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GENERALIZED KKM MAPS, MAXIMAL ELEMENTS AND ALMOST FIXED POINTS

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ABSTRACT. In the framework of generalized convex spaces, we show that generalized KKM maps can be regarded as ordinary KKM maps, and obtain some applications to equilibrium result, maximal element theorems, and almost fixed point theorems on multimaps of the Zima type.

1. Introduction

The KKM theory is the study of applications of various equivalent formulations of the classical Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM principle). At the beginning, the theory was mainly concerned with convex subsets of topological vector spaces. Later, it has been extended to convex spaces by Lassonde [15], and to spaces having certain families of contractible subsets (simply, C -spaces or H -spaces) by Horvath [7, 8]. Moreover, generalized notions of KKM maps were introduced by Kassay and Kolumbán [10] and Chang and Zhang [2] for convex spaces, and by Chang and Ma [1] and first author [11] for H -spaces.

On the other hand, in [17-20, 22-26], the second author introduced the concept of generalized convex (simply, G -convex) spaces as a common generalization of those of the usual convexity in a topological vector space, the C -spaces, and many abstract convexities which have been developed in connection with the fixed point theory and the KKM theory. Recently, there have appeared a large number of papers devoted mainly to the study on various equilibrium problems related G -convex spaces.

Our aim in this paper is to show that the generalized KKM maps can be viewed as KKM maps and to give some applications of them to certain equilibrium problems. In fact, using our basic results, we obtain an equilibrium theorem, maximal element theorems, and general fixed point theorems on multimaps of the Zima type.

In Section 3, we show that generalized KKM maps can be viewed as KKM maps on G -convex spaces. From this and the KKM theorem, we deduce that the

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set of functional values of a generalized KKM map has the finite intersection property. Section 4 deals with the relation between generalized KKM maps and certain convexities of corresponding extended real-valued functions. This is applied to deduce an equilibrium theorem. In Section 5, we use our result to generalize Tian's theorems on existence of maximal elements. In Section 6, we show that the notion of multimaps of the Zima type due to Hadžić [4, 5] can be extended to G -convex spaces, and obtain (almost) fixed point theorems on such type of multimaps.

2. Preliminaries

A *multimap* (or *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y ; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{F(x) \mid x \in A\}$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D .

A *generalized convex space* or a *G -convex space* $(X, D; \Gamma)$ consists of a topological space X , a nonempty set D , and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $N = \{z_0, z_1, \dots, z_n\} \in \langle D \rangle$, there exist a subset $\Gamma(N) = \Gamma_N$ of X and a continuous function $\phi_N : \Delta_n \rightarrow \Gamma(N)$ such that $J \subset \{0, 1, \dots, n\}$ implies $\phi_N(\Delta_J) \subset \Gamma(\{z_j \mid j \in J\})$, where $\Delta_n = \text{co}\{v_0, v_1, \dots, v_n\}$ is a standard n -simplex and $\Delta_J = \text{co}\{v_j \mid j \in J\}$. It is possible to assume $\phi_N(\Delta_n) = \Gamma_N$ for each $N \in \langle D \rangle$.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$ and, if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$. For a G -convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ -convex if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$. For details on G -convex spaces and examples, see [17-20, 22-26], where basic theory was extensively developed.

A well-known subclass of G -convex spaces due to Horvath [7, 8] can be generalized as follows:

A triple $(X, D; \Gamma)$ is called a *C -space* (or an *H -space*) if X is a topological space, D a nonempty subset of X , and $\Gamma : \langle D \rangle \multimap X$ a multimap such that $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D \rangle$ and each Γ_A is ω -connected (that is, n -connected for all $n \geq 0$).

For a G -convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$.

Let $\overline{\quad}$ and Int denote the closure and the interior, respectively.

The following KKM theorem for G -convex spaces is due to second author [19, 20]:

Theorem 1. *Let $(X, D; \Gamma)$ be a G -convex space and $F : D \multimap X$ a multimap such that*

- (1.1) *F has closed [resp. open] values; and*
- (1.2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property (More precisely, for each $N \in \langle D \rangle$, we have $\Gamma_N \cap \bigcap_{z \in N} F(z) \neq \emptyset$).

Further, if

$$(1.3) \quad \bigcap_{z \in M} \overline{F(z)} \text{ is compact for some } M \in \langle D \rangle,$$

then we have $\bigcap_{z \in D} \overline{F(z)} \neq \emptyset$.

3. Relation between Generalized KKM maps and KKM maps

Motivated by Kassay and Kolumbán [10], Park and Lee [27] defined generalized KKM maps on G -convex spaces as follows:

Let $(X, D; \Gamma)$ be a G -convex space and I a nonempty set. A map $F : I \multimap X$ is called a *generalized KKM map* provided that for each $N \in \langle I \rangle$, there exists a function $\sigma : N \rightarrow D$ such that $\Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle N \rangle$.

Examples. (1) A generalized KKM map $F : I \multimap X$ reduces to a KKM map if $I = D$ and σ is chosen to be the identity function 1_N for each $N \in \langle D \rangle$.

