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On Finite Intersection Properties in Abstract Convex Spaces

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Abstract

We deduce some basic results in the KKM theory on abstract convex spaces mainly in the forms of the finite intersection properties of various families of subsets. In the first half, we obtain generalizations of the KKM principle [9] and the intersection theorems of Sperner [25] and Alexandroff-Pasynkoff [1]. The second half concerns with generalizations of the finite intersection properties, coincidence or fixed point theorems, and a minimax inequality in Horvath [7].

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1 Introduction

The KKM theory [13] is the study of applications of various equivalent formulations of the classical Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) [9]. It was developed first by Ky Fan [3-5,12] on convex subsets of topological vector spaces, and then by Lassonde [10] on convex spaces and by Horvath [6,7] on C-spaces. More generally, the author introduced generalized convex spaces (simply, G-convex spaces) and established various results and new applications in the fixed point theory and the KKM theory on such spaces. For the literature, see [14-21,23] and references therein.

In a sequence of papers [6,7, and references therein], Horvath established a large number of important results on his C-spaces and their applications. Many of them are restated or generalized to corresponding ones on G-convex spaces;

see [19-21,23] and their references. However, there still remain some results of Horvath which can be improved. On the other hand, in our recent work [21], from our version of the KKM theorem for G-convex spaces, we deduced generalizations of the intersection theorems of Sperner [25] and Alexandroff-Pasynkoff [1], matching theorems of Lassonde [11] and Klee [8], various forms of the Fan-Browder type fixed point theorems [2], and applications to existence of maximal elements and approximate fixed points.

In the present paper, we introduce a new concept of abstract convex spaces which include the classical convexity spaces and our generalized convex spaces. In the first half of this paper, we deduce basic results in the KKM theory on abstract convex spaces mainly in the forms of the finite intersection properties of various families of subsets. Consequently, we obtain generalizations of some results in [21] to abstract convex spaces. Those are abstract forms of the KKM theorem and theorems of Sperner [25] and Alexandroff-Pasynkoff [1]. The second half concerns with generalizations of the finite intersection properties, coincidence or fixed point theorems, and a minimax inequality in Horvath [11].

2 Some equivalent forms of the Brouwer theorem

It is well-known that the Brouwer fixed point theorem, the Sperner (combinatorial) lemma, and the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem are mutually equivalent. For the literature, see [14,21,22].

Let $\Delta_n = v_0 v_1 \cdots v_n$ be an n-simplex and $\partial\Delta_n = \bigcup_{i=0}^{n} v_0 v_1 \cdots \widehat{v_i} \cdots v_n$ its boundary, that is, the union of $(n-1)$-faces of Δ_n.

The Brouwer fixed point theorem as follows is one of the most well-known and useful theorems in topology:

Theorem B. (Brouwer) *A continuous map $f : \Delta_n \to \Delta_n$ has a fixed point $x_0 = f(x_0) \in \Delta_n$.*

The following is the combinatorial lemma of Sperner [25]:

Lemma S. (Sperner) *Let K be a simplicial subdivision of an n-simplex $v_0 v_1 \cdots v_n$. To each vertex of K, let an integer be assigned in such a way that whenever a vertex u of K lies on a face $v_{i_0} v_{i_1} \cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, the number assigned to u is one of the integers $i_0, i_1, \cdots, i_k$. Then the total number of those n-simplexes of K, whose vertices receive all $n+1$ integers $0, 1, \cdots, n$, is odd. In particular, there is at least one such n-simplex.*

This was applied to obtain the closed version of the following in [9]:

Theorem KKM. *Let $F_i\ (0 \le i \le n)$ be $n+1$ closed [resp. open] subsets of an n-simplex $v_0 v_1 \cdots v_n$. If the inclusion relation*

$$v_{i_0} v_{i_1} \cdots v_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

holds for all faces $v_{i_0}v_{i_1}\cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$*, then* $\bigcap_{i=0}^n F_i \neq \emptyset$.

