and $\mathfrak{A}^\kappa_c$; see [5–6, 8, 13–16, 20–21, 38–39].

Subclasses of the admissible class $\mathfrak{A}^\kappa_c$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains being convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}^+_c$ of Lassonde, the class $\mathbb{V}^+_c$ of Park et al., approximable maps of Ben-El-Mechaiekh and Idizk, and many others.

Note that, for a G-convex space $(X, D; \Gamma)$ and any space Y, an admissible class $\mathfrak{A}^\kappa_c(X, Y)$ is a subclass of $\mathfrak{B}(X, Y)$ with some possible exceptions such as Kakutani maps. Some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}^\kappa_c$ were known [40]. Note that the connectivity map due to Nash and Girolo is such an example; see [21].

Example 3.2 For a convex space $(X \supset D;\ \Gamma)$, where $\Gamma = \mathrm{co}$ and ϕ_N is a homeomorphism, the class $\mathfrak{B}(X, Y)$ is originally given in [38] and investigated in [20–21].

Example 3.3 For a convex space X and a topological space Y, motivated by a work of the author, Chang and Yen [41] defined the class of maps $T : X \multimap Y$ having the KKM property as follows:

$T \in \mathfrak{K}(X, Y) \iff$ the family $\{S(x) : x \in X\}$ has the finite intersection property whenever $S : X \multimap Y$ has closed values and $T(\mathrm{co}\, N) \subset S(N)$ for each $N \in \langle X \rangle$.

For a convex space X, it is known that $\mathfrak{A}^\kappa_c(X, Y) \subset \mathfrak{K}(X, Y)$ and we observe that two subclasses $\mathfrak{B}$ and $\mathfrak{K}$ coincide in the class of all compact closed maps $T : X \multimap Y$ [36].

Generalizations of the class $\mathfrak{K}$ to G-convex spaces are possible; see [19].

Example 3.4 Ben-El-Mechaiekh et al. [32, 42] introduced the class $\mathbb{A}$ of approachable multimaps as follows:

Let X and Y be uniform spaces (with respective bases $\mathscr{U}$ and $\mathscr{V}$ of symmetric entourages). A multimap $T : X \multimap Y$ is said to be *approachable* whenever T admits a continuous W-approximative selection $s : X \to Y$ for each W in the basis $\mathscr{W}$ of the product uniformity on $X \times Y$; that is, $\mathrm{Gr}(s) \subset W[\mathrm{Gr}(F)]$, where

$$W[A] := \bigcup_{z \in A} W[z] = \{z' \in X \times Y : W[z'] \cap A \neq \emptyset\},$$

for any $A \subset X \times Y$, and

$$W[z] := \{z' \in X \times Y : (z, z') \in W\},$$

for $z \in X \times Y$.

A multimap $T : X \multimap Y$ is said to be *approximable* if its restriction $T|_K$ to any compact subset K of X is approachable.

It is known that if $(X \supset D;\ \Gamma)$ is a G-convex uniform space and Y is a uniform space, then any compact closed approachable map $F : X \multimap Y$ belongs to $\mathfrak{B}(X, Y)$; see [8, Lemma 3].

We give some examples of approachable maps $T : X \to Y$ as follows:

(1) Any selectionable multimap is approximable.

(2) A locally selectionable map T with convex values is approximable whenever Y is a convex subset of a t.v.s.

(3) A u.s.c. map T with nonempty convex values is approachable whenever X is paracompact and Y is a convex subset of a locally convex t.v.s.

(4) A u.s.c. map T with nonempty compact contractible values is approachable whenever X is a finite polyhedron.

(5) A u.s.c. map T with nonempty compact values having trivial shape (that is, contractible in each neighborhood in Y) is approachable whenever X is a finite polyhedron.

For (1) and (2), see [12]; and for (3)–(5), see [42]. The following has recently been known from (5):

Example 3.5 Let $(X, D; \Gamma)$ be a G-convex space and $F : X \multimap X$ a u.s.c. map. If F has nonempty compact values having trivial shape, then $F \in \mathfrak{B}(X, X)$.

Example 3.6 An important subclass of $\mathfrak{B}$ is the class of Φ-maps (or Fan–Browder maps) as follows:

Definition *Let Y be a topological space and $(X, D; \Gamma)$ a G-convex space. Then a map $T : Y \multimap X$ is called a Φ-map (or a Fan-Browder map) if there is a map $S : Y \multimap D$ such that*

(i) *For each $y \in Y$, $M \in \langle S(y) \rangle$ implies $\Gamma_M \subset T(y)$; and,*

(ii) $Y = \bigcup\{\operatorname{Int} S^-(z) : z \in D\}$.

Recall that Horvath [22] first defined a Φ-map when $(X; \Gamma)$ is a C-space.

For Φ-maps, we obtain the following selection theorem in view of [22–23; 2, Lemma 1]:

Lemma 3.1 *Let Y be a normal space, $(X, D; \Gamma)$ a G-convex space, and $S : Y \multimap D$ a map such that $Y = \bigcup\{\operatorname{Int} S^-(z) : z \in N\}$ for some $N \in \langle D \rangle$. Then there exists a continuous function $s : Y \to \Gamma_N$ such that $s(y) \in \Gamma(N \cap S(y))$ for all $y \in Y$. In fact, if $|N| = n+1$, then $s = \phi_N \circ p$, where $\phi_N : \Delta_n \to \Gamma_N$ and $p : Y \to \Delta_n$ are continuous functions.*

Note that Lemma 3.1 sharpens the compact case of [2, Theorem 3.2] and shows that every Φ-map $T : Y \multimap X$ belongs to $\mathbb{C}_c^\kappa(Y, X) \subset \mathfrak{A}_c^\kappa(Y, X)$. Therefore, if $X = Y$, then a Φ-map $T : X \multimap X$ belongs to $\mathfrak{B}(X, X)$.

Moreover, we need the following in [22, 2]:

Lemma 3.2 *Let Y be a paracompact space, $(X, D; \Gamma)$ an H-space, and $T : Y \multimap X$ a Φ-map. Then T has a continuous selection.*

In our previous works [6, 8], we gave some fixed point theorems for Φ-maps on G-convex spaces. The following is one of the simplest consequences of the KKM Theorem 2.1:

Theorem 3.3 *Let $(X,\ N;\ \Gamma)$ be a G-convex space, N a finite set, and $S : N \multimap X$, $T : X \multimap X$ be two maps satisfying*

(1) *For each $z \in N$, $S(z)$ is open [resp., closed];*

(2) *For each $y \in X$, $J \in \langle S^-(y) \rangle$ implies $\Gamma_J \subset T^-(y)$; and,*

(3) $X = S(N)$.

