REMARKS ON RECENT RESULTS IN ANALYTICAL FIXED POINT THEORY

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Abstract. We show that some fixed point theorems and related results in our previous works [4-8] need additional requirements for their validities. Some of the new correct results will appear in [9].

Based on the works of Cauty [1] and Dobrowolski [2,3], we claimed or announced in [4-8] to obtain some fixed point theorems on multimaps (maps) and related results. Recently, it is known that there are some gaps in the proofs in [1-3], and so some of our results might be groundless. Our aim in this paper is to clarify this and to give additional requirements which guarantee the validity of each of such results. We add some new related results in our forthcoming work [9].

1. The Schauder conjecture and other problems

A t.v.s. is a Hausdorff topological vector space E with a basis $\mathcal{V}$ of neighborhoods of the origin 0 of E.

In 2001, Robert Cauty [1] claimed the affirmative resolution of the Schauder conjecture as follows:

(I) [1] *Let E be a t.v.s., C a convex subset of E, and f a continuous function from C into C. If $f(C)$ is contained in a compact subset of C, then f has a fixed point.*

Later it is known that there is a gap in the proof of Statement (I).

An infinite dimensional compact convex subset K of a t.v.s. E is said to have the *simplicial approximation property* if for every $V \in \mathcal{V}$ there exists a finite dimensional compact convex set K_0 in K such that if S is any finite dimensional simplex in K then there exists a continuous function $\psi : S \to K_0$ with $\psi(x) - x \in V$ for all $x \in S$.

2000 *Mathematics Subject Classification.* 47H04, 47H10, 52A07, 54C60, 54H25, 55M20.

Key words and phrases. The Schauder conjecture, weakly admissible set, Klee approximable, Kakutani map, Fan-Browder map, approximable map, admissible class of multimaps.

(II) [2] *A compact convex subset of a metrizable t.v.s. has the simplicial approximation property.*

It is also known that there is a gap in the proof of Statement (II).
From Statement (II), we deduced the following in [5]:

(III) *A compact convex subset of a metrizable t.v.s. is admissible.*

From Statement (III), the author [4,5] deduced the affirmative resolutions to the compact AR problem and the Banach problem as follows:

(IV) *A compact convex subset of a metrizable t.v.s. is an AR.*

(V) *An infinite dimensional compact convex subset of a metrizable t.v.s. is homeomorphic to the Hilbert cube Q.*

Since Statements (III)-(V) are based on Statement (II), their validities are not sufficiently justified. In view of Statements (IV) and (V), we claimed in [4,5] that every *Roberts spaces* – that is, compact convex sets with no extreme points constructed by Roberts' method of needle point spaces – are AR and homeomorphic to the Hilbert cube Q.

2. Kakutani maps

A *polytope* P in a subset X of a t.v.s. E is a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset K of X and every $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

A nonempty subset K of E is said to be *Klee approximable* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope of E. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

Let X be a nonempty closed convex subset of a t.v.s. E. We say that X is *weakly admissible* (in the sense of Nhu and Arandelović) if for every $V \in \mathcal{V}$ there exist closed convex subsets $X_1, X_2, \cdots, X_n$ of X with $X = \mathrm{co}\,(\bigcup_{i=1}^n X_i)$ and continuous functions $f_i : X_i \to X \cap L, i = 1, 2, \cdots, n$, where L is a finite dimensional subspace of E, such that $\sum_{i=1}^n (f_i(x_i) - x_i) \in V$ for every $x_i \in X_i$ and $i = 1, 2, \cdots, n$.

A subset B of a t.v.s. E is said to be *convexly totally bounded* (simply, c.t.b.) if for every $V \in \mathcal{V}$, there exist a finite subset $\{x_i\}_{i=1}^n \subset B$ and a finite family of convex subsets $\{C_i\}_{i=1}^n$ of V such that $B \subset \bigcup_{i=1}^n (x_i + C_i)$. (In [6], we incorrectly stated that any convex and c.t.b. subset in a t.v.s. is admissible. This is not known yet.)

A *Kakutani map* is an upper semicontinuous (u.s.c.) multimap with nonempty compact convex values.