For the history of generalizations and applications of the open-valued version, see Park et al. [14,18,22].

The following intersection theorem follows from the KKM theorem, see [22]:

Theorem S. (Sperner) *Let* F_i $(0 \le i \le n)$ *be* $n+1$ *nonempty closed [resp. open] sets covering an* n*-simplex* $\Delta_n = v_0v_1\cdots v_n$*. If, for each* i*,* F_i *is disjoint from the* $(n-1)$*-face* $v_0v_1\cdots \widehat{v_i}\cdots v_n$*, then* $\bigcap_{i=0}^n F_i \neq \emptyset$.

The closed version of Theorem S is due to Sperner [25] and applied to prove the invariance of dimension, and the open version is due to Stromquist [26].

Theorem AP. (Alexandroff-Pasynkoff) *Let* X_i $(0 \le i \le n)$ *be* $n+1$ *closed [resp. open] sets covering an* n*-simplex* $\Delta_n = v_0v_1\cdots v_n$ *such that* $v_0\cdots\widehat{v_i}\cdots v_n \subset X_i$ *for each* i*. Then* $\bigcap_{i=0}^n X_i \neq \emptyset$.

The closed version of Theorem AP is due to Alexandroff and Pasynkoff [1] and applied to the essentiality of the identity map of the boundary of a simplex, and the open version is noted by Lassonde [11]; see also [22].

3 The KKM type theorems in abstract convex spaces

It was known that the KKM theorem holds for topological spaces with certain abstract convexity without any linear structure. One of the most general spaces having this property seems to be the generalized convex spaces due to the author.

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of a set Y; that is, a function with nonempty values $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$.

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

We introduce the following:

Definitions. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if, for any $A \in \langle X \cap D\rangle$, we have $\Gamma_A \subset X$; and for any subset $X \subset E$, the Γ-*convex hull* of X is defined as follows:

$$\Gamma\text{-co}\, X := \bigcap\{Z \subset E \mid Z \text{ is a } \Gamma\text{-convex subset of } E \text{ containing } X\}.$$

It is easily seen that $\Gamma\text{-co}\, X = \bigcup\{\Gamma\text{-co}\, N \mid N \in \langle X\rangle\}$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

An abstract convex space with any topology is called *an abstract convex topological space.*

If the reader prefers, abstract convex spaces can be called A-convex spaces.

Examples 1. Usually, a *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see [24], where the bibliography lists 283 papers. For any subset $X \subset E$, its $\mathcal{C}$-*convex hull* is defined and denoted by $\mathrm{Co}_{\mathcal{C}} X := \bigcap\{Y \in \mathcal{C} \mid X \subset Y\}$. We say that X is $\mathcal{C}$-*convex* if $X{=}\mathrm{Co}_{\mathcal{C}} X$. Now we can consider the map $\Gamma : \langle E \rangle \multimap E$ given by $\Gamma_A{:=}\mathrm{Co}_{\mathcal{C}} A$ for each $A \in \langle E \rangle$. Then $(E, \mathcal{C})$ becomes our abstract convex space $(E; \Gamma)$.

2. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J := \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n an n-simplex with vertices $v_0, v_1, \cdots, v_n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$. For details on G-convex spaces, see [14-21], where basic theory was extensively developed.

It is possible to assume $\Gamma_J = \phi_A(\Delta_J)$. In case to emphasize $X \supset D$, a G-convex space $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$.

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$.

3. If $X = D$ and each Γ_A is assumed to be contractible or, more generally, ω-connected (that is, n-connected for all $n \geq 0$) and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then a G-convex space $(X; \Gamma)$ becomes a C-*space* (or an H-*space*) due to Horvath [6,7].

4. If $X = D$ is a convex subset of a vector space and each Γ_A is the convex hull of $A \in \langle X \rangle$ equipped with the Euclidean topology, then a G-convex space $(X; \Gamma)$ becomes a *convex space* in the sense of Lassonde [10]. Note that any convex subset of a topological vector space is a convex space, but not conversely.