Then T has a fixed point $x_0 \in X$.

In [3], this is applied to obtain various forms of known Fan–Browder type theorems, the Ky Fan intersection theorem, and the Nash equilibrium theorem.

The following is the dual form of Theorem 3.3:

Theorem 3.4 *Let $(X, D; \Gamma)$ be a G-convex space and $S : X \multimap D$, $T : X \multimap X$ be maps such that*

(1) *For each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$;*

(2) *$S^-(z)$ is open [resp., closed] for each $z \in D$; and,*

(3) *$X = \bigcup\{S^-(z) : z \in N\}$ for some $N \in \langle D \rangle$.*

Then T has a fixed point $x_0 \in X$.

From Theorem 3.4, we have the following:

Theorem 3.5 *Let $(X \supset D; \Gamma)$ be a G-convex space and $A : X \multimap X$ be a multimap such that $A(x)$ is Γ-convex for each $x \in X$. If there exist $z_1, z_2, \ldots, z_n \in D$ and nonempty open [resp. closed] subsets $G_i \subset A^-(z_i)$ for $i = 1, 2, \ldots, n$ such that $X = \bigcup_{i=1}^n G_i$, then A has a fixed point.*

Theorem 3.6 *Let $(X, D; \Gamma)$ be a G-convex space and $S : X \multimap D$, $T : X \multimap X$ be maps such that*

(1) *For each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$; and,*

(2) *$X = \bigcup\{\operatorname{Int} S^-(z) : z \in N\}$ for some $N \in \langle D \rangle$.*

Then T has a fixed point.

From Theorems 3.3–3.6, most of the popular variations or generalizatons of the Fan–Browder theorem (in the forms of the compact or so-called non-compact versions) can be deduced.

For example, from Theorem 3.4, we have the following generalization of the Fan–Browder fixed point theorem:

Theorem 3.7 *Let $(X, D; \Gamma)$ be a compact G-convex space (that is, X is compact). Then any Φ-map $T : X \multimap X$ has a fixed point.*

Note that, in Theorems 3.3–3.7, X is not necessarily Hausdorff. Theorem 3.7 is originated from Browder [43] and has numerous applications.

4 Admissible G-convex Spaces

For more general purposes, we introduce a generalized version of our previous definition of the admissibility of domains of multimaps by switching it to the Klee approximability of their ranges, as follows:

Definition *A G-convex uniform space $(X, D; \Gamma; \mathscr{U})$ is a G-convex space such that $(X, \mathscr{U})$ is a uniform space with a basis $\mathscr{U}$ of the uniformity consisting of symmetric entourages. For each $U \in \mathscr{U}$, let*

$$U[x] = \{x' \in X : (x, x') \in U\}$$

be the U-ball around a given element $x \in X$.

We introduce a new class of G-convex spaces adequate to establish our fixed point theory:

Definition *For a G-convex uniform space $(X, D; \Gamma; \mathscr{U})$, a subset Y of X is said to be admissible (in the sense of Klee) if, for each nonempty compact subset K of Y and for each entourage $U \in \mathscr{U}$, there exists a continuous function $h : K \to Y$ satisfying*

(1) *$(x, h(x)) \in U$ for all $x \in K$;*

(2) *$h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$; and,*

(3) *There exists a continuous function $p : K \to \Delta_n$ such that $h = \phi_N \circ p$, where $\phi_N : \Delta_n \to \Gamma_N$ and $|N| = n + 1$.*

Example 4.1 A nonempty subset Y of a t.v.s. E is said to be admissible (in the sense of Klee) provided that, for every compact subset K of Y and every neighborhood V of the origin 0 of E, there exists a continuous function $h : K \to Y$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite-dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable function on $[0,1]$, the Hardy spaces H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarreled t.v.s. admitting Schauder basis, and others. Note also that any locally convex subset of an F-normable t.v.s. or any locally convex subset which is a finite union of closed convex subsets of a t.v.s. is admissible. For the details, see Hadžić [27], Weber [44–45], Hahn [46] and references therein.

We introduce another new concept.

Definition *Let $(X, D; \Gamma; \mathscr{U})$ be a G-convex uniform space. A subset K of X is said to be Klee approximable if, for each entourage $U \in \mathscr{U}$, there exists a continuous function $h : K \to X$ satisfying conditions* (1)–(3) *in the preceding definition. In particular, for a subset Y of X, K is said to be Klee approximable into Y whenever the range $h(K) \subset \Gamma_N \subset Y$ for some $N \in \langle D\rangle$ in condition* (2).

Example 4.2 Every nonempty compact Φ-set of a G-convex uniform space is Klee approximable, and every Φ-space $(X, D;\ \Gamma;\ \mathscr{U})$ is admissible; see Section 5.

Example 4.3 In a t.v.s. E, we give some examples of Klee approximable sets as follows:

(1) If a subset X of E is admissible (in the sense of Klee), then every compact subset K of X is Klee approximable into E.

(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.

(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.

(4) Any compact subset K of an admissible convex subset X of a t.v.s. is Klee approximable into X.

Note that (4)⇒(3).

Remark For a compact subset K of E, it is well known that $\operatorname{co} K$ is σ-compact and paracompact. Therefore, if E is metrizable, then the following is supplementary to (4):

Lemma 4.1 (Dobrowolski [47]) *For a σ-compact convex set C in a metrizable t.v.s. the following conditions are equivalent*:

(a) $C \in AR$;

(b) $C = \bigcup_1^\infty A_n$ *with* $A_n = \overline{A}_n \in AE(compact)$ *for* $n = 1, 2, \dots$;

(c) *C is admissible.*

Here AR denotes an absolute retract and AE (compact) the absolute extension property for compacta.

We have the following main result in this section:

Theorem 4.2 *Let $(X, D; \Gamma; \mathscr{U})$ be a G-convex uniform space and $F \in \mathfrak{B}(X, X)$ a multimap such that $F(X)$ is Klee approximable. Then F has the almost fixed point property (that is, for each $U \in \mathscr{U}$, F has a U-fixed point $x_U \in X$, that is, $F(x_U) \cap U[x_U] \neq \emptyset$).*

Furthermore if F is closed and compact, then F has a fixed point $x_0 \in X$ (that is, $x_0 \in F(x_0)$).