The following [9] shows some of the most general known partial resolutions of the Schauder conjecture:

Theorem 1. *Let X be a nonempty subset of a t.v.s. E. Then a compact Kakutani map $T : X \multimap X$ has a fixed point if one of the following conditions holds:*

(1) (Idzik) *X is convex and $\overline{T(X)}$ is c.t.b.*

(2) (Okon) *X is compact, convex and weakly admissible.*

(3) (Park) *$T(X)$ is Klee approximable into X.*

Note that (3) holds whenever $\overline{T(X)}$ is locally convex or X is convex and admissible (in the sense of Klee).

We say that a topological space X has the (*compact*) *fixed point property* (simply, f.p.p.) if any (compact) continuous selfmap $f : X \to X$ has a fixed point $x_0 \in X$.

We say that a subset X of a t.v.s. has the (*compact*) *Kakutani fixed point property* (simply, $\mathbb{K}$-f.p.p.) if any (compact) Kakutani map $T : X \multimap X$ has a fixed point.

Based on [1,2], the following were announced:

(VI) [3] *A compact convex subset X of a t.v.s. has the $\mathbb{K}$-f.p.p.*

(VII) [3] *A convex subset X of a metrizable t.v.s. has the compact $\mathbb{K}$-f.p.p.*

In view of Theorem 1, (VI) is valid whenever X is c.t.b. or weakly admissible, and that (VII) is valid whenever X is admissible or any compact subset of X is c.t.b. for any t.v.s. (not necessarily metrizable); see [4,6].

In [8], we claimed that the following follows from (VI):

(VIII) *Let X be a compact convex subset of a t.v.s. and U an open subset of X such that $0 \in U$. Let $T : \overline{U} \multimap X$ be a Kakutani map. Then one of the following holds*:

(i) *T has a fixed point.*

(ii) *There exists an $x \in \operatorname{Bd} U$ and a $\lambda \in (0, 1)$ such that x is a fixed point of λT.*

Note that (VIII) is true whenever X is c.t.b. or weakly admissible.

Similarly, from the correct form of Statement (VII), we can obtain a Leray-Schauder type alternative like (VIII).

3. The Fan-Browder maps

A multimap with nonempty convex values and open fibers is called a *Browder map*. The well-known *Fan-Browder fixed point theorem* states that a Browder map from a compact convex subset of a t.v.s. into itself has a fixed point.

For a subset X of a t.v.s. E, a multimap $T : X \multimap X$ is called a Φ-*map* (or *a Fan-Browder map*) if there exists a multimap $S : X \multimap X$ such that (1) for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$ and (2) $X = \bigcup\{\operatorname{Int} S^-(x) : x \in X\}$.

A subset X of a t.v.s. E is called a Φ-*space* if for each $U \in \mathcal{V}$, there is a Φ-map $T : X \multimap X$ such that $T(x) \subset x + U$ for each $x \in X$.

We say that a convex subset X of a t.v.s. has the (*compact*) Φ-*fixed point property* (simply, Φ-f.p.p.) if any (compact) Φ-map $T : X \multimap X$ has a fixed point.

Since any Φ-map from a paracompact space into a convex subset of a t.v.s. has a continuous selection, we have

Lemma 2. (Komiya) *Let X be a paracompact convex subset of a t.v.s. If X has the f.p.p., then it has the Φ-f.p.p.*

In 1992, Ben-El-Mechaiekh conjectured that the Fan-Browder theorem might be true if we assume that the Browder map is compact instead of the compactness of its domain. In [7,8], we incorrectly claimed that this conjecture is affirmative as follows:

(IX) *A convex subset X of a t.v.s. E has the compact Φ-f.p.p.*

This follows from the above-mentioned selection theorem and Statement (I).

Lemma 3. (Komiya) *Let X be a paracompact convex subset of a locally convex t.v.s. If X has the Φ-f.p.p., then it has the $\mathbb{K}$-f.p.p.*

It is well-known that, if K is a compact subset of a t.v.s., then $\operatorname{co} K$ is paracompact. In view of this fact, from Lemmas 2 and 3, we easily have

Theorem 4. *A convex subset X of a locally convex t.v.s. has* (1) *the compact f.p.p.,* (2) *the compact Φ-f.p.p., and* (3) *the compact $\mathbb{K}$-f.p.p.*

Recall that (1) is due to Hukuhara, (2) to Ben-El-Mechaiekh, and (3) to Himmelberg.