Definitions. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called *a* $\mathfrak{K}$-*map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we have

$$\mathfrak{K}(E,Z) \subset \mathfrak{KC}(E,Z) \cap \mathfrak{KO}(E,Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation $\mathrm{KKM}(E,Z)$ instead of $\mathfrak{KC}(E,Z)$.

The following is a simple observation:

Proposition 1. Let $(E,D;\Gamma)$ be an abstract convex space, Z a set, and $F : E \multimap Z$ a multimap. Then $F \in \mathfrak{K}(E,Z)$ if and only if for any multimap $G : D \multimap Z$ satisfying

(1.1) $F(\Gamma_N) \subset G(N)$ for any $N \in \langle D \rangle$,

we have $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.

Proof. For the necessity, from (1.1), for any $N \in \langle D \rangle$, we have $F(\Gamma_N) \subset F(E) \cap G(N) = \bigcup_{y \in N}\{F(E) \cap G(y)\}$. Since F is a $\mathfrak{K}$-map, the family $\{F(E) \cap G(y)\}_{y \in D}$ has the finite intersection property. The sufficiency is clear. □

Remark. If Z has any topology and if $F \in \mathfrak{KO}(E,Z)$ [resp., $F \in \mathfrak{KC}(E,Z)$], then we have to assume G is open-valued [resp. closed-valued].

For an abstract convex topological space, the following recovers the meaning of $1_E \in \mathfrak{KC}(E,E)$ [resp. $1_E \in \mathfrak{KO}(E,E)$]:

Corollary 1. *Let* $(E,D;\Gamma)$ *be an abstract convex topological space. Then the identity map* 1_E *belongs to* $\mathfrak{KC}(E,E)$ [*resp.* $1_E \in \mathfrak{KO}(E,E)$] *if and only if for any multimap* $G : D \multimap E$ *satisfying*

(1) *G has closed* [*resp. open*] *values, and*

(2) *G is a KKM map,*

$\{G(y)\}_{y \in D}$ *has the finite intersection property.*

Under an additional requirement, we have the whole intersection property for the map-values of a KKM map:

Corollary 2. *Let* $(E,D;\Gamma)$ *be an abstract convex topological space, the identity map* $1_E \in \mathfrak{KC}(E,E)$ [*resp.* $1_E \in \mathfrak{KO}(E,E)$], *and* $G : D \multimap E$ *a multimap satisfying* (1), (2) *in Corollary 1, and*

(3) $\bigcap_{z \in M} \overline{G(z)}$ *is compact for some* $M \in \langle D \rangle$.

Then we have

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Recall that $1_X \in \mathfrak{KC}(X,X) \cap \mathfrak{KO}(X,X)$ in a G-convex space $(X,D;\Gamma)$. This is due to Park [18,20,21] where a number of applications in various fields were given. Moreover, we have the following result for G-convex spaces in [23]:

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space and $G : D \multimap X$ a KKM map with closed [resp. open] values. Then $\{G(a)\}_{a \in D}$ has the finite intersection property. More precisely, for any $A \in \langle D \rangle$, we have*

$$\Gamma_A \cap \bigcap_{a \in A} F(a) \neq \emptyset.$$

Note that the first part of Theorem 1 follows from Corollary 2. But the precise part is not.

Recall that, at first, a G-convex space was defined under the additional isotonicity condition:

$$(*) \qquad \text{if } M, N \in \langle D \rangle \text{ and } M \subset N, \text{ then } \Gamma_M \subset \Gamma_N.$$

Condition $(*)$ holds for convex spaces due to Lassonde [10] or C-spaces due to Horvath [6,7], but not for G-convex spaces in general. Later, it is known that this restriction was superfluous (see [15-21]) in most applications.

However, for any G-convex space $(X, D; \Gamma)$, when $D = A$ is finite, by putting $\Gamma_J = \phi_A(\Delta_J)$, we may assume the isotonicity condition $(*)$.

