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REMARKS ON $\mathfrak{KC}$-MAPS AND $\mathfrak{KO}$-MAPS IN ABSTRACT CONVEX SPACES

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ABSTRACT. We study the mutual relations among multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ on abstract or generalized convex spaces. We show also that the examples given by Jeng, Huang, and Zhang [6] can be used to deduce more examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps. Finally, some historical remarks on classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ are added.

1. Introduction

In early 1990's, the author introduced the admissible class $\mathfrak{A}_c^\kappa$ of multimaps in the KKM theory on topological vector spaces. Since then there have appeared new classes of multimaps such as the KKM class, the S-KKM class, the 'better' admissible class $\mathfrak{B}$, and modifications of them. Those classes of multimaps were first applied to convex spaces and later to generalized convex spaces (simply, G-convex spaces).

Recently, in [18], we introduced a new concept of abstract convex spaces and multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ having certain KKM property. These new spaces and multimap classes are known to be adequate to establish the KKM theory. Within such new framework, in [20], we generalized and simplified known results of the theory on convex spaces, H-spaces, G-convex spaces, and others. It is noticed that the class $\mathfrak{KC}$ contains the KKM class and the S-KKM class.

In the present paper, our aim is to study the mutual relations among multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ in abstract or G-convex spaces. We show also that the interesting new examples of $\mathfrak{KC}$-maps given by Jeng, Huang, and Zhang [6] can be used to deduce more examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps. Finally, some historical remarks on multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ are added.

Section 2 deals with preliminaries on definitions and examples of abstract convex spaces and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$. In Section 3, we introduce the multimap class $\mathfrak{B}$ on G-convex spaces and some basic properties of normal spaces (not necessarily Hausdorff). Section 4 deals with the relations among three multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$. In Section 5, we are concerned with the examples of $\mathfrak{KC}$-maps given by Jeng, Huang, and Zhang [6] and comments on them. Actually, we show that their

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examples can be used to deduce more examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps. Finally, Section 6 deals with historical remarks on multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$.

2. Abstract convex spaces and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$

This section is concerned with definitions and examples of abstract convex spaces and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

Definitions. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

[co is reserved for the convex hull in topological vector spaces.] A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if, for any $A \in \langle X \cap D \rangle$, we have $\Gamma_A \subset X$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

If E is given a topology, then the abstract convex space $(E, D; \Gamma)$ is called *an abstract convex topological space.*

Examples of abstract convex spaces are given in [18,20].

Definitions. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. When $E = Z$, a *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called *a* $\mathfrak{K}$-*map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical.

Examples 2.1. Every abstract convex space in our sense has a map $F \in \mathfrak{K}(E, Z)$ for any nonempty set Z. In fact, for each $x \in E$, choose $F(x) := Z$ or $F(x) := \{z_0\}$ for some $z_0 \in Z$.

The following is a typical example of abstract convex spaces:

Definition. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

We have established a large amount of literature on G-convex spaces; see [19,20] and references therein.

Consider the following condition for a G-convex space $(X \supset D; \Gamma)$ as in [20]:

$(*)$ $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$; and, for each $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that $\phi_N(\Delta_n) = \Gamma_N$ and that $J \in \langle N \rangle$ implies $\phi_N(\Delta_J) = \Gamma_J$.

Definition. For an abstract convex topological space $(E, D; \Gamma)$, the *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$. A *KKM space* is an abstract convex topological space satisfying the KKM principle.

Examples 2.2. Even for generalized convex spaces $(X, D; \Gamma)$, in general, the identity map $1_X \notin \mathfrak{K}(X, X)$. We give examples of KKM spaces as follows:

1. If $X = \Delta_n$ is an n-simplex, D is the set of its vertices, $\Gamma = \text{co}$ is the convex hull operation, then the celebrated KKM principle says that $1_X \in \mathfrak{KC}(X, X)$. Later, it is known that $1_X \in \mathfrak{KO}(X, X)$.

2. If D is a nonempty subset of a topological vector space (simply, t.v.s.) E (not necessarily Hausdorff), Fan's KKM lemma [5] says that $1_E \in \mathfrak{KC}(E, E)$.

3. If X is a subset of a vector space, $D \subset X$ such that $\text{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology, then $(X, D; \Gamma)$ is called a *convex space*. When $X = D$ is convex, this concept reduces to the one due to Lassonde; see [18]. Note that any convex subset of a t.v.s. is a convex space, but not conversely.