4. Locally selectionable maps

For topological spaces X and Y, a multimap $T : X \multimap Y$ is said to be *selectionable* if it has a continuous selection $f : X \to Y$ (that is, $f(x) \in T(x)$ for all $x \in X$), and *locally selectionable* if for each $x_0 \in X$, there exist an open neighborhood V_0 of x_0 and a continuous function $f_0 : V_0 \to Y$ such that $f_0(x) \in T(x)$ for all $x \in V_0$.

The following is stated in [8] as a consequence of (I):

(X) *Let X be a convex subset of a t.v.s. E and $T : X \multimap X$ a locally selectionable map having convex values. If T is compact, then T has a fixed point.*

It is known that, if X is paracompact and Y is a convex subset of a t.v.s., then any Φ-map $T : X \multimap Y$ is locally selectionable and a locally selectionable map $T : X \multimap Y$ having convex values is selectionable. From this, we have

Theorem 5. *Let E be a t.v.s. whose nonempty paracompact convex subsets have the compact f.p.p., X a nonempty convex subset of E, and $T : X \multimap X$ a locally selectionable map having convex values. If T is compact, then it has a fixed point.*

Note that Theorem 5 corrects (IX) and (X).

5. Approximable maps

Let X and Y be subsets of t.v.s. E and F, respectively, and $T : X \multimap Y$ a multimap. Given two open neighborhoods U and V of the origin 0 of E and F, respectively, a

(U,V)-*approximative continuous selection* of T is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (T[(x+U)\cap X] + V) \cap Y \quad \text{for every } x \in X.$$

T is said to be *approachable* if it admits a (U,V)-approximative continuous selection for every U and V as above; and T *approximable* if its restriction $T|_K$ to any compact subset K of X is approachable. Note that an approachable map is always approximable.

We say that a subset X of a t.v.s. has the (*compact*) *approachable fixed point property* (simply, $\mathbb{A}$-f.p.p.) if any (compact) u.s.c. approachable map $T : X \multimap X$ has a fixed point; and the *approximable fixed point property* (simply, $\mathbb{A}^\kappa$-f.p.p.) if any u.s.c. approximable map $T : X \multimap X$ has a fixed point.

We stated the following in [8] as consequences of Statement (I):

(XI) *A convex subset X of a t.v.s. E has the compact $\mathbb{A}$-f.p.p.*

(XII) *Let X be a closed subset of a t.v.s. E such that $0 \in \operatorname{Int} X$ and $T : X \multimap E$ a compact closed approachable map. Then either*
(1) *T has a fixed point*; *or*
(2) *$\lambda x \in T(x)$ for some $\lambda > 1$ and $x \in \operatorname{Bd} X$.*

(XIII) *A compact convex subset X of a t.v.s. has the $\mathbb{A}^\kappa$-f.p.p.*

The validities of (XI)-(XIII) require certain restrictions as follows:

Theorem 6. *For a subset X of a t.v.s. E, the following are equivalent:*
(1) *X has the compact f.p.p.*
(2) *X has the compact $\mathbb{A}$-f.p.p.*
Moreover, (1) *and* (2) *follow from*
(3) *X has the compact $\mathbb{K}$-f.p.p.*

For the proof, see [9].

From Theorems 1 and 6, we immediately have the following corrected form of (XI):

Corollary 7. *A convex subset X of a t.v.s. E has the compact $\mathbb{A}$-f.p.p. whenever one of the following holds:*
(1) *X is admissible* (*in the sense of Klee*).
(2) *every compact subset of X is c.t.b.*
(3) *X is compact and weakly admissible.*

Note that (XII) also requires one of (1)-(3) of Corollary 7.

Recall that a subset X of a t.v.s. E is said to be *almost convex* if for each $U \in \mathcal{V}$ and for any finite subset $\{x_1, x_2, \cdots, x_n\}$ of X, there exists a finite subset $\{z_1, z_2, \cdots, z_n\}$ of X such that $z_i - x_i \in U$ for all i and $\operatorname{co}\{z_i\}_{i=1}^n \subset X$.

Corollary 8. *An almost convex subset X of a locally convex t.v.s. E has* (1) *the compact f.p.p.,* (2) *the compact $\mathbb{A}$-f.p.p., and* (3) *the compact $\mathbb{K}$-f.p.p.*

Recall that (3) is due to Park and Tan.

The following corrects (XIII).