From Proposition 1, we deduce generalizations of Theorems S and AP for abstract convex spaces, resp., as follows:

Theorem 2. *Let $(E, D; \Gamma)$ be an abstract convex topological space with $D = \{a_0, a_1, \cdots, a_n\}$ and $S : D \multimap X$ a multimap with nonempty closed [resp. open] values such that*

(i) $E = S(D)$ *and*

(ii) *for each* $i,\ 0 \leq i \leq n,\ S(a_i)$ *is disjoint from* $\Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_n\})$.

If the identity map $1_E \in \mathfrak{KC}(E, E)$ *[resp.* $1_E \in \mathfrak{KO}(E, E)$*], then* $\bigcap_{i=0}^n S(a_i) \neq \emptyset$.

Proof. It suffices to show that S is a KKM map. Let $N \in \langle D \rangle$. If $N = D$, then $\Gamma_N \subset E = S(N)$ by (i). Suppose that $N \subsetneq D$. Then there exists an index $j,\ 0 \leq j \leq n$, such that $a_j \notin N$. By (ii) and condition $(*)$, we have

$$S(a_j) \cap \Gamma_N \subset S(a_j) \cap \Gamma(\{a_0, \cdots, \widehat{a_j}, \cdots, a_n\}) = \emptyset.$$

However, we have $\Gamma_N \subset E = S(D) = \bigcup_{i=0}^n S(a_i)$ and hence

$$\Gamma_N \subset \bigcup \{S(a_i) \mid a_i \in N\} = S(N).$$

Now the conclusion follows from Corollary 1. □

In [21], from the KKM Theorem 1, we deduced the following generalization of Theorem S for G-convex spaces:

Corollary 3. *Let $(X, D; \Gamma)$ be a G-convex space with $D = \{a_0, a_1, \cdots, a_n\}$ and $S : D \multimap X$ a multimap with nonempty closed [resp. open] values such that*

(i) $X = S(D)$ *and*

(ii) *for each* $i,\ 0 \leq i \leq n,\ F(a_i)$ *is disjoint from* $\Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_n\})$.

Then $\bigcap_{i=0}^n S(a_i) \neq \emptyset$.

For $D = \{a_0, a_1, \cdots, a_n\}$, we denote as follows:

$$D_0 := \{a_0, \cdots, a_{n-1}\},$$
$$D_i := \{a_0, \cdots, \widehat{a_{i-1}}, \cdots, a_n\}$$

for $1 \leq i \leq n$.

Theorem 3. *Let $(E, D; \Gamma)$ be an abstract convex topological space with $D = \{a_0, a_1, \cdots, a_n\}$ and $T : D \multimap E$ a multimap with nonempty closed [resp. open] values such that*
(i) $E = T(D)$ *and*
(ii) $\Gamma_{D_i} \subset T(a_i)$ *for* $0 \leq i \leq n$.
If the identity map $1_E \in \mathfrak{KC}(E, E)$ [resp. $1_E \in \mathfrak{KO}(E, E)$], then $\bigcap_{i=0}^n T(a_i) \neq \emptyset$.

Proof. We show that T is a KKM map. Let $N \in \langle D \rangle$. If $N = D$, then $\Gamma_N \subset E = T(N)$ by (i). Suppose that $N \subsetneq D$. Then, by (ii) and condition $(*)$,

$$\Gamma_N \subset \Gamma_{D_i} \subset T(a_i) \text{ for some } a_i \in N,$$

and hence

$$\Gamma_N \subset \bigcup \{T(a_i) \mid a_i \in N\} = T(N).$$

Now the conclusion follows from Corollary 1. □

In [21], from the KKM Theorem 1, we deduced the following generalization of Theorem AP for G-convex spaces:

Corollary 4. *Let $(X, D; \Gamma)$ be a G-convex space with $D = \{a_0, a_1, \cdots, a_n\}$ and $T : D \multimap X$ a multimap with nonempty closed [resp. open] values such that*
(i) $X = T(D)$ *and*
(ii) $\Gamma_{D_i} \subset T(a_i)$ *for* $0 \leq i \leq n$.
Then $\bigcap_{i=0}^n T(a_i) \neq \emptyset$.