Note that a convex space $(X \supset D; \Gamma)$ satisfies the condition $(*)$.

4. For a G-convex space $(X, D; \Gamma)$, we showed that $1_X \in \mathfrak{KC}(X, X) \cap \mathfrak{KO}(X, X)$. Moreover, for a topological space Z, we noted that if $F : X \to Z$ is a continuous single-valued map or if $F : X \multimap Z$ has a continuous selection, then $F \in \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$.

For further examples and references, see [18,20]. Recently, it is known that the class of KKM spaces properly contains G-convex spaces.

3. Multimap classes $\mathfrak{B}$

In this section, we introduce the class $\mathfrak{B}$ on G-convex spaces and some basic properties of normal spaces (not necessarily Hausdorff).

Definition. Let $(X, D; \Gamma)$ be a G-convex space and Z a topological space. The *better admissible class* $\mathfrak{B}$ of multimaps from X into Z is defined as follows:

$F \in \mathfrak{B}(X, Z) \iff F : X \multimap Z$ is a map such that for any $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$ and any continuous function $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n)$.

We give some subclasses of $\mathfrak{B}$ as follows:

Examples 3.1. For topological spaces X and Y, an *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $F : X \multimap Y$ is one such that, for each nonempty compact subset K of X, there exists a map $G \in \mathfrak{A}_c(K, Y)$ satisfying $G(x) \subset F(x)$ for all $x \in K$; where $\mathfrak{A}_c$ consists of finite compositions of maps in a class $\mathfrak{A}$ of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $T \in \mathfrak{A}_c$ is u.s.c. with nonempty compact values; and
(iii) for any polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Here, a polytope P is a homeomorphic image of a standard simplex.

Subclasses of the admissible class $\mathfrak{A}_c^\kappa$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}_c^\sigma$ of Lassonde, the class $\mathbb{V}_c^\sigma$ of Park et al., approximable maps of Ben-El-Mechaiekh and Idzik, and many others.

Note that for a G-convex space $(X, D; \Gamma)$ and any space Y, an admissible class $\mathfrak{A}_c^\kappa(X, Y)$ is a subclass of $\mathfrak{B}(X, Y)$ with some possible exceptions such as Kakutani maps; see [21]. Some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ were known; for example, the connectivity map due to Nash and Girolo; see [14].

Examples 3.2. We give a general definition of Kakutani maps as follows:

Definition. Let Y be a topological space and $(X \supset D; \Gamma)$ a G-convex space. A map $F : Y \multimap X$ is called a *Kakutani map* if it is u.s.c. and has nonempty compact Γ-convex values.

For Kakutani maps the following are known [21]:

(1) Let $(X, D; \Gamma)$ be a G-convex space and $(Y \supset D'; \Sigma)$ a Hausdorff LG-space. If $F : X \multimap Y$ is a Kakutani map, then $F \in \mathfrak{B}(X, Y) \cap \mathfrak{KC}(X, Y)$.

(2) Let $(X; \Gamma; \mathcal{U})$ be a locally G-convex space. Then any Kakutani map $F : X \multimap X$ belongs to $\mathfrak{B}(X, X)$.

Examples 3.3. An important subclass of $\mathfrak{B}$ is the class of Φ-maps (or Fan-Browder maps) as follows:

Definition. Let Y be a topological space and $(X, D; \Gamma)$ a G-convex space. Then a map $T : Y \multimap X$ is called a Φ-*map* (or a *Fan-Browder map*) if there is a map $S : Y \multimap D$ such that
(i) for each $y \in Y$, $\mathrm{co}_\Gamma S(y) \subset T(y)$ [that is, $T(y)$ is Γ-convex relative to $S(y)$]; and
(ii) $Y = \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in D\}$.

The following is known [20]:

Lemma 3.1. *Let $(X, D; \Gamma)$ be a G-convex space and Z a topological space. Then*
(1) $\mathbb{C}(X, Z) \subset \mathfrak{A}_c^\kappa(X, Z) \subset \mathfrak{B}(X, Z)$;
(2) $\mathbb{C}(X, Z) \subset \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$; *and*
(3) [8] $\mathfrak{A}_c^\kappa(X, Z) \subset \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$ *if Z is Hausdorff.*

