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Fixed point theorems for better admissible multimaps on almost convex sets

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Abstract

We obtain new fixed point theorems on multimaps in the class $\mathfrak{B}^p$ defined on almost convex subsets of topological vector spaces. Our main results are applied to deduce various fixed point theorems, coincidence theorems, almost fixed point theorems, intersection theorems, and minimax theorems. Consequently, our new results generalize well-known works of Kakutani, Fan, Browder, Himmelberg, Lassonde, and others.

Keywords: T.v.s.; Multimap (map); Almost fixed point; Almost convex set; Klee approximable set; Map classes $\mathfrak{A}_c^\kappa$, $\mathfrak{B}$, $\mathfrak{B}^p$

1. Introduction

The fixed point theory of multimaps in topological vector spaces has numerous applications in many fields in mathematical sciences. Recently, there have appeared a number of fixed point theorems for new classes of multimaps defined on convex subsets of topological vector spaces. In fact, in our previous work [14], we unified fixed point theorems for such classes of compact closed multimaps defined mainly on convex subsets of topological vector spaces. Moreover, in [20], we obtained a new fixed point theorem for the 'better' admissible class $\mathfrak{B}^p$, where p stands for polytope.

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Our aim in the present paper is to derive new fixed point theorems on multimaps in the class $\mathfrak{B}^p$ defined on *almost convex* subsets of topological vector spaces. It is essential to note that a multimap in the class has an almost fixed point whenever its range is *Klee approximable* as in [20]. Our main results are, then, applied to deduce various fixed point theorems, coincidence theorems, almost fixed point theorems, intersection theorems, and minimax theorems. Consequently, our new results generalize well-known works of Kakutani [6], Fan [4], Browder [3], Himmelberg [5], Lassonde [7], and others; for the related history, see Ref. [16].

Section 2 deals with preliminaries on the better admissible classes $\mathfrak{B}$ and $\mathfrak{B}^p$ for which we give a basic fixed point Theorem 2.2 and some of its direct consequences. Here, we showed that if the range is Klee approximable into the domain, then the multimap in the class $\mathfrak{B}^p$ has an almost fixed point. In Section 3, we are mainly concerned with fixed point theorems on multimaps in the class $\mathfrak{B}^p$ defined on almost convex subsets of topological vector spaces more general than locally convex ones. Section 4 is concerned with existence theorems of coincidence points of maps in the class $\mathfrak{B}^p$ with continuous functions, Φ-maps, locally selectionable maps with convex values, or approximable maps. In Section 5, we generalize the almost fixed point theorem due to Lassonde [7] to the maps in $\mathfrak{B}^p$. Section 6 deals with generalizations of the Himmelberg fixed point theorem in [5] to acyclic maps and others.

2. Better admissible maps

All topological spaces are assumed to be Hausdorff otherwise explicitly stated. A t.v.s. means a topological vector space and $\mathcal{V}$ denotes a fundamental system of neighborhoods of the origin 0 of a t.v.s. E. The convex hull and closure operation are denoted by co and $\overline{}$, respectively. We follow the terminology and notations in our previous work [20].

For topological spaces X and Y, a multimap (or map) $F : X \multimap Y$ is assumed to have nonempty values and said to be *closed* if its graph

$$\mathrm{Gr}(F) := \{(x, y) \mid y \in F(x),\ x \in X\}$$

is closed in $X \times Y$, and *compact* if $F(X)$ is contained in a compact subset of Y.

A map $F : X \multimap Y$ is said to be *upper semicontinuous* (*u.s.c.*) if, for each closed set $B \subset Y$, $F^-(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}$ is closed in X; *lower semicontinuous* (*l.s.c.*) if, for each open set $B \subset Y$, $F^-(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c.

If $F : X \multimap Y$ is u.s.c. with compact values, then F is closed. The converse is true whenever Y is compact.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a t.v.s., convex ⇒ star-shaped ⇒ contractible ⇒ ω-connected ⇒ acyclic ⇒ connected, and not conversely. Note that the Cartesian product of a finite number of acyclic spaces is acyclic by the Künneth theorem.

For topological spaces X and Y, a map $F : X \multimap Y$ is called a *Kakutani map* whenever Y is a subset of a t.v.s. and F is u.s.c. with compact convex values; and an *acyclic map* whenever F is u.s.c. with compact acyclic values.

Let $\mathbb{V}(X, Y)$ be the class of all acyclic maps $F : X \multimap Y$, and $\mathbb{V}_c(X, Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces.