Ky Fan [3,5] noted that each of Theorems S and AP can be easily derived from the other. We show that this can be done for Theorems 2 and 3 as follows:

Theorem 2 $\Longrightarrow$ Theorem 3. Suppose that $\bigcap_{i=0}^n T(a_i) \neq \emptyset$. Let $S : D \multimap E$ be a map with open [resp. closed] values defined by

$$S(a_i) := E \setminus T(a_{i+1}) = T(a_{i+1})^c \text{ for } 0 \leq i < n,$$
$$S(a_n) := E \setminus T(a_0) = T(a_0)^c.$$

Then we have

$$\bigcup_{i=0}^n S(a_i) = \bigcup_{i=0}^n T(a_i)^c = [\bigcap_{i=0}^n T(a_i)]^c = E$$

and hence condition (i) of Theorem 2 holds. Moreover, since

$$\Gamma_{D_{i+1}} = \Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_n\}) \subset T(a_{i+1}) \quad \text{for } 0 \leq i < n$$

and

$$\Gamma_{D_0} = \Gamma(\{a_0, \cdots, a_{n-1}, \widehat{a_n}\}) \subset T(a_0),$$

we have

$$F(a_i) \cap \Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_n\}) = \emptyset \quad \text{for } 0 \leq i \leq n,$$

and hence condition (ii) of Theorem 2 holds. Therefore,

$$\bigcap_{i=0}^{n} S(a_i) = \bigcap_{i=0}^{n} T(a_i)^c = [\bigcup_{i=0}^{n} T(a_i)]^c = [T(D)]^c = \emptyset$$

by (i) of Theorem 3. This contradicts Theorem 2.

Theorem 3 $\Longrightarrow$ *Theorem 2.* Similarly we can prove.

4 The Horvath type finite intersection property

In this section, by applying Theorems 2 and 3, we generalize results in Section 1 of Horvath [7] on the finite intersection property.

We use the following notation: If $N \in \mathbb{N}$, then $\langle N^+ \rangle$ is the family of nonempty subsets of $N^+ := \{0, 1, \cdots, N\}$ and we denote $N + 1 = 0$.

Theorem 4. *Let X be an ω-connected space, $\{M_i\}_{i=0}^N$ a closed [resp. open] cover of X, and $\{F_i\}_{i=0}^N$ a family of ω-connected subspaces of X such that*

(1) *for each $i \in N^+$, $F_i \cap M_i = \emptyset$; and*

(2) *for each $J \in \langle N^+ \rangle$ with $|J| \leq N$, $\bigcap_{i \in J} F_i$ is not empty and ω-connected.*

Then $\bigcap_{i=0}^N M_i \neq \emptyset$.

Proof. Define $\Gamma : \langle N^+ \rangle \multimap X$ by

$$\Gamma_J = \bigcap_{i \in J} F_i \quad \text{for } J \in \langle N^+ \rangle.$$

Then $(X, N^+; \Gamma)$ becomes a G-convex space. We apply Corollary 3 with $D = N^+$, $a_i = i$, $F(a_i) = M_{i+1}$ for each $i \in N^+$ with $N + 1 = 0$. Then

$$\Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_N\}) := F_0 \cap \cdots \cap \widehat{F_i} \cap \cdots \cap F_N \subset F_{i+1}.$$

Since $F_i \cap M_i = \emptyset$ for each $i \in N^+$ by (1), we have $F_{i+1} \cap F(a_i) = \emptyset$ for each $i \in N^+$ with $N + 1 = 0$. Hence, for each $i \in N^+$, $F(a_i)$ is disjoint from $\Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_N\})$. Therefore, by Corollary 3, $\bigcap_{i=0}^N F(a_i) \neq \emptyset$ and hence $\bigcap_{i=0}^N M_i \neq \emptyset$. □

Remarks. 1. Horvath [7, Section 1, Corollary 1] is the closed case of Theorem 4 for contractible sets.