Proof Since $K := F(X)$ is Klee approximable into $(X,\ \mathscr{U})$, for each symmetric entourage $U \in \mathscr{U}$, there exists a continuous function $h : K \to X$ satisfying conditions (1)–(3) of the definition of Klee approximable subsets, and we have

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} K \xrightarrow{p} \Delta_n,$$

for some $N \in \langle D\rangle$ with $|N| = n+1$. Let $p' := p|_{F(\Gamma_N)}$. Since $F \in \mathfrak{B}(X, X)$, the composition $p' \circ (F|_{\Gamma_N}) \circ \phi_N : \Delta_n \multimap \Delta_n$ has a fixed point $a_U \in \Delta_n$. Let $x_U := \phi_N(a_U)$. Then

$$a_U \in (p' \circ F \circ \phi_N)(a_U) = (p' \circ F)(x_U)$$

and hence

$$x_U = \phi_N(a_U) \in (\phi_N \circ p' \circ F)(x_U).$$

Since $h = \phi_N \circ p$ by definition, we have

$$x_U = h(y_U) \quad \text{for some} \quad y_U \in (F|_{\Gamma_N})(x_U).$$

Therefore, for each entourage $U \in \mathscr{U}$, there exist points $x_U \in X$ and $y_U \in F(x_U)$ such that $(x_U, y_U) = (h(y_U), y_U) \in U$. So, for each U, there exist $x_U, y_U \in X$ such that $y_U \in F(x_U)$ and $y_U \in U[x_U]$.

Now suppose that F is closed and compact. Since $F(X)$ is relatively compact, we may assume that the net y_U converges to some $x_0 \in \overline{F(X)}$. Then, by the Hausdorffness of X, the net x_U also converges to x_0. Since the graph of F is closed in $X \times \overline{F(X)}$ and $(x_U, y_U) \in \mathrm{Gr}(F)$, we have $(x_0, y_0) \in \mathrm{Gr}(F)$ and hence we have $x_0 \in F(x_0)$. This completes our proof.

Theorem 4.3 *Let $(X, D; \Gamma; \mathscr{U})$ be an admissible G-convex space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Proof Note that $\overline{F(X)}$ is a compact subset of an admissible G-convex space, and hence is Klee approximable. Therefore, Theorem 4.2 works.

Corollary 4.4 *Let $(X, D; \Gamma; \mathscr{U})$ be a compact admissible G-convex space. Then any map $F \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

Since an admissible convex subset of a t.v.s. is an admissible G-convex space, we have the following from Theorem 4.3:

Corollary 4.5 *Let X be an admissible convex subset of a t.v.s. E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Corollary 4.5 was given in [21] where we listed more than sixty papers in chronological order, from which we could deduce the particular forms of Corollary 4.4. In particular, from Corollary 4.4, we obtain:

Corollary 4.6 *Let X be an admissible convex subset of a t.v.s. E. Then any compact map $F \in \mathbb{V}_c(X, X)$ (that is, a finite composition of acyclic maps) has a fixed point.*

Corollary 4.6 was given in [11, 48–49] and applied to Simons-type cyclic coincidence theorems for acyclic maps, the von Neumann-type intersection theorems for graphs of compact compositions of acyclic maps, the Nash-type equilibrium theorems, saddle point or minimax theorems, quasi-equilibrium problems, and quasi-variational inequalities, where most of the related convexity was replaced by acyclicity.

5 G-convex Φ-spaces

In this section, we deal with a subclass of the class of admissible G-convex spaces.

Definition *For a G-convex uniform space $(X, D; \Gamma; \mathscr{U})$, a subset Y of X is called a Φ-set if for each entourage $U \in \mathscr{U}$, there exists a Φ-map $T : Y \multimap X$ such that $\mathrm{Gr}(T) \subset U$ (that is, $T(y) \subset U[y]$ for all $y \in Y$). If X itself is a Φ-set, then it is called a Φ-space.*

Note that every subset Y of a Φ-space is a Φ-set.

Examples 5.1 Horvath [22] first defined a Φ-space for a C-space $(X; \Gamma)$ and gave examples as follows:

(1) A particular type of uniform spaces including locally convex t.v.s.;

(2) Convex metric spaces in the sense of Takahashi with a metric satisfying a certain property.

Example 5.2 An important subclass of Φ-sets is that of locally convex sets in a t.v.s. A nonempty subset K of a t.v.s. E is said to be *locally convex* if, for each $x \in K$ and each neighborhood U_x of x, there exists a neighborhood V_x of x such that $\mathrm{co}\,(V_x \cap K) \subset U_x$; see [50].

The following are examples of locally convex sets:

(1) Every nonempty subset of a locally convex t.v.s.;

(2) Every nonempty subset of a locally convex set in a t.v.s.

For other nontrivial examples of convex and locally convex subsets, see Hadžić [27]. Moreover, there is an example of a nonconvex, admissible, locally convex subset of a non-locally convex t.v.s.; see Hahn [46].

Proposition 5.1 *Every locally convex subset Y of a convex subset X of a t.v.s. E is a Φ-subset of X.*

Proof For any neighborhood V of 0 in E and for each $y \in Y$, let $V_y := y + V$. Since y is locally convex in X, we choose a neighborhood W_y of y such that $\operatorname{co}(W_y \cap Y) \subset V_y$. Define two maps $S : Y \multimap Y$ and $T : Y \multimap X$ by

$$T(y) := V_y \cap X \text{ and } S(y) := W_y \cap Y,$$

for each $y \in Y$. *Then*

(1) $M \in \langle S(y) \rangle$ implies $\operatorname{co} M \subset V_y \cap X = T(y)$;

(2) Since $y \in S(y)$ for each $y \in Y$,

$$y \in S^-(y) = \{z \in Y : z \in S(y)\} = \{z \in Y : z \in W_y\} = W_y \cap Y$$

is open in Y, and hence

$$Y = \bigcup_{y \in Y} W_y \cap Y = \bigcup_{y \in Y} \operatorname{Int} S^-(y).$$

Moreover, $T(y) \subset V_y = y + V$ for each $y \in Y$. Therefore, Y is a Φ-subset of X.

Remark The concept of locally convex subsets for G-convex spaces is said to be *of type* I by Kim [51] without assuming any uniformity. Therefore, her method in [51] is different from ours.

Example 5.3 Any subset of the Zima type in a G-convex uniform space $(X \supset D;\ \Gamma;\ \mathscr{U})$ such that every singleton is Γ-convex (that is, $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$) is a Φ-set; see Section 6.