A *polytope* P in a subset X of a t.v.s. E is a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

In 1996, the author introduced the 'better' admissible class $\mathfrak{B}$ of maps defined on a subset X of a t.v.s. E into a topological space Y as follows:

$F \in \mathfrak{B}(X, Y) \Leftrightarrow F : X \multimap Y$ is a map such that, for each polytope P in X and for any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

Subclasses of $\mathfrak{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$, the Aronszajn maps $\mathbb{M}$ (u.s.c. with R_δ values), the acyclic maps $\mathbb{V}$, the Powers maps $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the Fan–Browder maps (Φ-maps), locally selectionable maps $\mathbb{L}$ having convex values, u.s.c. approachable maps $\mathbb{A}$ (whose domains are uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}_c^\sigma$ of Lassonde, the class $\mathbb{V}_c^\sigma$ of Park et al., u.s.c. approximable maps of Ben-El-Mechaiekh and Idzik, and others. Those subclasses are examples of the admissible class $\mathfrak{A}_c^\kappa$ due to the author [9–12,22]. There are maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$, for example, the connectivity map due to Nash and Girolo; see [14]. Moreover, compact closed maps in the KKM class due to Chang and Yen and in the s-KKM class due to Chang et al. also belong to the class $\mathfrak{B}$. For details on $\mathfrak{B}$, see [13–16,20,21].

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset K of X and every $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E. Examples of admissible subsets can be seen in [14,16,20].

In 1997, we obtained the following:

Theorem 2.1. [13,14] *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

Moreover, in [14], we listed more than sixty papers in chronological order, from which we could deduce particular forms of Theorem 2.1. In [20], we obtained a generalized version of Theorem 2.1 by switching the admissibility of domain of the map to the Klee approximability of range.

A nonempty subset K of E is said to be *Klee approximable* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope of E. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

We give some examples of Klee approximable sets:

(1) If a subset X of E is admissible (in the sense of Klee), then every compact subset K of X is Klee approximable into E.
(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.
(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.
(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.
(5) Any compact subset K of an admissible convex subset X of a t.v.s. is Klee approximable into X.

Note that (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (5) will be generalized to an almost convex subset X in the next section.

We define a class $\mathfrak{B}^p$ of maps from a subset X of a t.v.s. E into a topological space Y as follows:

$F \in \mathfrak{B}^p(X,Y) \Leftrightarrow F : X \multimap Y$ is a map such that for any polytope P of X, there is a map $\Gamma \in \mathfrak{B}(P,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in P$.

We call $\mathfrak{B}^p$ also the 'better' admissible class. Note that any map $F \in \mathfrak{B}(X,Y)$ belongs to $\mathfrak{B}^p(X,Y)$. Recently it is known that any u.s.c. map with compact values having *trivial shape* (that is, contractible in each neighborhood) belongs to $\mathfrak{B}^p(X,Y)$; see [21].

The following generalizes Theorem 2.1 and the main theorem in [20]:

Theorem 2.2. *Let X and Y be subsets of a t.v.s. E such that $X \subset Y$ and $F : Y \multimap Y$ a map.*

(1) *If $F|_X \in \mathfrak{B}^p(X,Y)$ and $F(X)$ is Klee approximable into X, then $F|_X$ has the almost fixed point property (that is, for any $V \in \mathcal{V}$, $F|_X$ has a V-fixed point $x_V \in X$, that is, $F(x_V) \cap (x_V + V) \neq \emptyset$).*
(2) *Further if F is closed and $F|_X$ is compact, then F has a fixed point.*

Proof. (1) Let $V \in \mathcal{V}$. Since $F(X)$ is Klee approximable into X, there exist a continuous function $h : F(X) \to P$ for a polytope P in X such that $y - h(y) \in V$ for all $y \in F(X)$. Note that $F|_P : P \multimap F(X)$. Since $F|_X \in \mathfrak{B}^p(X,Y)$, there is a map $\Gamma \in \mathfrak{B}(P,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in P$. Since $\Gamma(P) \subset F(P) \subset F(X)$ and $h : F(X) \to P$, the composition $h\Gamma : P \multimap P$ has a fixed point $x_V \in h\Gamma(x_V)$. Let $x_V = h(y_V)$ for some $y_V \in \Gamma(x_V) \subset F(x_V) \subset F(X)$. We have $y_V - h(y_V) = y_V - x_V \in V$ and hence $y_V \in F(x_V) \cap (x_V + V)$.

(2) Since $\overline{F(X)}$ is a compact subset of Y, we may assume that the net y_V in $\overline{F(X)}$ converges to some $\hat{x} \in \overline{F(X)}$. Then the corresponding net x_V in X also converges to $\hat{x}$. Since the graph of F is closed and $(x_V, y_V) \in \mathrm{Gr}(F)$, we have $\hat{x} \in F(\hat{x})$. This completes our proof. □

Remark. In (1), E is not necessarily Hausdorff.