Lemma 3.2 [20, Theorem 16]**.** *Let $(X, D; \Gamma)$ be a G-convex space and Z a topological space.*
(1) *If Z is a Hausdorff space, then every compact map $F \in \mathfrak{B}(X, Z)$ belongs to $\mathfrak{KC}(X, Z)$.*
(2) *If $F : X \multimap Z$ is a closed map such that $F\phi_N \in \mathfrak{KC}(\Delta_n, Z)$ for any $N \in \langle D\rangle$ with the cardinality $|N| = n + 1$, then $F \in \mathfrak{B}(X, Z)$.*
(3) *In the class of closed maps defined on a G-convex space $(X \supset D; \Gamma)$ satisfying condition $(*)$ into a space Z, a map $F \in \mathfrak{KC}(X, Z)$ belongs to $\mathfrak{B}(X, Z)$.*

Remark. There are non-closed maps in $\mathfrak{B}(X, X) \cap \mathfrak{KC}(X, X)$ for any G-convex space $(X \supset D; \Gamma)$; for example, a Fan-Browder map.

We recall the following properties of normal spaces; see Bourbaki [1] and Lefschetz [9].

Lemma 3.3. *A topological space X is normal iff every point-finite open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ of X is shrinkable, that is, an open cover $\mathcal{V} = \{V_\alpha \mid \alpha \in I\}$ exists such that $\overline{V}_\alpha \subset U_\alpha$ for each $\alpha \in I$.*

Lemma 3.4. *Let X be a normal space and let $\{A_1, \ldots, A_n\}$ a finite family of closed subsets of X. Then there exists a ("thickening") family of open sets of X such that $A_i \subset U_i$ for all $1 \le i \le n$, and for every $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, we have*

$$A_{i_1} \cap \cdots \cap A_{i_k} = \emptyset \quad \textit{iff} \quad U_{i_1} \cap \cdots \cap U_{i_k} = \emptyset.$$

The following continuous selection theorem for multimaps with noncompact domain is given in view of [15, Lemma 1]:

Lemma 3.5. *Let X be a normal space, $(Y, D; \Gamma)$ a G-convex space, and $S : X \multimap D$ a map such that $X = \bigcup\{\text{Int } S^-(y) \mid y \in A\}$ for some $A \in \langle D \rangle$. Then there exists a continuous function $s : X \to \Gamma_A$ such that $s(x) \in \Gamma(A \cap S(x))$ for all $x \in X$. In fact, if $|A| = n+1$, then $s = \phi_A \circ p$, where $\phi_A : \Delta_n \to \Gamma_A$ and $p : X \to \Delta_n$ are continuous functions.*

Proof. Let $A := \{y_0, y_1, \dots, y_n\} \in \langle D \rangle$. Since X is normal, there exists a partition of unity $\{p_i\}_{i=0}^n$ subordinated to the open cover $\{\text{Int } S^-(y_i)\}_{i=0}^n$. Define a continuous function $p : X \to \Delta_n$ by $p(x) := \sum_{i=0}^n p_i(x)e_i$. Let $s := \phi_A \circ p$ and, for each $x \in X$, let $J_x := \{y_i \mid p_i(x) \neq 0\} \subset A$. Then

$$s(x) = \phi_A \circ p(x) = \phi_A(\sum_{y_i \in J_x} p_i(x)e_i) \in \phi_A(\Delta_{J_x}).$$

Hence, we have

$$y_i \in J_x \Leftrightarrow p_i(x) \neq 0 \Rightarrow x \in S^-(y_i) \Leftrightarrow y_i \in S(x) \cap A.$$

So $s(x) \in \phi_A(\Delta_{J_x}) \subset \Gamma(A \cap S(x))$ for each $x \in X$. This completes our proof.

4. Relations among multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$

The relations among three multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ are investigated in this section.

For topological spaces X, Y and a map $F : X \multimap Y$, let $\overline{F} : X \multimap Y$ be a map defined by $\overline{F}(x) := \overline{F(x)}$ for all $x \in X$.

The following enables us to give lots of examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps:

Theorem 4.1. *Let $(E, D; \Gamma)$ be an abstract convex topological space, Z a topological space, and $F, F' : X \multimap Z$ multimaps such that $F(x) \subset F'(x)$ for each $x \in X$.*
(1) If $F \in \mathfrak{KC}(E, Z)$, then $F' \in \mathfrak{KC}(E, Z)$.
(2) If $F \in \mathfrak{KO}(E, Z)$, then $F' \in \mathfrak{KO}(E, Z)$.
(3) $F \in \mathfrak{KC}(E, Z)$ if and only if $\overline{F} \in \mathfrak{KC}(E, Z)$.
(4) If $F \in \mathfrak{KO}(E, Z)$, then $\overline{F} \in \mathfrak{KO}(E, Z)$.