Example 5.4 A locally G-convex space $(X \supset D; \Gamma; \mathscr{U})$ is a Φ-space; see Section 7.

We give another example of the subclasses of admissible G-convex spaces:

Proposition 5.2 *Every Φ-space $(X, D; \Gamma; \mathscr{U})$ is admissible. More precisely, for a G-convex uniform space $(X, D; \Gamma; \mathscr{U})$, every nonempty compact Φ-subset K of X is Klee approximable.*

Proof Since K is a Φ-set, for each $U \in \mathscr{U}$, there exist multimaps $S : K \multimap D$ and $T : K \multimap X$ such that

(i) For each $y \in K$, $M \in \langle S(y) \rangle$ implies $\Gamma_M \subset T(y)$;

(ii) $K = \bigcup \{\operatorname{Int} S^-(z) : z \in D\}$; and,

(iii) $\operatorname{Gr}(T) \subset U$.

Since K is compact, it follows from Lemma 3.1 that T has a continuous selection $h : K \to X$ such that

(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$ with $|N| = n + 1$; and,

(3) There exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ such that $h = \phi_N \circ p$.

Moreover, $h(x) \in T(x)$ for all $x \in K$ implies

(1) $(x, h(x)) \in \operatorname{Gr}(T) \subset U$ for all $x \in K$.

Therefore, K is Klee approximable and hence $(X, D; \Gamma; \mathscr{U})$ is admissible.

From Lemma 3.2 and Theorem 4.3, we have the following:

Theorem 5.3 *Let $(X, D; \Gamma; \mathscr{U})$ be an admissible H-space such that X is paracompact. Then any compact Φ-map $T : X \multimap X$ has a fixed point.*

Proof By Lemma 3.2, T has a continuous selection $f : X \to X$. Since $f(X) \subset T(X)$, f is compact and hence has a fixed point by Theorem 4.3.

From Proposition 5.2 and Theorem 5.3, we have

Corollary 5.4 *Let $(X, D; \Gamma; \mathscr{U})$ be an H-space. If it is also a paracompact Φ-space, then any compact Φ-map $T : X \multimap X$ has a fixed point.*

Some applications of Corollary 5.4 were given in [22, 9].

From Proposition 5.2 and Theorem 4.2, we have the following:

Theorem 5.5 *Let $(X, D; \Gamma; \mathscr{U})$ be a G-convex uniform space and $F \in \mathfrak{B}(X, X)$ be a map such that $\overline{F(X)}$ is a compact Φ-subset of X. If F is closed, then F has a fixed point.*

Since every locally convex set is a Φ-set, we have the following:

Corollary 5.6 *Let X be a nonempty convex subset of a t.v.s. Then any compact closed map $F \in \mathfrak{B}(X, X)$, such that $\overline{F(X)}$ is locally convex, has a fixed point.*

Note that Rzepecki [50] first obtained a particular form of Corollary 5.6 for a single-valued continuous function.

From Theorem 5.5 and Proposition 5.2, we have the following in [5, Theorem 2]:

Corollary 5.7 *Let $(X, D; \Gamma; \mathscr{U})$ be a Φ-space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Particular forms of Corollary 5.7 were known by Horvath [22], and Park and Kim [13]. Later, Ben-El-Mechaiekh et al. [32, Theorem 4.2] obtained a particular form of Corollary 5.7 for approachable multimaps. In [5, 8–10, 48–49], it was shown that Corollary 5.7 subsumes a large number of fixed point theorems related to approachable maps on G-convex spaces, acyclic maps on locally G-convex spaces, and Kakutani maps on Φ-spaces or on hyperconvex spaces.

For a non-closed map, we have the following:

Corollary 5.8 *Let $(X, D; \Gamma; \mathscr{U})$ be a compact Φ-space and $F \in \mathfrak{A}_c^\kappa(X, X)$. Then F has a fixed point.*

Now we give an example of a Φ-space as follows:

Theorem 5.9 *Let $(X \supset D; \Gamma)$ be a metric G-convex space such that*

(1) *D is dense in X; and,*

(2) *Every open ball is Γ-convex.*

Then $(X \supset D; \Gamma)$ is a Φ-space and every compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.

Proof Let $\varepsilon > 0$ and $S : X \multimap D$, $T : X \multimap X$ be defined by

$$S(x) = \{y \in D : d(x, y) < \varepsilon\}, \quad T(x) = \{y \in X : d(x, y) < \varepsilon\}$$

for $x \in X$. Then each $T(x)$ is Γ-convex and hence $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$. Moreover, since D is dense, for each $x \in X$, there exists a $y \in D$ such that $d(x, y) < \varepsilon$. Therefore, $\{x \in X : d(x, y) < \varepsilon\} = S^-(y) = \operatorname{Int} S^-(y)$ for $y \in D$, and hence

$$X = \bigcup_{y \in D} \{x \in X : d(x, y) < \varepsilon\} = \bigcup \{\operatorname{Int} S^-(y) : y \in D\}.$$

Therefore, T is a Φ-map. Note that, for any $x \in X$ and $y \in T(x)$, we have $d(x, y) < \varepsilon$. Therefore, for any entourage U of the metric uniformity, we have a Φ-map $T : X \multimap X$ such that $\operatorname{Gr}(T) \subset U$. This shows that $(X \supset D; \Gamma)$ is a Φ-space, and F has a fixed point by Corollary 5.7.

Remarks (1) Theorem 5.9 is a generalization of Horvath [23, Corollary 4.4].

(2) Particular versions of Theorem 5.9 were known in [10].

From Proposition 5.2, Theorems 5.3 and 5.9, we have

Corollary 5.10 *Let $(X \supset D; \Gamma)$ be a metric H-space such that D is dense in X and every open ball is Γ-convex. Then every compact Φ-map $F : X \multimap X$ has a fixed point.*

6 *G*-convex Spaces of the Zima Type

This section deals with a subclass of the class of Φ-spaces. We begin with the following almost fixed point theorem:

Theorem 6.1 *Let $(X \supset D; \Gamma; \mathscr{U})$ be a G-convex uniform space, and K a totally bounded subset of X such that $D \cap K$ is dense in K. Let $T : X \multimap X$ be a u.s.c. [resp., an l.s.c.] multimap* *such that $T(x) \cap K \neq \emptyset$ for each $x \in X$. Let $U \in \mathscr{U}$. Suppose that there exists $V \in \mathscr{U}$ such that, for each $x \in X$, we have*

$$N \in \langle \{y \in D : T(x) \cap V[y] \neq \emptyset\} \rangle \Rightarrow \Gamma_N \subset \{y \in X : T(x) \cap U[y] \neq \emptyset\}.$$

Then T has a U-fixed point $x_U \in X$ (that is, $T(x_U) \cap U[x_U] \neq \emptyset$).