For $X = Y$, Theorem 2.2 reduces to the following main theorem of [20]:

Corollary 2.3. *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}^p(X,X)$ a compact closed map. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

It is well known that, for each locally finite open cover of a normal topological space, there is a partition of unity subordinated to it. From this fact, as in [21], we can deduce the following selection theorem, where Δ_n is the standard n-simplex:

Lemma 2.4. *Let K be a normal topological space, X a convex subset of a t.v.s. E, and $S : K \multimap X$ a map such that*

(1) *for each $x \in K$, $S(x)$ is convex*; *and*
(2) *$K = \bigcup_{i=0}^{n} \mathrm{Int}\, S^-(y_i)$ for some $\{y_i\}_{i=0}^n \subset X$.*

Then there exist continuous functions $g : \Delta_n \to \mathrm{co}\{y_i\}_{i=0}^n$ and $\phi : K \to \Delta_n$ such that $f = g\phi : K \to \mathrm{co}\{y_i\}_{i=0}^n$ is a continuous selection of S, that is, $f(x) \in S(x)$ for all $x \in K$.

For a topological space X and a subset Y of a t.v.s. E, a map $T : X \multimap Y$ is called a *Φ-map* or a *Fan–Browder map* provided that there exists a map $S : X \multimap Y$ such that

(1) for each $x \in X$, $\operatorname{co} Sx \subset Tx$; and
(2) $X = \bigcup\{\operatorname{Int} S^-(y) \mid y \in Y\}$.

Corollary 2.5. *Let X be a subset of a t.v.s. E and $T : X \multimap E$ a Φ-map. If $T(X)$ is Klee approximable into X, then T has the almost fixed point property.*

Proof. For any polytope P in X, it is covered by a finite number of $\{\operatorname{Int} S^-(y)\}$ by (2). Hence by Lemma 2.4, $T|_P$ has a continuous selection $f : P \to E$. Hence $T \in \mathfrak{B}^p(X, E)$. Since $T(X)$ is Klee approximable into X, by Theorem 2.2(1), T has the almost fixed point property. □

3. Fixed point theorems on almost convex sets

The following concept is well known; see [5]:

A subset X of a t.v.s. E is said to be *almost convex* if for any $V \in \mathcal{V}$ and for any finite subset $A := \{x_1, x_2, \ldots, x_n\}$ of X, there exists a subset $B := \{y_1, y_2, \ldots, y_n\}$ of X such that $y_i - x_i \in V$ for each $i = 1, 2, \ldots, n$ and $\operatorname{co} B \subset X$.

Note that for $V \in \mathcal{V}$ and for a finite dimensional subspace L of E generated by A and B, there is a symmetric convex open neighborhood V_L of 0 in L such that $V_L \subset L \cap V$ and $y_i - x_i \in V_L$ for each $i = 1, 2, \ldots, n$.

Moreover, in an almost convex set X, any finite subset is Klee approximable into X.

We give some examples of almost convex sets:

(1) Any convex subset is almost convex.
(2) If we delete a certain subset of the boundary of a closed convex set, then we obtain an almost convex set.
(3) Let $\mathcal{C}[0, 1]$ be the Banach space of all continuous real functions defined on the unit interval $[0, 1]$ and $\mathcal{P}[0, 1]$ its dense subset consisting of all polynomials. Then any subset of $\mathcal{C}[0, 1]$ containing $\mathcal{P}[0, 1]$ is almost convex. In general, by the various forms of the Stone–Weierstrass approximation theorem, we have a lot of examples of almost convex sets.

We give the following new example of Klee approximable subsets:

Lemma 3.1. *Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.*

Proof. Let $V \in \mathcal{V}$. Then there exists $W \in \mathcal{V}$ such that $W + W + W \subset V$.

(1) Since Y is admissible, there exists a continuous function $h_1 : K \to Y$ such that $y - h_1(y) \in W$ for all $y \in K$ and $h_1(K) \subset L$ for a finite dimensional subspace L of E. Since L is finite dimensional, there is a symmetric convex open neighborhood W_L of 0 in L such that $W_L \subset L \cap W$. Let us define a multimap $F : K \multimap L$ by $F(y) := h_1(y) + W_L$ for $y \in K$. Then each $F(y)$ is nonempty convex and open, $F^-(h_1(y)) := \{z \in K \mid h_1(y) \in F(z)\} = h_1^{-1}(h_1(y) - W_L)$ is open, and $y \in h_1^{-1}(h_1(y) - W_L) = F^-(h_1(y))$ for each $y \in K$. Therefore $K \subset \bigcup_{y \in K} F^-(h_1(y))$. Since K is compact, there exist $y_1, y_2, \ldots, y_n \in K$ such that $K \subset \bigcup_{i=1}^n F^-(h_1(y_i))$. Since K is normal, by Lemma 2.4, $F : K \multimap L$ has a continuous selection $p : K \to \operatorname{co}\{h_1(y_i)\}_{i=1}^n$ such that $p(y) \in F(y)$ for each $y \in K$. Moreover, $y - p(y) \in y - F(y) = y - h_1(y) + W_L \subset W + W$.