(2) A counter-example to the converse is given in [2] as follows: Let $E = \overline{\mathbb{R}}$, $X = [-2, 2]$, and $G : X \multimap E$ be defined by $G(x) = [-(1 + x^2/5), 1 + x^2/5]$ for each $x \in X$. Since $\bigcup_{x \in X} G(x) = [-9/5, 9/5]$, we have $x \notin G(x)$ for each $x \in [-2, -9/5) \cup (9/5, 2]$. This shows that G is not a KKM map from X into 2^E . But G is a generalized KKM map. In fact, if for any $N \in \langle X \rangle$, we take $\sigma : N \rightarrow [-1, 1]$ by $\sigma(x) = x/2$ for $x \in N$. Then for any $M \in \langle N \rangle$ we have $\text{co } \sigma(M) \subset [-1, 1] \subset G(M)$.

However, note that the above generalized KKM map can be viewed as a KKM map of a generalized convex space $(X; \Gamma)$ where $\Gamma(N) := \text{co } \sigma(N)$ for $N \in \langle X \rangle$.

The following theorem shows that every generalized KKM map can be regarded as a KKM map of a generalized convex space.

Theorem 2. *Let I be an index set and $(X, D; \Gamma)$ a G -convex space. If a map $F : I \multimap X$ is generalized KKM, then F is a KKM map on a G -convex space $(X, I; \Gamma^F)$ where the map $\Gamma^F : \langle I \rangle \multimap X$ is defined by $\Gamma_J^F = \Gamma^F(J) := \bigcup \{ \Gamma_{\sigma(J)} \mid \sigma : J \rightarrow D \text{ is a function satisfying } \Gamma_{\sigma(M)} \subset F(M) \text{ for } M \in \langle J \rangle \}$ for each $J \in \langle I \rangle$.*

Proof. For each $J = \{i_0, i_1, \dots, i_n\} \in \langle I \rangle$, there exists a function $\sigma : J \rightarrow D$ such that for any $M = \{i_{j_0}, \dots, i_{j_k}\} \subset J$, we have $\Gamma_{\sigma(M)} \subset F(M)$. Since $\sigma(i_j)$'s are not necessarily mutually distinct, for each $j = 0, 1, \dots, n$, we may write $\sigma(i_j) = z_{\tilde{\sigma}(j)} \in D$ where the cardinality of $\{\tilde{\sigma}(j) \mid j = 0, 1, \dots, n\}$ is $l + 1$. Let $\Delta_n = \{ \sum_{i=0}^n \gamma_i v_i \mid \gamma_i \geq 0, \sum_{i=0}^n \gamma_i = 1 \}$ and $\Delta_l = \{ \sum_{j=0}^l \alpha_j v_j \mid \alpha_j \geq 0, \sum_{j=0}^l \alpha_j = 1 \}$. Define a function $g : \Delta_n \rightarrow \Delta_l$ by $g(\gamma_0, \dots, \gamma_n) = (\alpha_0, \dots, \alpha_l)$ where $\alpha_j = \sum_{i \in \tilde{\sigma}^{-1}(j)} \gamma_i$. Then g is well-defined and continuous. Note that $g(\Delta_M) = \Delta_{\sigma(M)}$ where $\Delta_M = \text{co}\{v_{j_0}, \dots, v_{j_k}\}$ and $\Delta_{\sigma(M)} = \text{co}\{v_{\tilde{\sigma}(j_0)}, \dots, v_{\tilde{\sigma}(j_k)}\}$.

On the other hand, by the definition of a G -convex space, there exists an $f \in \mathbb{C}(\Delta_I, \Gamma_{\sigma(J)})$ such that $\sigma(M) = \{z_{\bar{\sigma}(j_0)}, \dots, z_{\bar{\sigma}(j_k)}\} \subset \sigma(J)$ implies $f(\Delta_{\sigma(M)}) \subset \Gamma_{\sigma(M)}$. Define $\phi_J : \Delta_n \rightarrow \Gamma_J^F$ by $\phi_J(v) = f \circ g(v)$ for all $v \in \Delta_n$. Since $\Gamma_{\sigma(J)} \subset \Gamma_J^F$, ϕ_J is well defined and continuous, and $M \subset J$ implies $\phi_J(\Delta_M) = f \circ g(\Delta_M) = f(\Delta_{\sigma(M)}) \subset \Gamma_{\sigma(M)} \subset \Gamma_M^F$. Therefore $(X, I; \Gamma^F)$ is a G -convex space.

Clearly, for each $J \in \langle I \rangle$, we have $\Gamma_J^F \subset F(J)$ and hence F is KKM on $(X, I; \Gamma^F)$. □

Remarks. (1) If we define the map $\Gamma^F : \langle I \rangle \rightarrow X$ by $\Gamma_M^F = \Gamma^F(M) := F(M)$ for each $M \in \langle I \rangle$, then F is a KKM map on a G -convex space $(X, I; \Gamma^F)$, also.