Proof We may assume that all members of $\mathscr{U}$ are open [resp., closed]. Then there exists an open [resp., a closed] member V of $\mathscr{U}$ satisfying the hypothesis. Note that, for each $x \in X$, $V[x]$ is a neighborhood of x. Since K is totally bounded and $D \cap K$ is dense in K, there exists an $M := \{y_1, \ldots, y_n\} \in \langle D \cap K \rangle$ such that $K \subset \bigcup_{y \in M} V[y]$.

For each $y_i \in M$, let $F(y_i) := \{x \in X : T(x) \cap V[y_i] = \emptyset\}$. Since T is u.s.c. [resp., l.s.c.], each $F(y_i)$ is open [resp., closed]. Moreover, since $T(X) \cap K \subset \bigcup_{i=1}^n V[y_i]$, we have

$$\bigcap_{i=1}^{n} F(y_i) \subset \left\{ x \in X : T(x) \cap \bigcup_{i=1}^{n} V[y_i] = \emptyset \right\} = \emptyset.$$

We will apply Theorem 2.1 to the G-convex space $(X \supset M; \Gamma)$. Since the conclusion of Theorem 2.1 does not hold, $F : M \multimap X$ can not be a KKM map; that is, there exist an $N \in \langle M \rangle$ and an $x_U \in \Gamma_N$ such that $x_U \notin F(N) = \bigcup_{y \in N} F(y)$. Hence $T(x_U) \cap V[y] \neq \emptyset$ for all $y \in N$, and

$$N \subset \{y \in D : T(x_U) \cap V[y] \neq \emptyset\}.$$

Then

$$x_U \in \Gamma_N \subset \{y \in X : T(x_U) \cap U[y] \neq \emptyset\}.$$

Hence $T(x_U) \cap U[x_U] \neq \emptyset$.

Remarks (1) Note that, in the above proof, if $\Gamma_N \subset D$ for each $N \in \langle D \rangle$, it is sufficient to assume that, for each $x \in D$ and each $U \in \mathscr{U}$, the set $\{y \in D : T(x) \cap U[y] \neq \emptyset\}$ is Γ-convex.

(2) In Theorem 6.1, X is not necessarily Hausdorff.

(3) Theorems 6.1 is motivated by Hadžić [29, Theorem 1].

Theorem 6.2 *Under the hypothesis of Theorem* 6.1, *furthermore if T is closed and compact, then T has a fixed point.*

Proof For each $U \in \mathscr{U}$, there exist $x_U, y_U \in X$ such that $y_U \in T(x_U)$ and $y_U \in U[x_U]$. Since $T(X)$ is relatively compact, we may assume that the net y_U converges to some $x_0 \in \overline{T(X)}$. Then the corresponding net x_U also converges to x_0 by the Hausdorffness of X. Since the graph of T is closed in $X \times \overline{T(X)}$ and $(x_U, y_U) \in \mathrm{Gr}(T)$, we have $x_0 \in T(x_0)$. This completes our proof.

Motivated by Theorems 6.1 and 6.2, we introduce the following:

Definition *For a G-convex uniform space $(X \supset D;\ \Gamma;\ \mathscr{U})$, a subset Y of X is said to be of the Zima type* (*or of the Zima–Hadžić type*) *if $D \cap Y$ is dense in Y and for each $U \in \mathscr{U}$ there exists a $V \in \mathscr{U}$ such that, for each $N \in \langle D \cap Y \rangle$ and any Γ-convex subset A of Y, we have*

$$A \cap V[z] \neq \emptyset \quad \forall z \in N \Rightarrow A \cap U[x] \neq \emptyset \quad \forall x \in \Gamma_N.$$

Example 6.1 (1) Hadžić [24] defined that a nonempty subset K of a t.v.s. E is of the Zima type whenever for any $U \in \mathscr{V}$, there exists a $V \in \mathscr{V}$ satisfying $\mathrm{co}(V \cap (K - K)) \subset U$, where $\mathscr{V}$ is a neighborhood system of the origin of E.

Note that any nonempty subset of a locally convex t.v.s. is of the Zima type, and that there exists a subset of the Zima type in a non-locally convex t.v.s.; see Hadžić [27–29].

(2) For a C-space, our definition reduces to that of Hadžić [29].

Example 6.2 Motivated by a well-known work of Idzik [33] on convexly totally bounded (c.t.b. for short) sets, Weber [44–45] defined the following:

A subset K of a t.v.s. E is said to be *strongly convexly totally bounded* (s.c.t.b.) if for every neighborhood V of $0 \in E$ there exist a convex subset C of V and a finite subset N of K such that $K \subset N + C$.

The following is known:

Lemma 6.3 (Weber [45, Corollary 2.8]) *Let K be a compact convex subset of a t.v.s. (E,τ) and $F = \operatorname{span} K$. Then the following conditions are equivalent*:

(1) *K is s.c.t.b.*;

(2) *K is of Zima type*;

(3) *K is locally convex*;

(4) *K is affinely embeddable in a locally convex t.v.s.*;

(5) *E admits a Hausdorff locally convex linear topology $\sigma = \sigma(E, E')$, which induces on F a finer topology than τ such that $\sigma|_K = \tau|_K$.*

Example 6.3 For an LG-space $(X \supset D;\ \Gamma;\ \mathscr{U})$, any nonempty subset Y of X is of the Zima type; see Section 8.

We show that a certain set of the Zima type is a Φ-set as follows:

Proposition 6.4 *Let $(X \supset D;\ \Gamma;\ \mathscr{U})$ be a G-convex uniform space such that every singleton is Γ-convex. Then any subset Y of the Zima type in X is a Φ-set.*

Proof For each $U \in \mathscr{U}$, let $V \in \mathscr{U}$ be the one satisfying the definition of the Zima type. Define $S : Y \multimap D$ and $T : Y \multimap X$ by

$$S(y) := \{z \in D : (z, y) \in V\} \text{ and } T(y) := \{x \in X : (x, y) \in U\},$$

for $y \in Y$. Since $D \cap Y$ is dense in Y, $S(y)$ is not empty.