Proof. (1) Let $F \in \mathfrak{KC}(E, Z)$ and let $G : D \multimap Z$ be a closed-valued KKM map with respect to F'; that is,

$$F'(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle.$$

Then

$$F(\Gamma_A) \subset F'(\Gamma_A) \subset G(A) \qquad \text{for all } A \in \langle D \rangle.$$

Since $F \in \mathfrak{KC}(E, Z)$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. Therefore $F' \in \mathfrak{KC}(E, Z)$.

(2) Similar.

(3) For 'if' part, let $\overline{F} \in \mathfrak{KC}(E, Z)$ and let $G : D \multimap Z$ be a closed-valued KKM map with respect to F; that is,

$$F(\Gamma_A) \subset G(A) \qquad \text{for all } A \in \langle D \rangle.$$

Since $G(A)$ is closed, we immediately have

$$F(\Gamma_A) \subset \overline{F}(\Gamma_A) \subset \overline{F(\Gamma_A)} \subset \overline{G(A)} = G(A) \qquad \text{for all } A \in \langle D \rangle.$$

Since $\overline{F} \in \mathfrak{KC}(E, Z)$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. Therefore $F \in \mathfrak{KC}(E, Z)$.

The 'only if' part follows from (1).

(4) This follows from (2).

Example 4.1. Let $(X, D; \Gamma)$ be a G-convex space and Z a Hausdorff convex space. Then any Kakutani map belongs to $\mathfrak{A}_c^\kappa(X, Z) \subset \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$ by Lemma 3.1(3). Now, by Theorem 4.1(3), any u.s.c. map $F : X \multimap Z$ having nonempty convex values belongs to $\mathfrak{KC}(X, Z)$ whenever Z is compact.

Theorem 4.2. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, and $F : E \multimap Z$. Suppose that for any $A \in \langle D \rangle$ with $|A| = n + 1$, the set $F(\Gamma_A)$ in its induced topology is a normal space. If $F \in \mathfrak{KC}(E, Z)$, then $F \in \mathfrak{KO}(E, Z)$. The converse also holds.*

Proof. Let $F \in \mathfrak{KC}(E, Z)$ and $G : D \multimap Z$ an open-valued KKM map with respect to F. Let $A \in \langle D \rangle$. Then $F(\Gamma_A) \subset G(A)$. Since $\{F(\Gamma_A) \cap G(y)\}_{y \in A}$ is an open cover of the normal space $F(\Gamma_A)$, by Lemma 3.3, there exists an open cover $\{V_y\}_{y \in A}$ of $F(\Gamma_A)$ such that $\overline{V_y} \subset G(y)$, where $\overline{V_y}$ is the intersection of $F(\Gamma_A)$ and a closed subset of Z and we denote this closed set by $\overline{V_y}$. Let $H : D \multimap Z$ be a map defined by

$$H(y) := \begin{cases} \overline{V}_y & \text{for } y \in A; \\ Z & \text{for } y \in D \setminus A. \end{cases}$$

Then H is a closed-valued KKM map with respect to F. Since $F \in \mathfrak{KC}(E, Z)$, $\{H(y)\}_{y \in D}$ has the finite intersection property. Hence $\bigcap_{y \in A} G(y) \supset \bigcap_{y \in A} H(y) \neq \emptyset$. Therefore, $\{G(y)\}_{y \in D}$ has the finite intersection property. This establishes the first assertion.

Conversely, let $F \in \mathfrak{KO}(E, Z)$ and $G : D \multimap Z$ a closed-valued KKM map with respect to F. Let $A \in \langle D \rangle$. Then $F(\Gamma_A) \subset G(A)$. Since $\{F(\Gamma_A) \cap G(y)\}_{y \in A}$ is a finite family of closed subsets of the normal space $F(\Gamma_A)$, by Lemma 3.4, there exists a family $\{U_y\}_{y \in A}$ of open subsets of $F(\Gamma_A)$ such that $G(y) \subset U_y$ for all $y \in A$ and for every $J \subset A$, we have $\bigcap_{z \in J} U_z \neq \emptyset$ iff $\bigcap_{z \in J} G(z) \neq \emptyset$. Let $H : D \multimap Z$ be a map defined by

$$H(y) := \begin{cases} U_y & \text{for } y \in A; \\ Z & \text{for } y \in D \setminus A. \end{cases}$$

Then H is an open-valued KKM map with respect to F as the first case. Since $F \in \mathfrak{KO}(E, Z)$, $\{H(y)\}_{y \in D}$ has the finite intersection property. Hence $\bigcap_{y \in A} U_y \neq \emptyset$. This is equivalent to $\bigcap_{y \in A} G(y) \neq \emptyset$. Therefore, $\{G(y)\}_{y \in D}$ has the finite intersection property. This establishes the second assertion.