(2) If F is a KKM map on a G -convex space $(X, D; \Gamma)$, then $(X, I; \Gamma^F)$ is a G -convex space where the map $\Gamma^F : \langle I \rangle \rightarrow X$ is defined by $\Gamma_M^F = \Gamma^F(M) := F(M)$ for each $M \in \langle I \rangle$.

From Theorems 1 and 2, we obtain the finite intersection property for functional values of a generalized KKM map:

Theorem 3. *Let $(X, D; \Gamma)$ be a G -convex space, I a nonempty set, and $F : I \rightarrow X$ a map with closed [resp. open] values. If F is a generalized KKM map, then the family of its values has the finite intersection property (More precisely, for each $N \in \langle I \rangle$, there exists an $N' \in \langle D \rangle$ such that $\Gamma_{N'} \cap \bigcap_{z \in N} F(z) \neq \emptyset$).*

Remark. The converse holds whenever $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$; see [27, Theorem 2].

From Theorem 3, we obtain the following KKM type theorem for generalized KKM maps with closed values:

Theorem 4. *Let I be a set, $(X, D; \Gamma)$ a G -convex space, and $F : I \rightarrow X$ a map. Suppose that there exists a nonempty compact subset K of X such that*

(4.1) $\bigcap_{z \in I} F(z) = \bigcap_{z \in I} \overline{F(z)}$ [that is, F is transfer closed-valued];

(4.2) \overline{F} is a generalized KKM map; and

(4.3) either

(i) $\bigcap_{z \in M} \overline{F(z)} \subset K$ for some $M \in \langle I \rangle$; or

(ii) if $X \supset D$ and, for each $J \in \langle I \rangle$ and each function $\sigma : J \rightarrow D$, there exists a Γ -convex subset L_N of X containing $N = \sigma(J)$ such that $L_N \cap \bigcap_{z \in J} \overline{F(z)} \subset K$.

Then $K \cap \bigcap_{z \in I} F(z) \neq \emptyset$.

Proof. Case (i). Since $\overline{F} : I \rightarrow X$ is closed-valued, $\{\overline{F(z)}\}_{z \in I}$ has the finite intersection property by Theorem 3. Since $\bigcap_{z \in M} \overline{F(z)}$ is compact, so we have $\bigcap_{z \in I} \overline{F(z)} \neq \emptyset$. Therefore, (4.1) ensures the conclusion.

Case (ii). Suppose that $K \cap \bigcap_{z \in I} \overline{F(z)} = \emptyset$; that is, $K \subset \bigcup_{z \in I} (X \setminus \overline{F(z)})$. Since each $X \setminus \overline{F(z)}$ is open and K is compact, there exists a $J \in \langle I \rangle$ such that $K \subset \bigcup_{z \in J} (X \setminus \overline{F(z)})$. Since $\overline{F} : I \rightarrow X$ is generalized KKM, there exists

a function $\sigma : J \rightarrow D$ such that $\Gamma_{\sigma(M)} \subset \overline{F(M)}$ for each $M \in \langle J \rangle$. Let $N = \sigma(J) \in \langle D \rangle$ and L_N be the set in (ii). Define $F' : J \multimap L_N$ by $F'(z) = \overline{F(z)} \cap L_N$ for each $z \in J$. Then each $F'(z)$ is closed in L_N . For each $M \in \langle J \rangle$, since $\Gamma_{\sigma(M)} \subset \overline{F(M)}$, we have

$$\Gamma_{\sigma(M)} \subset \overline{F(M)} \cap L_N \subset F'(M).$$

Therefore, F' is a generalized KKM map with closed values, and we have

$$\bigcap_{z \in J} F'(z) = L_N \cap \bigcap_{z \in J} \overline{F(z)} \neq \emptyset.$$

Let $x \in L_N \cap \bigcap_{z \in J} \overline{F(z)}$. Then $x \in K$ by (ii), $K \subset \bigcup_{z \in J} (X \setminus \overline{F(z)})$ and hence $x \notin \overline{F(z)}$ for some $z \in J$, which is a contradiction. Therefore, we must have $K \cap \bigcap_{z \in I} \overline{F(z)} \neq \emptyset$. By condition (4.1), the conclusion holds. \square

Remarks. (1) If $X = K$ itself is compact, then the conclusion holds without assuming (i) or (ii).

(2) In (ii), L_N does not need to be compact.

(3) This kind of KKM theorem originates from Tian [32, Theorem 2]. If $(X; \Gamma)$ is an H -space, Theorem 4 improves Kim [11, Theorem 1] and Chang and Ma [1, Theorem 1]. If $(X, D; \Gamma)$ is an H -space and $I = D$, Theorem 4 reduces to Park [16, Theorem 1]. Further, if $X = D$ is a convex subset of a topological vector space, Theorem 4 improves Chang and Zhang [2, Theorem 3.1] and Kassay and Kolumbán [10, Theorem 3.1].