Corollary 9. *A compact convex subset X of a t.v.s. E has the $\mathbb{A}^\kappa$-f.p.p. whenever one of the following holds:*
(1) *X has the f.p.p.*
(2) *X is c.t.b.*
(3) *X is weakly admissible.*

It is known by Ben-El-Mechaiekh et al. that, for a compact uniform space X and a convex subset Y of a locally convex t.v.s. E, a Kakutani map $T : X \multimap Y$ is approximable. For non-locally convex t.v.s., we have

Corollary 10. *Let X be a compact subset of a t.v.s. E. Then the following are equivalent:*
(1) *X has the f.p.p.*
(2) *X has the $\mathbb{A}^\kappa$-f.p.p.*
Moreover, (1) *and* (2) *follow from*
(3) *X has the $\mathbb{K}$-f.p.p.*

Examples. For a compact convex subset X of a t.v.s. E, we give known cases when (1)-(3) of Corollary 10 hold as follows:
1. For a Euclidean space E, (1) is due to Brouwer and (1) $\Rightarrow$ (3) to Kakutani by a different method.
2. For a normed vector space E, (1) is due to Schauder and (3) to Bohnenblust and Karlin.
3. For a locally convex t.v.s. E, (1) is due to Tychonoff, (2) to Ben-El-Mechaiekh, and (3) to Fan and Glicksberg, independently.
4. For a t.v.s. E having sufficiently many linear functionals, (1) is due to Fan and (3) to Granas and Liu.
5. For an admissible set X, (1) is due to Hahn and Pötter and (2) and (3) to Park.
6. If X is locally convex, (1) is due to Rzepecki and (3) to Idzik..
7. For a c.t.b. set X, (1) and (3) are particular cases of a result of Idzik.
8. Further if X is a Φ-space, (1) is due to Horvath and (3) to Park.
9. For a weakly admissible set X, (1) is due to Nhu and Arandelović and (3) to Okon.
10. For a Roberts space X, (1) is due to Nhu et al. and (3) to Okon.

6. Admissible classes of multimaps

In 1996, the author introduced the 'better' admissible class $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for each polytope P in X and for any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point,

where X is a subset of a t.v.s. and Y is a topological space.

We give some subclasses of $\mathfrak{B}$ as follows:

For topological spaces X and Y, an *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $F : X \multimap Y$ is one such that, for each nonempty compact subset K of X, there exists a map $G \in$

$\mathfrak{A}_c(K, Y)$ satisfying $G(x) \subset F(x)$ for all $x \in K$; where $\mathfrak{A}_c$ consists of finite compositions of maps in a class $\mathfrak{A}$ of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $T \in \mathfrak{A}_c$ is u.s.c. with nonempty compact values; and
(iii) for any polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

There are lots of examples of $\mathfrak{A}$ and $\mathfrak{A}_c^\kappa$.

Subclasses of the admissible class $\mathfrak{A}_c^\kappa$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$, Browder maps, Φ-maps, selectionable maps, locally selectionable maps having convex values, the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the u.s.c. approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}_c^\sigma$ of Lassonde, the class $\mathbb{V}_c^\sigma$ of Park et al., u.s.c. approximable maps of Ben-El-Mechaiekh and Idizk, and others.

Note that for a subset X of a t.v.s. and any space Y, an admissible class $\mathfrak{A}_c^\kappa(X, Y)$ is a subclass of $\mathfrak{B}(X, Y)$. Some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ were known. Note that the connectivity map due to Nash and Girolo is such an example.

In 2004, we obtained the following:

Theorem 11. *Let E be a t.v.s. and X a nonempty subset of E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ such that $F(X)$ is Klee approximable into X has a fixed point.*

From Theorem 11, we have the following earlier result:

Corollary 12. *Let X be an admissible convex subset of a t.v.s. E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

In view of Statement (II) or (III), we claimed the following in [8]:

(XIV) *Let K be a compact convex subset of a metrizable t.v.s. E. Then any $T \in \mathfrak{A}_c^\kappa(K, K)$ has a fixed point.*

(XV) *Let K be a compact convex subset of a metrizable t.v.s. E. Then any closed $T \in \mathfrak{B}(K, K)$ has a fixed point.*

In view of Theorem 11 and Corollary 12, Statements (XIV) and (XV) hold for any t.v.s. if K is admissible.

Final remark. Statements (III)-(XV) are consequences of (I), (II) or (VII) with the aid of other results. Each of them can be regarded as conjectures as like as (I).

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