We show that $N \in \langle S(y)\rangle$ implies $\Gamma_N \subset T(y)$ for each $y \in Y$. In fact, for each $z \in N$,

$$z \in S(y) \Rightarrow y \in V[z] \Rightarrow \{y\} \cap V[z] \neq \emptyset,$$

which implies, for all $x \in \Gamma_N$,

$$\{y\} \cap U[x] \neq \emptyset \Rightarrow y \in U[x] \Rightarrow x \in T(y),$$

since $\{y\}$ is Γ-convex.

Furthermore, since $S(y) \neq \emptyset$ for $y \in Y$ and $y \in S^-(z) = V[z]$ for some $z \in D$, we have $Y = \bigcup\{\operatorname{Int} S^-(z) : z \in D\}$.

Therefore, we have a Φ-map $T : Y \multimap X$ such that $\operatorname{Gr}(T) \subset U$. This shows that Y is a Φ-set.

From Theorem 5.5 and Proposition 6.4, we have the following:

Theorem 6.5 *Let $(X \supset D;\ \Gamma;\ \mathscr{U})$ be a G-convex uniform space such that every singleton is Γ-convex and $F \in \mathfrak{B}(X, X)$ such that $\overline{F(X)}$ is of the Zima type. If F is closed and compact, then F has a fixed point.*

Remark It is known that, in a t.v.s. E, a subset A is of the Zima type if and only if its closure $\overline{A}$ is of the Zima type; see [52].

From Theorems 6.1 and 6.2, we have the following:

Theorem 6.6 *Let $(X \supset D;\ \Gamma;\ \mathscr{U})$ be a G-convex uniform space. Let $T : X \multimap X$ be a u.s.c. [resp., an l.s.c.] multimap with nonempty Γ-convex values such that $T(X)$ is totally bounded and of the Zima type. Then T has the almost fixed point property.*

Remark In Theorem 6.6, X is not necessarily Hausdorff.

Here, we give a general definition of Kakutani maps as follows:

Definition *Let Y be a topological space and $(X \supset D;\ \Gamma)$ a G-convex space. A map $F : Y \multimap X$ is called a Kakutani map if it is u.s.c. and has nonempty compact Γ-convex values.*

From Theorem 6.6, we have the following fixed point theorems for Kakutani maps:

Theorem 6.7 *Let $(X \supset D;\ \Gamma;\ \mathscr{U})$ be a G-convex uniform space. Let $T : X \multimap X$ be a compact Kakutani map such that $T(X)$ is of the Zima type. Then T has a fixed point.*

Theorem 6.7 generalizes the results of Hadžić [29].

Corollary 6.8 *Let X be a convex subset of a t.v.s. Then any compact Kakutani map $T : X \multimap X$ has a fixed point in X whenever $T(X)$ is of the Zima type.*

Remarks (1) Hadžić [24, 26] obtained Corollary 6.8 under the restriction that X is closed. A number of consequences and applications of her result were given in Hadžić [24, 26] and Hadžić and Gajić [30].

(2) Hadžić [25] obtained a particular form of Corollary 6.8 for a compact convex subset X of a metrizable t.v.s. E.

(3) Hadžić [28] obtained a particular form of Corollary 6.8 for a subset X of the Zima type in a complete t.v.s. E, and applied her result to some economic problems.

(4) Recall that Corollary 6.8 is a consequence of the Idzik theorem [33].

For more results on the Zima type in t.v.s., see [53] and references therein.

7 Locally G-convex Spaces

This section deals with another subclass of the class of Φ-spaces.

Definition *A G-convex uniform space $(X \supset D;\ \Gamma;\ \mathscr{U})$ is said to be locally G-convex if D is dense in X and, for each $U \in \mathscr{U}$, there exists a $V \in \mathscr{U}$ such that $V \subset U$ and, for each $x \in X$,*

$$N \in \langle V[x] \cap D \rangle \Rightarrow \Gamma_N \subset U[x].$$

Remarks (1) This concept was first considered by Bielawski [31] and later by Ben-El-Mechaiekh et al. [32].

(2) In particular, if the U-ball $U[x]$ itself is Γ-convex for each $x \in X$, then $(X \supset D;\ \Gamma;\ \mathscr{U})$ is locally G-convex. In such a case, every singleton is Γ-convex since X is Hausdorff, $\{x\} = \bigcap_{U \in \mathscr{U}} U[x]$ and the intersection of Γ-convex subsets is Γ-convex.

Example 7.1 Any convex subset of a locally convex t.v.s. is a locally G-convex space. Note that the concept of local G-convexity does not generalize that of a local convexity of a subset of a t.v.s. defined in Example 5.2.

Example 7.2 Every LG-space is locally G-convex if every singleton is Γ-convex; see Section 8.

We give another example of Φ-spaces as follows:

Proposition 7.1 *For a locally G-convex space $(X \supset D;\ \Gamma;\ \mathscr{U})$, any nonempty subset Y of X is a Φ-set.*

Proof Let $U \in \mathscr{U}$ and $V \subset U$ be as in the definition of locally G-convex spaces. We may assume V is open. Define multimaps $S : Y \multimap D$ and $T : Y \multimap X$ by

$$S(y) := \{z \in D : (y, z) \in V\} \quad \text{and} \quad T(y) := \{x \in X : (y, x) \in U\},$$

for $y \in Y$. Since D is dense in X, we have $\emptyset \neq S(y) \subset T(y)$ for $y \in Y$. For each $y \in Y$, $N \in \langle V[y] \cap D \rangle = \langle S(y) \rangle$ implies $\Gamma_N \subset U[y] = T(y)$. Moreover, $S^-(z)$ is open for each $z \in D$ and $Y = \bigcup\{S^-(z) : z \in D\}$ since D is dense in X. Note that $\mathrm{Gr}(T) \subset U$. Therefore, Y is a Φ-set.

From Proposition 7.1 and Theorem 5.5, we have the following:

Theorem 7.2 *Let $(X \supset D;\ \Gamma;\ \mathscr{U})$ be a locally G-convex space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Particular forms of Theorem 7.2 are known by Ben-El-Mechaiekh et al. [32, Corollary 4.7] and [42, Corollary 4.6] for an approachable map and by Horvath [22] and Bielawski [31] for a continuous function.

Corollary 7.3 *Let X be a convex subset of a locally convex t.v.s. E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

This was first obtained in [38].

Corollary 7.4 *Let X be a convex subset of a locally convex t.v.s. E. Then any compact acyclic map $F : X \multimap X$ has a fixed point.*

This was first obtained in [54] as a generalization of the Himmelberg theorem [55] and applied to abstract variational inequalities, minimax inequalities, geometric properties of convex sets, and other problems. This has been generalized step by step in a number of works of the author and, finally, Theorem 4.2 is the most general form we have.