The following is needed in Section 5:

Theorem 5. *Let I be a set, $(X \supset D; \Gamma)$ a G -convex space, and $F : I \multimap X$ a map satisfying (4.1) and (4.2) in Theorem 4. Suppose that there exists a nonempty compact subset K of X such that $\bigcap_{z \in M} \overline{F(z)} \subset K \cap D$ for some $M \in \langle I \rangle$. Then $D \cap \bigcap_{z \in I} F(z) \neq \emptyset$.*

Proof. By the same argument in the proof of case (i) in Theorem 4,

$$\bigcap_{z \in I} F(z) = \bigcap_{z \in I} \overline{F(z)} \neq \emptyset.$$

And if $x \in \bigcap_{z \in I} F(z)$, then $x \in \bigcap_{z \in I} \overline{F(z)} \subset K \cap D \subset D$. Hence $D \cap \bigcap_{z \in I} F(z) \neq \emptyset$. \square

4. Analytic formulations of generalized KKM maps

It is well-known that the KKM theory has many applications to equilibrium problems. Some applicability of our results are based on the fact that generalized KKM maps are closely related to certain convexity (or concavity) of extended real-valued functions.

Let I be a nonempty set, $(X, D; \Gamma)$ a G -convex space, and $f : I \times X \rightarrow \overline{\mathbb{R}}$, $g : X \times I \rightarrow \overline{\mathbb{R}}$ functions. Let $\gamma \in \mathbb{R}$. We say that

- (i) f is *generalized γ -quasiconvex* [resp. *generalized γ -quasiconcave*] in the first variable $z \in I$ if for each $N \in \langle I \rangle$, there exists a function $\sigma : N \rightarrow D$ such that $\emptyset \neq M \subset N$ implies $\gamma \leq \max_{z \in M} f(z, x)$ [resp. $\gamma \geq \min_{z \in M} f(z, x)$] for all $x \in \Gamma_{\sigma(M)}$; and
- (ii) g is *generalized γ -quasiconvex* [resp. *generalized γ -quasiconcave*] in the second variable z if the function $h : I \times X \rightarrow \overline{\mathbb{R}}$ defined by $h(z, x) = g(x, z)$ for all $(z, x) \in I \times X$ is *generalized γ -quasiconvex* [resp. *generalized γ -quasiconcave*] in the first variable $z \in I$.

Note that, $f(z, x)$ is *generalized γ -quasiconvex* in the first variable z if and only if $f(z, x) - \gamma$ is *generalized 0-quasiconvex* in the first variable. Also note that $f(z, x)$ is *generalized γ -quasiconvex* in the first variable z if and only if $-f(x, z)$ is *generalized $-\gamma$ -quasiconcave* in the second variable z .

If $I = D$ and $\sigma = i_N : N \hookrightarrow D$, the inclusion, for each $N \in \langle D \rangle$, then the above *generalized γ -quasiconvexity* [resp. *generalized γ -quasiconcavity*] is called *diagonally γ -quasiconvexity* [resp. *diagonally γ -quasiconcavity*].

If γ is replaced by $f(x, x)$ for all $x \in \Gamma_N$ and $D = X$, then the above *diagonally γ -quasiconvexity* [resp. *diagonally γ -quasiconcavity*] is called *diagonally quasiconvexity* [resp. *diagonally quasiconcavity*]; that is, if for any finite $N \subset X$ and for all $x \in \Gamma_N$, $f(x, x) \leq \max_{z \in N} f(z, x)$ [resp. $f(x, x) \geq \min_{z \in N} f(z, x)$] holds, then f is said to be *diagonally quasiconvex* [resp. *diagonally quasiconcave*] in the first variable z .

Examples. (1) For a convex subset X of a topological vector space, the concept of *generalized γ -quasiconvexity* is first introduced by Chang and Zhang [2].

(2) For a convex subset X of a topological vector space and $I = D \subset X$, the concept of *diagonally γ -quasiconcavity* is first introduced by Tian [32] and more general than the concept of *diagonally quasiconcavity* due to Zhou and Chen [34].

(3) For $X = D$, our *generalized γ -quasiconvexity* [resp. *generalized γ -quasiconcavity*] reduces to the *γ -generalized G -quasiconvexity* [resp. *γ -generalized G -quasiconcavity*] due to Tan [31, Definition 1.8].

(4) If $X = D$ is a hyperconvex metric space and I is a nonempty finite subset of X , Kirk, Sims, and Yuan [14] defined the *hyper γ -generalized quasiconvexity* [resp. *the corresponding quasiconcavity*] as above in (2). Note that *hyperconvex metric spaces* are particular types of C -spaces [9].