(2) Since X is a dense subset of Y and $A := \{h_1(y_i) \mid i = 1, 2, \ldots, n\}$ is a finite subset of Y, there exists a subset $B := \{x_i \mid i = 1, 2, \ldots, n\}$ of X such that $x_i - h_1(y_i) \in W$ for each i. Since

X is almost convex, there exists a subset $C := \{u_i \mid i = 1, 2, \ldots, n\}$ of X such that $u_i - x_i \in W$ for each i and $\operatorname{co} C \subset X$. Recall that $p(y) \in \operatorname{co} A$ for each $y \in K$. For the finite dimensional subspace M of E generated by $A \cup B \cup C$, there is a symmetric convex open neighborhood W_M of 0 in M such that $W_M + W_M \subset M \cap W$ and $x_i - h_1(y_i) \in W_M$, $u_i - x_i \in W_M$ for all i. Now the function $h_2' : A \to C$ defined by $h_2' h_1(y_i) := u_i$ for each i can be extended affinely to a continuous function $h_2 : \operatorname{co} A \to \operatorname{co} C$. Since W_M is convex, we have $v - h_2(v) \in W_M + W_M \subset W$ for all $v \in \operatorname{co} A$. In fact, for $v = \sum_{i=1}^n \alpha_i h_1(y_i)$ with $0 \leqslant \alpha_i \leqslant 1$ and $\sum_{i=1}^n \alpha_i = 1$, we have

$$v - h_2(v) = \sum_{i=1}^n \alpha_i h_1(y_i) - \sum_{i=1}^n \alpha_i u_i = \sum_{i=1}^n \alpha_i \big[h_1(y_i) - x_i + x_i - u_i\big].$$

Hence, $v - h_2(v) \in W_M + W_M \subset M \cap W$.

(3) Let $h := h_2 p : K \to \operatorname{co} C$. In order to claim that K is Klee approximable into X, it is sufficient to show $y - h(y) \in V$ for each $y \in K$. In fact, by both of the last statements of steps (1) and (2), we have $y - h(y) = y - p(y) + p(y) - h_2 p(y) \in W + W + W \subset V$. This completes our proof. □

For $X = Y$, Lemma 3.1 reduces to the following:

Corollary 3.2. *Every compact subset K of an almost convex admissible subset X of a t.v.s. E is Klee approximable into X.*

The following is a new example of admissible subsets:

Corollary 3.3. *Let X be an almost convex subset of a locally convex t.v.s. E. Then X is admissible.*

Proof. Since X is almost convex, it is a dense subset of $\operatorname{co} X$. Moreover, the closure of an almost convex set is convex. Therefore, $Y := \overline{X} = \operatorname{co} \overline{X}$ is a convex subset of a locally convex t.v.s. E and hence, admissible. Let K be a nonempty compact subset of X. Then it is a compact subset of Y, and hence, Klee approximable into X by Lemma 3.1. Therefore, X is admissible. □

From Theorem 2.2 and Lemma 3.1, we have the following main theorem of this section:

Theorem 3.4. *Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then any closed map $F : Y \multimap Y$ such that $F|_X \in \mathfrak{B}^p(X, Y)$ is compact has a fixed point.*

Proof. By Lemma 3.1, being a compact subset of an admissible subset Y, $K := \overline{F(X)}$ is Klee approximable into X. Since $F|_X \in \mathfrak{B}^p(X, Y)$ is compact and F is closed, the conclusion follows from Theorem 2.2. □

Corollary 3.5. *Let X be an almost convex admissible subset of a t.v.s. E and $F \in \mathfrak{B}^p(X, X)$ a compact closed map. Then F has a fixed point.*

Corollary 3.6. *Let X be an almost convex admissible subset of a t.v.s. E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

This is a rather surprising result since it generalizes Theorem 2.1, which contains results in more than sixty papers previously appeared; see [14]. Therefore, the convexity assumption on the domains of multimaps in these results can be replaced by the almost convexity.

From Corollaries 3.3 and 3.5, we have the following:

Theorem 3.7. *Let X be an almost convex subset of a locally convex t.v.s. E and $F \in \mathfrak{B}^p(X, X)$ a compact closed map. Then F has a fixed point.*

One of the most simple known example is that every compact continuous selfmap on an almost convex subset in an Euclidean space has a fixed point. This generalizes the Brouwer fixed point theorem.

Moreover, since the class $\mathfrak{B}^p(X, X)$ contains a large number of special types of maps, we can apply Theorem 3.7 to them. Here we give a well-known generalization of the Kakutani theorem [6]:

Corollary 3.8. *Let X be an almost convex subset of a locally convex t.v.s. E and $F : X \multimap X$ a compact Kakutani map. Then F has a fixed point.*

4. Coincidence theorems

For multimaps $F : X \multimap Y$ and $G : Y \multimap X$, a *coincidence point* $(x_0, y_0) \in X \times Y$ is the one satisfying $y_0 \in F(x_0)$ and $x_0 \in G(y_0)$ (that is, $x_0 \in X$ is a fixed point of $GF : X \multimap X$).

An equivalent definition is as follows: For multimaps $F : X \multimap Y$ and $G : X \multimap Y$, a *coincidence point* $x_0 \in X$ is the one satisfying $F(x_0) \cap G(x_0) \neq \emptyset$ (that is, $x_0 \in X$ is a fixed point of $G^- F : X \multimap X$).