2. Note that condition (1) can be replaced by the following weaker condition: (1)′ for all $i \in N^+$ with $N+1=0$, $M_i \cap \bigcap_{j \neq i+1} F_j = \emptyset$.

3. Note that Theorem 4 generalizes Theorem S. In fact, let $N := n, X := \Delta_n, M_i := F_i$, and $F_i := v_0 \cdots \widehat{v_i} \cdots v_n$ for each i. Then condition (1) is assumed. Moreover, for each J as in (2), $\bigcap_{i \in J} F_i$ is convex and contains a vertex v_k for some $k \in N^+ \setminus J$. Hence condition (2) holds. Therefore, Theorem S follows from Theorem 4.

Theorem 5. *Let X be an ω-connected space, $\{M_i\}_{i=0}^N$ a closed [resp. open] cover of X, and $\{F_i\}_{i=0}^N$ a family of ω-connected subspaces of X such that*
(1) *for each $i \in N^+$, $F_i \subset M_i$; and*
(2) *for each $J \in \langle N^+ \rangle$ with $|J| \leq N$, $\bigcap_{i \in J} F_i$ is not empty and ω-connected.*
Then $\bigcap_{i=0}^N M_i \neq \emptyset$.

Proof. If $J \in \langle N^+ \rangle$ with $|J| \leq N$, let $\Gamma_J = \bigcap_{i \in J} F_i$; and if $J = N^+$, let $\Gamma_J := X$. Then $(X, N^+; \Gamma)$ is a C-space and hence a G-convex space. We apply Corollary 4 with $D = N^+,\ a_i = i,\ T(a_i) = M_i$ for each $i \in N^+$ with $N+1=0$. Then
(i) $X = T(D)$ since $\{M_i\}$ covers X.
(ii) For each $j \in N^+$ and $J = D_j = \{a_0, \cdots, \widehat{a_{j-1}}, \cdots, a_N\}$, we have $\Gamma_{D_j} = \bigcap_{i \neq j-1} F_i \subset F_j \subset M_j$ by (1). Note that $\Gamma_{D_0} = \bigcap_{i=0}^{N-1} F_i \subset F_0 \subset M_0$.
Therefore, by Corollary 4, $\bigcap_{i=0}^N T(a_i) \neq \emptyset$ and $\bigcap_{i=0}^N M_i \neq \emptyset$. □

Remarks 1. Horvath [7, Section 1, Proposition 2] obtained the closed cover case of Theorem 5 under the additional restriction that X is normal.

2. Note that condition (1) can be replaced by the following weaker condition: (1)′ $F_0 \cap \cdots \cap \widehat{F_{i-1}} \cap \cdots \cap F_N \subset M_i$ for each $i \in N^+$.

3. Note that Theorem 5 extends Theorem AP. In fact, let $N := n, X := \Delta_n, M_i := X_i$, and $F_i := v_0 \cdots \widehat{v_i} \cdots v_n \subset X_i$ for each i. Then Theorem AP follows from Theorem 5.

Corollary 5. *Let X be a topological space, and $\{F_i\}_{i=0}^N$ a family of closed [resp. open] ω-connected subspaces of X such that for each $J \in \langle N^+ \rangle$,*
(1) *$\bigcup_{i \in J} F_i$ is ω-connected; and*
(2) *$\bigcap_{i \in J} F_i$ is empty or ω-connected.*
Then $\bigcap_{i=0}^N F_i \neq \emptyset$.