The following [27, Theorem 6] shows the equivalency of certain concavity [resp. convexity] of extended real functions and the related *generalized KKM maps*:

Theorem 6. *Let I be a nonempty set, $(X, D; \Gamma)$ a G -convex space, $f : I \times X \rightarrow \overline{\mathbb{R}}$, and $\gamma \in \mathbb{R}$. Then the followings are equivalent:*

- (6.1) *The multimap $F : I \multimap X$, defined by $F(z) = \{x \in X \mid f(z, x) \leq \gamma\}$ [resp. $F(z) = \{x \in X \mid f(z, x) \geq \gamma\}$] for all $z \in I$, is a *generalized KKM map*.*

(6.2) f is generalized γ -quasiconcave [resp. generalized γ -quasiconvex] in the first variable z .

From Theorem 6, we obtain the following equilibrium result implying mini-max inequalities, variational inequalities, and so on:

Theorem 7. *Let I be a nonempty set, $(X, D; \Gamma)$ a G -convex space, $f : I \times X \rightarrow \overline{\mathbb{R}}$, and $\gamma \in \mathbb{R}$. Suppose that there exists a nonempty compact subset K of X such that*

- (7.1) *for each $z \in I$, $\{x \in X \mid f(z, x) \leq \gamma\}$ is [transfer] closed (for example, $x \mapsto f(z, x)$ is lower semicontinuous);*
- (7.2) *f is generalized γ -quasiconcave in the first variable z ; and*
- (7.3) *either*
 - (i) *there exists a set $M \in \langle I \rangle$ such that $\bigcap_{z \in M} \overline{\{x \in X \mid f(z, x) \leq \gamma\}} \subset K$;*
 - or
 - (ii) *if $X \supset D$ and, for each $J \in \langle I \rangle$ and each function $\sigma : J \rightarrow D$, there exists a Γ -convex subset L_N of X containing $N = \sigma(J)$ such that $L_N \cap \bigcap_{z \in J} \overline{\{x \in X \mid f(z, x) \leq \gamma\}} \subset K$.*

Then there exists an $x_0 \in K$ such that $f(z, x_0) \leq \gamma$ for all $z \in I$.

Proof. Let us define a map $F : I \multimap X$ by $F(z) = \{x \in X \mid f(z, x) \leq \gamma\}$ for $z \in I$. Then, by (7.1), F is [transfer] closed-valued. By Theorem 6, (7.2) implies that F is a generalized KKM map and so is \overline{F} . Therefore, by Theorem 4, $K \cap \bigcap_{z \in I} F(z) \neq \emptyset$. Hence there exists an $x_0 \in K$ such that $x_0 \in F(z)$ or $f(z, x_0) \leq \gamma$ for all $z \in I$.

This completes our proof. □

Examples. (1) For closed convex subsets I and $X = D$ of Hausdorff topological vector spaces, Theorem 7 reduces to Chang and Zhang [2, Theorem 3.4].

(2) Tan [31, Theorem 3.1] obtained a particular case of Theorem 7 for $X = D$ under some superfluous restrictions.

(3) Tan’s result was modified by Kirk, Sims, and Yuan [14, Theorem 2.8] for a hyperconvex metric space X .

5. Maximal elements

In the literature, there are two approaches to nontransitive-nontotal preference theory: one through “weak” (i.e., reflexive) preferences (see Sonnenschein [30] and Shafer and Sonnenschein [29]) and the other through “strict” (i.e., irreflexive) preferences (see Schmeidler [28], Gale and Mas-Colell [3], and Yannelis and Prabhakar [33]). The distinction becomes important when preferences are not total. Kim and Richter [13] made the connection between the weak preference approach and the strict preference approach.

In order to apply those concepts in preference theory to Economics, we can use some results in the KKM theory. Basic equilibrium results in the theory

enable us to obtain a class of existence theorems on maximal elements of binary relations, price equilibria, and solutions of complementarity problems.

In this section, we obtain existence theorems of maximal elements for certain preference relations and price equilibria under some conditions from our theory of generalized KKM maps.

The (weak or strict) preference relation is defined on Z and is a subset of $Z \times Z$. Here Z may be considered as a *consumption space*. Let \succeq be the *weak preference relation*. An element $(x, y) \in \succeq$ is written as $x \succeq y$ and read as “ x is at least as good as y ”. Let \succ be the *strict preference relation*. An element $(x, y) \in \succ$ is written as $x \succ y$ and read as “ x is strictly preferred to y ”. We may think $x \succ y$ if and only if $x \succeq y$ and $x \neq y$. For each x , the *weakly upper*, *weakly lower*, *strictly upper*, and *strictly lower contour sets* (or *sections*) of x are denoted by $U_w(x) = \{y \in Z \mid y \succeq x\}$, $L_w(x) = U_w^-(x) = \{y \in Z \mid x \succeq y\}$, $U_s(x) = \{y \in Z \mid y \succ x\}$, and $L_s(x) = \{y \in Z \mid x \succ y\}$, respectively.