In this section, we derive existence theorems of coincidence points of maps in the class $\mathfrak{B}^p$ with continuous functions, Φ-maps, locally selectionable maps with convex values, or approximable maps. Those coincidence theorems generalize or unifies known results.

We begin with the following generalization of results in [15]:

Theorem 4.1. *Let X be a subset of a t.v.s. E and Y a compact space. Let $G \in \mathfrak{B}^p(X, Y)$ be a closed map and $g \in \mathbb{C}(Y, X)$. If $(gG)(X)$ is Klee approximable into X, then G and g have a coincidence point, that is, there exists a point $(x_0, y_0) \in X \times Y$ such that $y_0 \in G(x_0)$ and $x_0 = g(y_0)$.*

Proof. Since $G \in \mathfrak{B}^p(X, Y)$ is closed and compact and $g \in \mathbb{C}(Y, X)$, their composition gG is closed and compact. We show that $gG \in \mathfrak{B}^p(X, X)$. In fact, since $G \in \mathfrak{B}^p(X, Y)$, for any polytope P in X, there exists a map $\Gamma \in \mathfrak{B}(P, Y)$ such that $\Gamma(x) \subset G(x)$ for each $x \in P$. Note that for any continuous function $f : (gG)(P) \to P$, the composition $fg(G|_P) = (fg)(G|_P) : P \multimap P$ has a fixed point since $G|_P$ has a selection $\Gamma \in \mathfrak{B}(P, Y)$. Therefore $gG \in \mathfrak{B}^p(X, X)$. Since $(gG)(X)$ is Klee approximable into X, by Corollary 2.3, gG has a fixed point $x_0 \in X$, that is, $x_0 \in (gG)(x_0)$. Then $x_0 = g(y_0)$ for some $y_0 \in G(x_0)$. □

In view of Corollary 3.5, we have

Corollary 4.2. *Let X be an almost convex admissible subset of a t.v.s. E and Y a compact space. Let $G \in \mathfrak{B}^p(X, Y)$ be a closed map and $g \in \mathbb{C}(Y, X)$. Then G and g have a coincidence point.*

For particular forms of Corollary 4.2, see [15].

Theorem 4.3. *Let X be a convex subset of a t.v.s. E and Y a normal topological space. Let $F \in \mathfrak{B}^p(X,Y)$ and $S,T : Y \multimap X$ maps such that*

(1) *for each $y \in Y$, $\operatorname{co} S(y) \subset T(y)$; and*
(2) *there exists a finite subset N of X such that*

$$X = \bigcup \{ \operatorname{Int} S^-(x) \mid x \in N \}.$$

Then F and T have a coincidence point.

Proof. By Lemma 2.4, there exist a continuous selection $g : Y \to X$ of T and a polytope P of X such that $g(Y) \subset P \subset X$. Since $F \in \mathfrak{B}^p(X,Y)$ and $g \in \mathbb{C}(Y,P)$, there is a map $\Gamma \in \mathfrak{B}(P,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in P$. Note that $f := g|_{\Gamma(P)} : \Gamma(P) \to P$ is continuous. Since $\Gamma \in \mathfrak{B}(P,X)$, $f\Gamma : P \multimap P$ has a fixed point $x_0 \in P \subset X$ such that

$$x_0 \in (f\Gamma)(x_0) = (g|_{\Gamma(P)})\big(\Gamma(x_0)\big) \subset (gF)(x_0).$$

Hence $x_0 = g(y_0)$ for some $y_0 \in F(x_0)$. On the other hand, $x_0 = g(y_0) \in T(y_0)$ since g is a selection of T. This completes our proof. □

Corollary 4.4. *Let X be a convex subset of a t.v.s. E and Y a normal topological space. Let $F : X \multimap Y$ be an acyclic map and $S,T : Y \multimap X$ multimaps satisfying conditions* (1) *and* (2) *of Theorem* 4.3. *Then F and T have a coincidence point.*

A particular form of Corollary 4.4 for a Kakutani map F was given by Browder [4, Theorem 7], which is the origin of Theorem 4.3.

For topological spaces X and Y, a multimap $T : X \multimap Y$ is said to be *selectionable* if it has a continuous selection $f : X \to Y$ (that is, $f(x) \in T(x)$ for all $x \in X$), and *locally selectionable* if for each $x_0 \in X$, there exist an open neighborhood V_0 of x_0 and a continuous function $f_0 : V_0 \to Y$ such that $f_0(x) \in T(x)$ for all $x \in V_0$; see [17].

Any continuous function is selectionable and there are lots of examples of selectionable maps due to Michael and others. Any selectionable map is locally selectionable.

For a subset X of a t.v.s. E and a topological space Y, we define a class of multimaps as follows:

$T \in \mathbb{M}^*(Y,X) \Leftrightarrow T : Y \multimap X$ is a map such that $T|_K$ has a continuous selection $s : K \to X$ for each nonempty compact subset K of Y such that $s(K) \subset P$ for some polytope P of X.