Proof. We use induction on $|J|$. If $|J| = 1$, then the intersection is nonempty by hypothesis. If $|J| = k+1$, let $X_J := \bigcup_{i \in J} F_i$. Then X_J is an ω-connected subspace of X. Note that $\{F_i\}_{i \in J}$ is a closed [resp. open] cover of X_J such that $\bigcap_{i \in J'} F_i \neq \emptyset$ if $J' \subsetneq J$. Now apply Theorem 5 to conclude that $\bigcap_{i \in J} F_i \neq \emptyset$. □

Remark. Horvath [7, Section 1, Corollary 2] obtained the closed case of Corollary 5 under the additional restriction that X is normal.

Corollary 6. *Let V be a vector space with the finite topology, and $\{C_i\}_{i=0}^N$ a family of closed [resp. open] convex subsets of V. If*

(1) *for each $i \in N^+$, $\bigcap_{j \neq i} C_j \neq \emptyset$; and*

(2) *$\bigcup_{i=0}^N C_i$ is convex.*

Then $\bigcap_{i=0}^N C_i \neq \emptyset$.

Proof. For each $i \in N^+$, choose $x_i \in \bigcap_{j \neq i} C_j$, and let $X := \mathrm{co}\,\{x_0, x_1, \cdots, x_N\}$ and $F_i = M_i = X \cap C_i$ and now apply Theorem 5. □

Remark. Corollary 6.2 is due to Klee [8] and also known as the Berge intersection theorem; see [7].

From Corollary 5, we have the following:

Corollary 7. *Let V be a topological vector space and $\{C_i\}_{i=0}^N$ a family of nonempty closed [resp. open] convex subsets of V. If for each $J \in \langle N^+ \rangle$, $\bigcup_{i \in J} C_i$ is ω-connected, then $\bigcap_{i=0}^N C_i \neq \emptyset$.*

Remark. Horvath [7, Section 1, Corollary 4] obtained the closed case of Corollary 7 under the additional restriction that V is normal.

5 Some Horvath type fixed point theorems

In this section, we generalize some (not all) fixed point theorems in Section 4 of Horvath [7].

We follow the notations in [7]: Let $A : X \multimap Y$ be a multimap between sets. Define the map $A^* : Y \multimap X$ by $A^*(y) := X \setminus A^-(y)$. Obviously $A^{**} = A$; $x \in A^*(y)$ if and only if $y \notin A(x)$; and if $B : X \multimap Y$ is another map such that $B \subset A$ then $A^* \subset B^*$.

For $Y = D$, the following reduces to Horvath's amusing lemma [7, Section 4, Lemma 1]:

Lemma 1. *Let Y be a set, $D \subset Y$, $\alpha : \langle D \rangle \multimap Y$, and $A : Y \multimap Y$ a map such that*

(1) *for each $y \in Y$, $y \in A(y)$; and*

(2) *for each $y \in Y$ and $J \in \langle D \rangle$, if $J \subset A^*(y)$ then $\alpha(J) \subset A^*(y)$.*

Then for each $J \in \langle D \rangle$, $\alpha(J) \subset A(J)$.

Proof. Suppose $u \notin A(J)$. Then, for each $y \in J$,

$$u \notin A(y) \Rightarrow y \notin A^-(u) \Rightarrow y \in Y \setminus A^-(u) = A^*(u).$$

Hence, by (2), $\alpha(J) \subset A^*(u)$. On the other hand, since $u \in A(u)$ by (1), we have $u \notin A^*(u)$. Therefore $u \notin \alpha(J)$. □

Lemma 2. *Let Y be a set, $D \subset Y$, $\alpha : \langle D \rangle \multimap Y$, and $B : Y \multimap Y$ a map such that*

(1) *there exists a $J_0 \in \langle D \rangle$ such that $\alpha(J_0) \cap \bigcap_{y \in J_0} B^-(y) \neq \emptyset$; and*

(2) *for each $y \in Y$ and $J \in \langle D \rangle$, if $J \subset B(y)$ then $\alpha(J) \subset B(y)$.*

Then there exists a $\overline{y} \in Y$ such that $\overline{y} \in B(\overline{y})$.