In some cases, not all points in Z can be chosen. A subset $B \subset Z$ is called as a *choice set* which may be considered as the upper bound set for feasible set. A weak preference relation \succeq is said to *have a greatest element on the subset B of Z* if there exists a point $x^* \in B$ such that $x^* \succeq x$ for all $x \in B$, or equivalently $\bigcap_{x \in B} U_w(x) \cap B \neq \emptyset$. The above x^* is called a *greatest element*. A strict preference relation \succ is said to *have a maximal element on the subset B of Z* if there exists a point $x^* \in B$ such that for any $x \in B$, $x \succ x^*$ does not hold; that is $U_s(x^*) \cap B = \emptyset$. This x^* is called a *maximal element*.

Let Y be a convex subset of Hausdorff topological vector space E and let $\emptyset \neq X \subset Y$. In [32], Tian defined the following convexities:

$F : X \multimap Y$ is said to be *FS-convex* if for every $M \in \langle X \rangle$, $\text{co}M \subset F(M)$, where *FS* represents Fan and Sonnenschein.

$F : Y \multimap Y$ is said to be *SS-convex* if $x \notin \text{co}F(x)$ for all $x \in Y$, where *SS* represents Shafer and Sonnenschein.

Motivated by the above definitions, we define the following:

For a G -convex space $(X, D; \Gamma)$, a map $F : D \multimap X$ is said to be *GFS-convex* on D if for every $N \in \langle D \rangle$, there exists a function $\sigma : N \rightarrow D$ such that for any $M \subset N$, $\Gamma_{\sigma(M)} \subset F(M)$.

The following theorem shows the existence of a greatest element for a weak preference relation:

Theorem 8. *Let $(Z \supset B; \Gamma)$ be a G -convex space, and \succeq defined on Z a weak preference relation. Suppose that there exists a nonempty compact subset K of Z such that*

- (8.1) U_w is transfer closed-valued on B ;
- (8.2) $\overline{U_w}$ is GFS-convex on B ; and
- (8.3) $\bigcap_{x \in M} \overline{U_w}(z) \subset B \cap K$ for some $M \in \langle B \rangle$.

Then \succeq has a greatest element on B .

Proof. Define a map $F : B \multimap Z$ by $F(x) := U_w(x)$ for each $x \in B$. Then (8.2) shows that \overline{F} is a generalized KKM map. And (8.3) reduces to the condition of Theorem 5. So, we deduce that $B \cap (\bigcap_{x \in B} F(x)) = B \cap (\bigcap_{x \in B} U_w(x)) \neq \emptyset$. \square

Example. If Z is a convex subset of a Hausdorff topological vector space and $\overline{U_w}$ is FS -convex on B , then Theorem 8 reduces to [32, Theorem 5].

For a G -convex space $(X, D; \Gamma)$, a map $F : X \multimap D$ is called an SS -map if for every $M \in \langle D \rangle$ and $x_0 \in \Gamma_M$, $x_j \notin F(x_0)$ for some $x_j \in M$. Tian [32] called this concept by generalized SS -convex on a convex subset of a Hausdorff topological vector space.

A map $F : X \multimap D$ on a G -convex space $(X, D; \Gamma)$ is called a GSS -map if, for every $N \in \langle D \rangle$, there exists a function $\sigma : N \rightarrow D$ such that for any $M \subset N$ and $x_0 \in \Gamma_{\sigma(M)}$, $x_j \notin F(x_0)$ for some $x_j \in M$.

Remark. If σ is the identity function, then a GSS -map $F : X \multimap D$ becomes an SS -map.

Let X and Y be two topological spaces. A multimap $F : X \multimap Y$ is said to be *transfer open-valued* on X if for every $x \in X$, $y \in F(x)$ implies that there exists a point $x' \in X$ such that $y \in \text{Int } F(x')$.

Note that $F : X \multimap Y$ is transfer open-valued on X if and only if the multimap $G : X \multimap Y$, defined by $G(x) = Y \setminus F(x)$ for every $x \in X$, is transfer closed-valued on X . See [32].

Theorem 9. Let $(Z \supset B; \Gamma)$ be a G -convex space and \succ a strict preference relation on Z . Suppose that there exists a nonempty compact subset K of Z such that

- (9.1) L_s is transfer open-valued on B ;
- (9.2) $\text{Int } U_s$ is a GSS -map; and
- (9.3) either
 - (i) $\bigcap_{x \in M} \{y \in Z \mid x \not\succeq y\} \subset K$ for some $M \in \langle B \rangle$; or
 - (ii) for each $M \in \langle B \rangle$ and each function $\sigma : M \rightarrow B$, there exists a Γ -convex subset L_N of Z containing $N = \sigma(M)$ such that

$$L_N \cap \bigcap_{x \in M} \overline{\{y \in Z \mid x \not\succeq y\}} \subset K.$$

Then \succ has a maximal element on B .