From the proof of Theorem 4.3, we have the following:

Theorem 4.5. *Let X be a convex subset of a t.v.s. E and Y a topological space. Let $F \in \mathfrak{B}^p(X,Y)$ be a compact map and $T \in \mathbb{M}^*(Y,X)$. Then F and T have a coincidence point.*

For particular forms of Theorem 4.5, see [12].

The following is given [17]:

Lemma 4.6. *Let X be a convex subset of a t.v.s. E and Y a paracompact topological space. Then*

(1) *any Φ-map $T : Y \multimap X$ is locally selectionable*; *and*
(2) *any locally selectionable map $T : Y \multimap X$ having convex values is selectionable.*

Theorem 4.7. *Let X be an almost convex admissible subset of a t.v.s. E_1 and Y a convex subset of a t.v.s. E_2. Let $G \in \mathfrak{B}^p(X, Y)$ be a compact closed map and $H : Y \multimap X$ a locally selectionable map with convex values. Then G and H have a coincidence point.*

Proof. Since G is compact, there exists a compact subset K of Y such that $G(X) \subset K$. Since K is compact, it is known that $\mathrm{co}\, K$ is paracompact. Then, by Lemma 4.6, $H|_{\mathrm{co}\, K}$ has a continuous selection $h : \mathrm{co}\, K \to X$. We claim that $hG \in \mathfrak{B}^p(X, X)$. In fact, for each polytope P in X and for any continuous function $f : (hG)(P) \to P$, we have $fh : G(P) \to P$ and $(fh)(G|_P) = f((hG)|_P) : P \multimap P$ has a fixed point since $G \in \mathfrak{B}^p(X, Y)$. This shows $hG \in \mathfrak{B}^p(X, X)$. Since X is almost convex and admissible, by Corollary 3.5, the compact closed map hG has a fixed point $x_0 \in X$, that is, $x_0 \in (hG)(x_0)$. Then $x_0 = h(y_0)$ for some $y_0 \in G(x_0) \subset K \subset Y$. Moreover, $x_0 = h(y_0) \in H(y_0)$. This completes our proof. □

Let X and Y be subsets of t.v.s. E and F, respectively, and $T : X \multimap Y$ a map. Given two open neighborhoods U and V of the origin 0 of E and F, respectively, a (U, V)-*approximative continuous selection* of T is a continuous function $s : X \to Y$ satisfying

$$s(x) \in \big(T\big[(x + U) \cap X\big] + V\big) \cap Y \quad \text{for every } x \in X.$$

T is said to be *approachable* if it admits a (U, V)-approximative continuous selection for every U and V as above; and T *approximable* if its restriction $T|_K$ to any compact subset K of X is approachable; see [1,2]. Note that an approachable map is always approximable.

Let $\mathbb{A}^\kappa(X, Y)$ denote the class of all u.s.c. approximable maps $T : X \multimap Y$ with compact values, and $\mathbb{A}^\kappa_c(X, Y)$ the class of all finite compositions $T : X \multimap Y$ of u.s.c. approximable maps with compact values, where the intermediate spaces are subsets of t.v.s. Recall that the class $\mathbb{A}^\kappa_c$ is an example of the admissible class $\mathfrak{A}^\kappa_c$ and the better admissible class $\mathfrak{B}^\kappa$ due to the author [10,12,14].

For approximable maps having compact domains, we need the following:

Lemma 4.8. [2] *Let X be a compact subset of a t.v.s., Y a subset of a t.v.s., and Γ a closed subset of $X \times Y$. Then the following statements are equivalent*:

(1) $\mathrm{Gr}(f) \cap \Gamma \neq \emptyset$ *for each* $f \in \mathbb{C}(X, Y)$;
(2) $\mathrm{Gr}(T) \cap \Gamma \neq \emptyset$ *for each* $T \in \mathbb{A}^\kappa_c(X, Y)$.

Theorem 4.9. *Let X be an almost convex admissible subset of a t.v.s. E_1 and Y a compact subset of a t.v.s. E_2. Let $G \in \mathfrak{B}^p(X, Y)$ be a closed map and $H \in \mathbb{A}^\kappa_c(Y, X)$. Then G and H have a coincidence point.*

Proof. For any $f \in \mathbb{C}(Y, X)$, we know that $fG \in \mathfrak{B}^p(X, X)$ is compact and closed as in the proof of Theorem 4.1. Since X is almost convex and admissible, fG has a fixed point by Corollary 3.5, and hence $\mathrm{Gr}(f) \cap \mathrm{Gr}(G^-) \neq \emptyset$. Since $\mathrm{Gr}(G^-)$ is a closed subset of $Y \times X$, by Lemma 4.8 with interchanging X and Y, $\mathrm{Gr}(H) \cap \mathrm{Gr}(G^-) \neq \emptyset$. Hence there exists a point $(y_0, x_0) \in Y \times X$ such that $x_0 \in H(y_0)$ and $x_0 \in G^-(y_0)$ or $y_0 \in G(x_0)$. This completes our proof. □

The following Simons type cyclic coincidence theorem for acyclic maps is an application of our fixed point theorems.