8 LG-spaces

In this section, we introduce a particular subclass of locally G-convex spaces:

Definition *A G-convex uniform space $(X \supset D; \Gamma; \mathscr{U})$ is called an LG-space if D is dense in X and, for each $U \in \mathscr{U}$, the U-neighborhood*

$$U[A] := \{x \in X : A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset X$ is Γ-convex.

Similarly, we can define an LH-space.

Remarks (1) This concept is first introduced in [8].

(2) A singleton is not necessarily Γ-convex in an LG-space.

Example 8.1 For a C-space $(X; \Gamma)$, an LG-space reduces to an LC-space [22,23] (or a locally C-convex space [56]). Any nonempty convex subset X of a locally convex t.v.s. E is an obvious example of an LC-space $(X; \Gamma)$ with $\Gamma_A = \operatorname{co} A$ for $A \in \langle X \rangle$. For other examples, see [22, 56].

Example 8.2 A G-convex space $(X \supset D; \Gamma)$ is called an *LG-metric space* if X is equipped with a metric d such that (1) D is dense in X; (2) for any $\varepsilon > 0$, the set $\{x \in X : d(x, C) < \varepsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex; and, (3) open balls are Γ-convex. This concept generalizes that of LC-metric spaces due to Horvath [22].

Example 8.3 A metric space (H, d) is said to be *hyperconvex* if, for any collection of points $\{x_\alpha\}$ of H and for any collection $\{r_\alpha\}$ of nonnegative reals such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, we have

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset,$$

where $B(x, r)$ denotes the closed ball with center $x \in H$ and radius $r > 0$.

For any nonempty bounded subset A of H, its *convex hull* $\operatorname{Co} A$ is defined by

$$\operatorname{Co} A = \bigcap \{B : B \text{ is a closed ball containing } A\}.$$

Let $\mathscr{A}(H) = \{A \subset H : A = \operatorname{Co} A\}$; that is, $A \in \mathscr{A}(H)$ iff A is an intersection of closed balls.

For fixed point theorems on hyperconvex metric spaces, see [57] and references therein.

The following is known:

Lemma 8.1 (Horvath [23]) *Any hyperconvex metric space (H, d) is a complete metric LC-space $(H; \Gamma)$ with $\Gamma_A = \operatorname{Co} A$ for each $A \in \langle H \rangle$.*

Note that most of the above examples of LG-spaces are locally G-convex spaces. In fact, we have:

Proposition 8.2 *Every LG-space $(X \supset D;\ \Gamma;\ \mathscr{U})$ is locally G-convex if every singleton is Γ-convex.*

Proof For each symmetric entourage $U \in \mathscr{U}$ and any $x \in X$,

$$\begin{aligned} U[x] &= \{x' \in X : (x, x') \in U\} \\ &= \{x' \in X : x \in U^-[x']\} \\ &= \{x' \in X : \{x\} \cap U^-[x'] \neq \emptyset\}. \end{aligned}$$

Since $\{x\}$ is Γ-convex and $(X \supset D;\ \Gamma;\ \mathscr{U})$ is an LG-space, $U[x]$ is Γ-convex. Therefore, $(X \supset D;\ \Gamma;\ \mathscr{U})$ is locally G-convex.

From Proposition 8.2 and Theorem 7.2, we have the following:

Theorem 8.3 *Let $(X \supset D;\ \Gamma;\ \mathscr{U})$ be an LG-space such that singletons are Γ-convex. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Recall that any nonempty subset of a locally convex t.v.s. is of the Zima type. This can be generalized as follows:

Proposition 8.4 *For an LG-space* $(X \supset D;\ \Gamma;\ \mathscr{U})$*, any nonempty subset* Y *of* X *is of the Zima type.*

Proof For each $U \in \mathscr{U}$, choose $V := U$ in the definition of the Zima type. Then, for any $N \in \langle D \rangle$ and any Γ-convex subset A of X, $U[A] := \{x \in X : A \cap U[x] \neq \emptyset\}$ is Γ-convex since $(X \supset D;\ \Gamma;\ \mathscr{U})$ is an LG-space. Hence, $N \subset U[A]$ implies $\Gamma_N \subset U[A]$. Hence Y is of the Zima type.

For LG-spaces, Theorems 6.6 and 6.7 reduce to the following:

Theorem 8.5 *Let* $(X \supset D;\ \Gamma;\ \mathscr{U})$ *be an LG-space and* $T : X \multimap X$ *a u.s.c.* [*resp., an l.s.c.*] *multimap with nonempty* Γ*-convex values such that* $T(X)$ *is totally bounded. Then* T *has the almost fixed point property.*

Remarks (1) Note that X is not necessarily Hausdorff.

(2) Note that, as was shown in Theorem 6.1, if $\Gamma_N \subset D$ for each $N \in \langle D \rangle$, it is sufficient to assume that T has Γ-convex values on D, not necessarily on the whole X.

Theorem 8.6 *Let* $(X \supset D;\ \Gamma;\ \mathscr{U})$ *be an LG-space and* $T : X \multimap X$ *be a compact Kakutani map. Then* T *has a fixed point.*

Remark This is the main result of [10] and a consequence of Theorem 6.7 in view of Proposition 8.4. A very particular form of Theorem 8.5 was given by Horvath [23] for a compact metric LC-space.

The following is not comparable to Theorem 8.6:

Proposition 8.7 (Komiya [58]) *Let* $(X \supset D;\ \Gamma;\ \mathscr{U})$ *be a paracompact LG-space such that any* Φ*-map* $F : X \multimap X$ *has a fixed point. Then any Kakutani map* $G : X \multimap X$ *has a fixed point.*

In order to give another proof of Theorem 8.6 whenever the singletons are Γ-convex, we need the following:

Lemma 8.8 (Ben-El-Mechaiekh et al. [32, Proposition 3.9]) *Let* $(X \supset D;\ \Gamma)$ *be an LG-space and* Y *a compact uniform space. Then any u.s.c. multimap* $F : Y \multimap X$ *with nonempty* Γ*-convex values is an approximable map.*

From Lemma 8.8, we have the following:

Lemma 8.9 *Let* $(X \supset D;\ \Gamma;\ \mathscr{U})$ *be an LG-space. Then any Kakutani map* $F : X \multimap X$ *belongs to* $\mathfrak{B}(X, X)$.