Proof. Let $F(x) = Z \setminus L_s(x)$. Then the set of maximal element is $\{x \in B \mid U_s(x) = \emptyset\} = \bigcap_{x \in B} F(x)$. Since F is transfer closed-valued by (9.1), we need to show that \overline{F} is a generalized KKM map. Suppose to the contrary, for some $M \in \langle B \rangle$, and for every $\sigma : M \rightarrow B$, there exists a point $x_0 \in \Gamma_{\sigma(M)}$ which is not in $\overline{F(x_j)} = Z \setminus \text{Int } L_s(x_j)$ for all $x_j \in M$. Then $x_0 \in \text{Int } L_s(x_j)$, so $x_j \in \text{Int } U_s(x_0)$ for all $x_j \in M$, which contradict (9.2). Since condition (9.3) implies condition (4.3) of Theorem 4, we can conclude $\bigcap_{x \in B} F(x) \neq \emptyset$. So there exists an $x^* \in B$ such that $U_s(x^*) = \emptyset$. \square

Remark. Tian [32, Theorem 6] obtained Theorem 9 when $\text{Int } U_s$ is an SS -map and Z is a convex subset of a Hausdorff topological vector space with an extra compactness condition.

6. Almost fixed point theorems in generalized convex uniform spaces of the Zima type

A G -convex uniform space $(X, D; \Gamma; \mathcal{U})$ is a G -convex space $(X \supset D; \Gamma)$ such that D is dense in X and (X, \mathcal{U}) is a Hausdorff uniform space, where \mathcal{U} is a basis of the uniformity.

Let $(X, D; \Gamma; \mathcal{U})$ be a G -convex uniform space and K a nonempty subset of X . We say that K is of the Zima type whenever for every $V \in \mathcal{U}$ there exists a $U \in \mathcal{U}$ such that for every $A \in \langle D \rangle$ and every Γ -convex subset M of K the following implication holds:

$$M \cap U[z] \neq \emptyset, \quad \forall z \in A \Rightarrow M \cap V[u] \neq \emptyset, \quad \forall u \in \Gamma_A,$$

where $U[z] = \{x \in X \mid (z, x) \in U\}$.

Examples. (1) Hadžić [4] defined that a nonempty subset K of a topological vector space E is of the Zima type whenever for any $V \in \mathcal{V}$, there exists a $U \in \mathcal{V}$ satisfying $\text{co}(U \cap (K - K)) \subset V$, where \mathcal{V} is a neighborhood system of the origin of E .

Note that any nonempty subset of a locally convex topological vector space is of the Zima type, and that there exists a subset of the Zima type in a non-locally convex topological vector space; see Hadžić [4].

(2) For an H -space, our definition reduces to that of Hadžić [5].

A G -convex uniform space $(X, D; \Gamma; \mathcal{U})$ is called an LG -space whenever for each $V \in \mathcal{U}$, $\{x \in X \mid C \cap V[x] \neq \emptyset\}$ is Γ -convex if $C \subset X$ is Γ -convex.

Lemma 1. For an LG -space $(X, D; \Gamma; \mathcal{U})$, any nonempty subset K of X is of the Zima type.

Proof. Consider the case $U = V \in \mathcal{U}$ in the definition of the Zima type. For every $A \in \langle D \rangle$ and every Γ -convex subset M of K , suppose that $M \cap U[z] \neq \emptyset$ for every $z \in A$. Since

$$A \subset \{x \in D \mid M \cap U[x] \neq \emptyset\}$$

and $\{x \in X \mid M \cap U[x] \neq \emptyset\}$ is Γ -convex by the definition of LG -spaces, we have

$$\Gamma_A \subset \{x \in X \mid M \cap U[x] \neq \emptyset\}$$

and hence $M \cap U[u] \neq \emptyset$ for every $u \in \Gamma_A$. \square

Theorem 10. Let $(X, D; \Gamma; \mathcal{U})$ be a G -convex uniform space and K a totally bounded subset of X . Let $T : X \rightarrow X$ be a l.s.c. [resp. u.s.c.] map such that $T(x)$ is Γ -convex and $T(x) \cap K \neq \emptyset$ for each $x \in X$. If $T(X)$ is of the Zima type, then for each $V \in \mathcal{U}$, T has a V -almost fixed point $x_* \in X$; that is, $T(x_*) \cap V[x_*] \neq \emptyset$.

Proof. For each $V \in \mathcal{U}$, there exists an open set $W \in \mathcal{U}$ such that $W \subset \overline{W} \subset V$. Since K is totally bounded and D is dense in X , we have a set $M = \{y_0, y_1, \dots, y_n\} \in \langle D \rangle$ such that $K \subset \bigcup_{y \in M} W[y] \subset \bigcup_{y \in M} V[y]$.