Let $\mathbb{Z}_{n+1} := \{0, 1, \ldots, n\}$ with $n+1$ interpreted as 0.

Theorem 4.10. *Let $n \geqslant 0$ and, for each $i \in \mathbb{Z}_{n+1}$, let Y_i be a subset in a t.v.s. E_i, and $V_i \in \mathbb{V}(Y_i, Y_{i+1})$ be compact. If $Y := \prod_{i=0}^n Y_i$ is Klee approximable into itself in the t.v.s. $E := \prod_{i=0}^n E_i$, then there exists $y := (y_i)_{i=0}^n \in Y$ such that $y_{i+1} \in V_i(y_i)$ for all $i \in \mathbb{Z}_{n+1}$.*

Proof. Case 1 ($n=0$). This follows from Corollary 2.3 for $\mathbb{V}_c \subset \mathfrak{B}^p$.

Case 2 ($n \geqslant 1$). Define $V : Y \multimap Y$ by

$$V(y) := V_n(y_n) \times V_0(y_0) \times \cdots \times V_{n-1}(y_{n-1})$$

for $y = (y_i)_{i=0}^n \in Y$. Note that $V \in \mathbb{V}(Y, Y)$ by the Künneth theorem and that V is compact. Moreover, $V(Y)$ is Klee approximable into Y. Therefore, by Corollary 2.3 or case 1, V has a fixed point $x \in V(x)$. This completes our proof. □

Particular forms of Theorem 4.10 were given by Simons, Lassonde, and Park; see [18].

5. The Lassonde type almost fixed point theorems

In this section, we generalize an almost fixed point theorem due to Lassonde [7] and obtain a fixed point theorem for the class $\mathfrak{B}^p$ of maps on locally convex t.v.s.

Theorem 5.1. *Let X and C be nonempty convex subsets of a locally convex t.v.s. E, and $F \in \mathfrak{B}^p(X, X+C)$ a compact closed map. Suppose that one of the following conditions holds*:

(i) *X is closed and C is compact.*
(ii) *X is compact and C is closed.*
(iii) *$C = \{0\}$.*

Then there is $\hat{x} \in X$ such that $F(\hat{x}) \cap (\hat{x} + C) \neq \emptyset$.

Proof. Let V be an open convex neighborhood of the origin 0 in E, and Y a compact set such that $F(X) \subset Y \subset X + C$. Define $G : X \to 2^Y$ by $G(x) := (x + C + V) \cap Y$ for $x \in X$. Then each $G(x)$ is open in Y and $G^-(y) = (y - C - V) \cap X$ is convex for each $y \in Y$. Moreover, since $Y \subset X + C$, for every $y \in Y$, there exists $x \in X$ such that $y \in x + C + V$, that is, $\{G(x) \mid x \in X\}$ covers Y. Therefore, by Lemma 2.4 and Theorem 4.5, F and $G^- : Y \multimap X$ has a coincidence point, that is, there exist $x_V \in X$ and $y_V \in Y$ such that $y_V \in F(x_V) \cap G(x_V)$, that is, $y_V - x_V \in C + V$. In other words, we obtain the assertion:

($*$) for each neighborhood V of 0 in E,

$$(F - i)(X) \cap (C + V) \neq \emptyset,$$

where $i : X \to E$ is the inclusion. Now we consider cases (i)–(iii).

Case (i). Since X is closed, so is $(F - i)(X)$. Since C is compact and E is regular, ($*$) implies $(F - i)(X) \cap C \neq \emptyset$, that is, there exists $\hat{x} \in X$ such that $F(\hat{x}) \cap (\hat{x} + C) \neq \emptyset$.

Case (ii). Since $(F - i)(X)$ is compact and C is closed, the same conclusion follows as in case (i).

Case (iii). Since F is u.s.c., for each neighborhood V of 0 in E, there exist $x_V, y_V \in X$ such that $y_V \in F(x_V)$ and $y_V - x_V \in V$. Since $F(X)$ is relatively compact, we may assume that y_V converges to some $\hat{x}$. Then x_V also converges to $\hat{x}$. Since the graph of F is closed in $X \times \overline{F(X)}$, we have $\hat{x} \in F(\hat{x})$.

This completes our proof. □

Remark. Lassonde [7] first obtained the particular form of Theorem 5.1 for the class $\mathbb{K}$. Since then it is generalized by the present author to $\mathbb{V}$, $\mathbb{V}_c$, $\mathfrak{A}_c$, and $\mathfrak{B}$ by step by step; see [8,14,16].

The case (iii) is a particular form of Theorem 3.7 and reduces to Himmelberg [5, Theorem 2] when F is a Kakutani map.

6. The Himmelberg type fixed point theorems

In this section, as applications of Theorem 3.4, we deduce generalized versions of results of Himmelberg in [5]. For simplicity, we will state them for maps having acyclic values or values of trivial shape.