Proof In the definition of $F \in \mathfrak{B}(X, X)$, the map $F|_{\Gamma_N}$ can be replaced by $F|_{\phi_N(\Delta_n)} : \phi_N(\Delta_n) \multimap F(\Gamma_N) \subset X$. Since $\phi_N(\Delta_n)$ is compact and $(X \supset D; \Gamma)$ is an LG-space, by Lemma 8.8, $F|_{\phi_N(\Delta_n)}$ is approximable. Then, for any continuous function $p : F(\Gamma_N) \to \Delta_n$, the composition $p \circ F \circ \phi_N : \Delta_n \multimap \Delta_n$ is a compact closed approximable map and hence has a fixed point; see [42]. Therefore, $F \in \mathfrak{B}(X, X)$.

From Lemma 8.9 and Theorem 8.3, we have Theorem 8.6 whenever the singletons are Γ-convex.

In the remainder of this section, we are concerned with C-spaces $(X; \Gamma)$; that is, each Γ_A is ω-connected for $A \in \langle X \rangle$.

Lemma 8.10 (Ben-El-Mechaiekh and Oudadess [59]) *Let* X *be a paracompact space,* $Z \subset X$ *with* $\dim_X Z \leq 0$*,* $B \subset X$ *countable,* $(Y; \Gamma)$ *a complete LC-metric space such that* $\Gamma_{\{y\}} = \{y\}$ *for all* $y \in Y$*, and* $T : X \multimap Y$ *an l.s.c. map having nonempty values such that* $T(x)$ *is closed for* $x \notin B$ *and* $\overline{T(x)}$ *is* Γ*-convex for* $x \notin Z$*. Then* T *has a continuous selection* $f : X \to Y$*; that is,* $f(x) \in T(x)$ *for all* $x \in X$*.*

For simplicity, we consider only the case $B = \emptyset$; see [59, Theorem 3] and [60, Theorem 3].

From Theorem 8.6 and Lemma 8.10, we have the following:

Theorem 8.11 *Let $(X;\Gamma)$ be a paracompact LC-space such that $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$, Y a compact LC-metric subset of X, and $Z \subset X$ with $\dim_X Z \leq 0$. Let $T : X \multimap Y$ be an l.s.c. map with nonempty closed values such that $T(x)$ is Γ-convex for $x \notin Z$. Then T has a fixed point.*

Proof By Lemma 8.10, T has a continuous selection $f : X \to Y$. Now, by applying Theorem 8.6, f has a fixed point $x_0 \in Y$; that is, $x_0 = f(x_0) \in T(x_0)$.

Remark In [3], further fixed point results for l.s.c. multimaps in LC-metric spaces are given.

9 Historical Remarks

A large number of fixed point theorems in t.v.s. and in G-convex spaces are unified and generalized in the present paper and [9, 12, 21, 61]; for the references, see mainly [4, 21, 53, 61].

(1) The celebrated Brouwer fixed point theorem in 1912 was generalized by Schauder (1930), Tychonov (1935), Hukuhara (1950), and Fan (1964). The Kakutani fixed point theorem in 1941 was generalized by Bohnenblust and Karlin (1950), Fan (1952), Glicksberg (1952), Himmelberg (1972), Granas and Liu (1986), and Park (1988). These are all for compact Kakutani maps on convex subsets of particular types of t.v.s., and particular forms of our Corollary 7.4.

(2) There are fixed point theorems for non-selfmaps (defined on convex subsets into the whole space) having certain boundary conditions (for example, the so-called Rothe condition, inwardness, outwardness, etc.).

For single-valued continuous functions, the Brouwer theorem was generalized by Knaster, Kuratowski and Mazurkiewicz (1929), Rothe (1938), Halpern (1965), Halpern and Bergman (1968), Fan (1969), Reich (1972), Sehgal and Singh (1983), Kaczynski (1983), Roux and Singh (1989), Sehgal, Singh and Whitfield (1990).

For convex-valued multimaps, upper-semicontinuity was extended to upper-demicontinuity, upper-hemicontinuity, and generalized upper-hemicontinuity. For such multimaps having certain boundary conditions, the Kakutani theorem was generalized by Browder (1968), Fan (1969, 1984), Glebov (1969), Halpern (1970), Cellina (1970), Reich (1972, 1978), Cornet (1975), Lasry and Robert (1975), Simons (1986), Shih and Tan (1987, 1988), Jiang (1988), Park (1988–93), Ding and Tan (1992), and Yuan, Smith and Lou (1998).

Those results are completely unified and generalized to a single result in Park [61].

(3) Motivated by a work of Zima (1977) on paranormed spaces (not necessarily locally convex), Rzepecki (1979) obtained a theorem for a compact continuous self-function whose range is locally convex. Later Hadžić (1981–87) introduced sets of the Zima type and obtained several fixed point theorems for compact Kakutani maps whose ranges are of the Zima type; see [24–30]. These are generalized by Corollary 5.6.

A well-known work of Idzik [33] introduced convexly totally bounded sets and generalized results mentioned in Subsection (1) and those of Zima, Rzepecki, and Hadžić. But any generalization of Idzik's result did not appear yet.

(4) The Fan–Browder fixed point theorem (1968) in [43] has numerous generalizations and applications in the KKM theory and equilibria theory. Theorems 3.5, 5.3, and others in [9] are examples of most general forms of the Fan–Browder theorem. They include Theorems 3.3–3.7 in the present paper as particular cases.

(5) Fixed point theorems for multimaps in t.v.s. were further generalized by Corollary 4.5 that first appeared in [21], which includes earlier results due to O'Neill (1957), Schaefer (1959), Nikaido (1959), Klee (1960), Powers (1970), Hahn and Pötter (1974), Krauthausen (1976), and others; see [21].

(6) Since the class $\mathfrak{B}$ contains a large number of subclasses of multimaps, Theorem 4.2 extends numerous results in Subsections (1), (5), and the first half of Subsection (3). Moreover, certain results on C-spaces due to Horvath [22–23], Hadžić [24–30] and on G-convex spaces due to Ben-El-Mechaiekh et al. [32] and others are included in Theorem 4.2.

(7) There are fixed point theorems on condensing maps defined on convex subsets of t.v.s. due to Darbo (1955), Sadovskii (1967), Lifšic and Sadovskii (1968), Himmelberg et al. (1969), Daneš (1970), Furi and Vignoli (1970), Nussbaum (1971), Reich (1971), Reinermann (1971), Mehta et al. (1997), and others. Those are also unified and extended in our previous work [12].

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