Proof. Let $A^* = B$ or $A = B^*$. Then the negation of the conclusion of Lemma 1 is condition (1). Since condition (2) of Lemmas 1 and 2 are identical, we obtain the negation of condition (1) of Lemma 1. Hence $\overline{y} \notin A(\overline{y})$ for some $\overline{y} \in Y$. Hence $\overline{y} \in A^*(\overline{y}) = B(\overline{y})$. □

Theorem 6. *Let $(E \supset D; \Gamma)$ be an abstract convex topological space and $R, S : E \multimap E$ two maps such that*

(1) *for each $y \in E$, $R(y)$ is closed [resp. open] and $y \in S(y) \subset R(y)$;*

(2) *for each $y \in E$, $S^*(y)$ is Γ-convex; and*

(3) *$\overline{R(y_0)}$ is compact for some $y_0 \in D$.*

If the identity map 1_E belongs to $\mathfrak{KC}(E, E)$ [resp. $1_E \in \mathfrak{KO}(E, E)$], then $\bigcap_{y \in D} \overline{R(y)} \neq \emptyset$.

Proof. By Lemma 1, $\Gamma_J \subset S(J)$ for each $J \in \langle D \rangle$ and hence S and R are KKM maps. Therefore, by Corollary 2, the family $\{\overline{R(y)} \mid y \in D\}$ has the finite intersection property. Since $\overline{R(y_0)}$ is compact for some $y_0 \in D$, the conclusion follows. □

Remarks 1. The reader might prefer to assume that each $R(y)$ is compactly closed, but this does not generalize anything; see [17].

2. Theorem 6 generalizes Horvath [7, Section 4, Theorem 2].

By taking $A = R^*$ and $B = S^*$ we obtain at once the following fixed point theorem:

Theorem 7. *Let $(E \supset D; \Gamma)$ be an abstract convex topological space and $A, B : Y \multimap Y$ two maps such that*

(1) *for each $y \in E$, $A^-(y)$ is open [resp. closed], $A(y) \neq \emptyset$ and $A(y) \subset B(y)$;*

(2) *for each $y \in E$, $B(y)$ is Γ-convex; and*

(3) *$\overline{E \setminus A^-(y_0)}$ is compact for some $y_0 \in D$.*

If the identity map 1_E belongs to $\mathfrak{KC}(E, E)$ [resp. $1_E \in \mathfrak{KO}(E, E)$], then there is a $\overline{y} \in E$ such that $\overline{y} \in B(\overline{y})$.

6 A Ky Fan type minimax inequality

From Theorem 6, we obtain the following form of the minimax inequality of Ky Fan as in Horvath [7, Section 5, Proposition 1].

Theorem 8. *Let $(E \supset D; \Gamma)$ be an abstract convex topological space and $f, g : E \times E \to \mathbb{R}$ two functions such that*

(1) *$f \leq g$;*

(2) *for each* $y \in E$, $x \mapsto g(x,y)$ *is l.s.c.;*
(3) *for each* $x \in E$, $\{y \in E \mid f(x,y) > 0\}$ *is* Γ*-convex; and*
(4) $\{x \in E \mid g(x,y_0) \leq 0\}$ *is compact for some* $y_0 \in D$.

If the identity map 1_E *belongs to* $\mathfrak{KC}(E,E)$ *[resp.* $1_E \in \mathfrak{KO}(E,E)$*], then the following alternative holds:*
(1) *there is an* $x_0 \in E$ *such that* $\sup_{y \in E} g(x_0,y) \leq 0$*; or*
(2) *there is a* $y_0 \in E$ *such that* $f(y_0,y_0) > 0$.

Proof. Use Theorem 6 with

$$R(y) := \{x \in E \mid g(x,y) \leq 0\}$$

and

$$S(y) := \{x \in E \mid f(x,y) \leq 0\}.$$

Then the conclusion follows. □

Remark. Theorem 8 works for G-convex spaces.

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