Case (i). T is l.s.c.: For each $y_i \in M$, define

$$F(y_i) := \{x \in X \mid T(x) \cap W[y_i] = \emptyset\}.$$

Since T is l.s.c., each $F(y_i)$ is closed. Since $T(x) \cap K \neq \emptyset$ for each $x \in X$, we have $T(x) \cap \bigcup_{y \in M} W[y] \neq \emptyset$ and hence

$$\bigcap_{i=1}^n F(y_i) \subset \{x \in X \mid T(x) \cap \bigcup_{i=1}^n W[y_i] = \emptyset\} = \emptyset.$$

Apply Theorem 1 to the G -convex space $(X, M; \Gamma)$, where $\Gamma = \Gamma|_{\langle M \rangle} : \langle M \rangle \multimap X$. Since $\{F(y)\}_{y \in M}$ does not have the finite intersection property, the closed-valued map $F : M \multimap X$ is not a KKM map. Therefore there exists an $A \in \langle M \rangle$ such that $\Gamma_A \not\subseteq F(A)$; that is, there exists an $x_V \in \Gamma_A$ such that $x_V \notin F(A)$. Hence $T(x_V) \cap W[y] \neq \emptyset$ for all $y \in A$. Since $T(X)$ is of the Zima type, $T(x_V) \cap V[x_V] \neq \emptyset$.

Case (ii). T is u.s.c.: For each $y_i \in M$, define

$$F(y_i) := \{x \in X \mid T(x) \cap \overline{W}[y_i] = \emptyset\}.$$

Since T is u.s.c., each $F(y_i)$ is open. We can follow the proof of Case (i). □

This completes our proof. □

Remarks. (1) Note that in the above proof, if $\Gamma_A \subset D$ for each $A \in \langle D \rangle$, then $x_V \in \Gamma_A \subset D$ in the above proof; and hence it is sufficient to assume that T has G -convex values on D , not necessarily on the whole X .

(2) Hadžić [5, Theorem 1] is a particular case of our Theorem 10.

Corollary 10.1. *Let $(X, D; \Gamma; \mathcal{U})$ be a G -convex uniform space and $T : X \multimap X$ a compact u.s.c. map with nonempty closed Γ -convex values. If $T(X)$ is of the Zima type, then T has a fixed point $x_* \in T(x_*)$.*

Proof. By Theorem 10, for each $V \in \mathcal{U}$, there exists $x_V, y_V \in X$ such that $y_V \in T(x_V)$ and $y_V \in V[x_V]$. Since $\overline{T(X)}$ is relatively compact, we may assume that y_V converges to some $x_* \in \overline{T(X)}$. Since X is Hausdorff, x_V also converges to x_* . Since T is u.s.c. with closed values, the graph of T is closed in $X \times T(X)$, and hence we have $x_* \in T(x_*)$.

This completes our proof. □

Remark. If $(X, D; \Gamma; \mathcal{U})$ is an LG -space, then Corollary 10.1 reduces to the following [21, Theorem 2].

Corollary 10.2. *Let $(X, D; \Gamma; \mathcal{U})$ be an LG -space and $T : X \multimap X$ a compact u.s.c. map with nonempty closed Γ -convex values. Then T has a fixed point.*

Proof. Note that $T(X)$ is of the Zima type by Lemma 1. Now, the conclusion follows from Corollary 10.1. □

In order to give an example of Theorem 10, we introduce a notion due to Himmelberg [6].

A nonempty subset Y of a topological vector space E is said to be *almost convex* if for any $V \in \mathcal{V}$, where \mathcal{V} is a neighborhood system of the origin 0 in E , and for any finite set $\{y_1, y_2, \dots, y_n\} \subset Y$, there exists a finite set $\{z_1, z_2, \dots, z_n\} \subset Y$ such that, for each $i = 1, 2, \dots, n$, $z_i - y_i \in V$ and $\text{co}\{z_1, z_2, \dots, z_n\} \subset Y$.

We give a new example of G -convex spaces :

Lemma 2. *Let X be a subset of a topological vector space E . If X has an almost convex subset Y , then X can be made into a G -convex space.*

Proof. Choose $V \in \mathcal{V}$. For any $A = \{y_1, y_2, \dots, y_n\} \in \langle Y \rangle$, there exists a $B = \{z_1, z_2, \dots, z_n\} \in \langle Y \rangle$ such that $z_i - y_i \in V$ for all $i = 1, 2, \dots, n$ and $\text{co} B \subset Y \subset X$. Let $\Gamma : \langle Y \rangle \rightarrow X$ be defined by $\Gamma_A = \text{co} B$ as above. Then $(X, Y; \Gamma)$ becomes a G -convex space. \square

From Theorem 10 with Remark (1) and Lemma 2, we have the following:

Theorem 11. *Let X be a subset of a topological vector space E , Y an almost convex dense subset of X , and K a totally bounded subset of X . Let $T : X \rightarrow X$ be a l.s.c. [resp. u.s.c.] map such that $T(y)$ is convex for each $y \in Y$ and $T(x) \cap K \neq \emptyset$ for each $x \in X$. If $T(X)$ is of the Zima type, then for each $V \in \mathcal{V}$, T has a V -almost fixed point $x_* \in X$; that is, $T(x_*) \cap (x_* + V) \neq \emptyset$.*

Proof. From Lemma 2, we have a G -convex space $(X, Y; \Gamma; \mathcal{U})$ with the uniformity \mathcal{U} such that Y is dense in X . Therefore, Theorem 11 follows from Theorem 10. \square

From Theorem 11, we can deduce several fixed point theorems as in [12].

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