Theorem 6.1. *Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Let $G : Y \multimap Y$ be a compact closed map such that $G(x)$ is acyclic* (*respectively has trivial shape*) *for all $x \in X$. Then G has a fixed point.*

Proof. If $G|_X$ has acyclic values (or values of trivial shape), we have $G|_X \in \mathfrak{B}^p(X, Y)$. Then the conclusion follows from Theorem 3.4. □

Corollary 6.2. *Let X be an almost convex dense subset of a closed subset Y of a locally convex t.v.s. E. Let $G : Y \multimap Y$ be a compact u.s.c. multimap with closed values such that $G(x)$ is acyclic for all $x \in X$. Then G has a fixed point.*

Proof. Since $Y = \overline{Y} = \overline{X}$ and the closure of an almost convex set is a convex set, Y is a convex subset of a locally convex t.v.s. E. Hence, by Corollary 3.3, Y itself is admissible. Now the conclusion follows from Theorem 3.4. □

Remark. If Y is compact, then Corollary 6.2 reduces to Himmelberg [5, Theorem 1]; see also [19, Corollary 1]. Note that our proof is quite different from the one in [5] or [19].

Corollary 6.3. *Let X be an almost convex admissible subset of a t.v.s. E. Then any compact closed map $G : X \multimap X$ such that $G(x)$ is acyclic* (*respectively has trivial shape*) *for all $x \in X$ has a fixed point.*

Proof. The conclusion follows from Theorem 6.1 for $X = Y$. □

Remark. If X is a convex subset of a locally convex t.v.s. E and G has convex values, then Corollary 6.3 reduces to Himmelberg [5, Theorem 2]. For a convex subset X of a locally convex t.v.s. E, Corollary 6.3 is obtained in [8] as a particular form of Theorem 5.1.

A lot of applications of [5, Theorem 2] are known.

As in Himmelberg [5], we conclude from Theorem 6.1 the following generalization of Ky Fan's results in [4]:

Theorem 6.4. *Let $\{E_i\}_{i=1}^n$ be a family of t.v.s. For each i, let X_i be an almost convex dense subset of an admissible subset Y_i of E_i, and K_i a compact subset of Y_i. Let $\{A_i\}_{i=1}^n$ be a family of closed subsets of $Y := \prod_{i=1}^n Y_i$ such that, for each i, the section $A_i(y^i)$ is acyclic (respectively has trivial shape) for all $y^i \in X^i := \prod_{j\neq i} X_j$ and nonempty for all $y^i \in Y^i := \prod_{j\neq i} Y_j$. Then $\bigcap_{i=1}^n A_i \neq \emptyset$.*

Proof. Each A_i is a map from Y^i to K_i. Define $F_i : Y \multimap K_i$ by $F_i(y) := A_i(y^i)$, where y^i is the projection of y on Y^i. Then F_i is u.s.c. with closed values, and hence closed. Define $F : Y \multimap Y$ by $F(y) := \prod_{i=1}^n F_i(y)$. Then it can be checked that F has closed graph, that Y is admissible, that X is almost convex and dense in Y, that $F(y)$ is acyclic (respectively has trivial shape) for all $y \in X$, and that $F(y) \neq \emptyset$ for all $y \in Y$. Moreover, F is compact. Thus, by Theorem 6.1, F has a fixed point. It belongs to each A_i. □

Remarks.

(1) If each $A_i(y^i)$ is assumed to be convex and each E_i is locally convex, then Theorem 6.4 holds for a family not-necessarily finite. In this case, Theorem 6.4 is due to Himmelberg [5, Theorem 3].
(2) More generally, if X is Klee approximable into Y, then a simplified form of Theorem 6.4 holds.

As in [5, Theorem 4], we can deduce from Theorem 6.4 the following:

Theorem 6.5. *Let Y_1, Y_2 be compact admissible subsets of t.v.s. E_1, E_2, respectively, and X_1, X_2 be almost convex dense subsets of Y_1, Y_2, respectively. Let f be a continuous real function defined on $Y_1 \times Y_2$ such that for any $x_1 \in X_1$, $y_2 \in X_2$, the sets*

$$\Big\{x \in Y_1 \;\Big|\; f(x, y_2) = \max_{\xi \in Y_1} f(\xi, y_2)\Big\}$$

and

$$\Big\{y \in Y_2 \;\Big|\; f(x_1, y) = \min_{\eta \in Y_2} f(x_1, \eta)\Big\}$$

are acyclic (respectively has trivial shape), then

$$\max_{x\in Y_1} \min_{y\in Y_2} f(x, y) = \min_{y\in Y_2} \max_{x\in Y_1} f(x, y).$$

Remark. When Y_1, Y_2 are subsets of locally convex t.v.s. E_1, E_2, respectively, and if we assume the convexity instead of the acyclicity, Theorem 6.5 reduces to [5, Theorem 4].

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