Remark. Closely examining the above proof, for the converse case, we may assume the set $\overline{F(\Gamma_A)}$ in its induced topology is a normal space. Therefore we have

Corollary 4.3. *Let $(E, D; \Gamma)$ be an abstract convex space and Z a normal space. Then $\mathfrak{KO}(E, Z) \subset \mathfrak{KC}(E, Z)$.*

Similarly, we have

Theorem 4.4. *Let $(X, D; \Gamma)$ be a G-convex space, Z a topological space, and $F : X \multimap Z$. Suppose that for any $A \in \langle D \rangle$ with $|A| = n + 1$, the set $K_A := F(\phi_A(\Delta_n))$ in its induced topology is a normal space. If $F \in \mathfrak{KC}(X, Z)$, then $F \in \mathfrak{KO}(X, Z)$. The converse also holds.*

Proof. Let $F \in \mathfrak{KC}(X, Z)$ [resp. $F \in \mathfrak{KO}(X, Z)$] and $G : D \multimap Z$ an open-valued [resp. a closed-valued] KKM map with respect to F. Let $A \in \langle D \rangle$. Then, follow the proof of Theorem 4.2 by replacing $F(\Gamma_A)$ by K_A.

Example 4.2. If Z is Hausdorff, for any compact-valued u.s.c. map $F : X \multimap Z$, K_A is compact and hence normal.

Theorem 4.5. *Let $(X, D; \Gamma)$ be a G-convex space and Z a topological space. Then any map $F \in \mathfrak{B}(X, Z)$ such that for any $A \in \langle D \rangle$ with $|A| = n + 1$, the set $K_A := F(\phi_A(\Delta_n))$ in its induced topology is a normal space, belongs to $\mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$.*

Proof. Let $G : D \multimap Z$ be a closed-valued KKM map with respect to F. Suppose that the family $\{G(x) \mid x \in D\}$ does not have the finite intersection property, that is, there exists an $A := \{y_0, y_1, \dots, y_n\} \in \langle D \rangle$ such that $K_A := F(\phi_A(\Delta_n)) \subset F(\Gamma_A) \subset G(A)$ and $\bigcap_{i=0}^{n} G(y_i) = \emptyset$. Then $K_A = \bigcup_{i=0}^{n} (K_A \setminus G(y_i))$. Let $S : K_A \multimap A \subset D$ with $S^{-}(y_i) := K_A \setminus G(y_i)$. Since K_A is normal, by Lemma 3.5, there exists a continuous function $p : K_A \to \Delta_n$ by $p(z) := \sum_{i=0}^{n} p_i(z) e_i$. Since $F \in \mathfrak{B}(X, Z)$, the composition $p \circ F \circ \phi_A : \Delta_n \multimap \Delta_n$ has a fixed point $a_0 \in p \circ F \circ \phi_A(a_0)$. For a $z \in p^{-1}(a_0) \subset F \circ \phi_A(a_0) \neq \emptyset$ and $J_z := \{i \mid p_i(z) \neq 0\}$, we have $i \in J_z$ iff $z \in K_A \setminus G(y_i)$, and so $z \in \bigcap_{i \in J_z} K_A \setminus G(y_i)$. Note that $z \in F \circ \phi_A(\Delta_{J_z}) \subset \bigcup_{i \in J_z} G(y_i)$, which is a contradiction. This shows $F \in \mathfrak{KC}(X, Z)$. Now, by Theorem 4.2, $F \in \mathfrak{KO}(X, Z)$.

Remark. Note that [4, Lemma 2.5] is for the case $F \in \mathfrak{KC}(X, Z)$ of Theorem 4.4, and we followed its proof. A similar proof works for the following:

Theorem 4.6. *Let $(X, D; \Gamma)$ be a G-convex space and Z a topological space. Then those elements $F \in \mathfrak{B}(X, Z)$ such that for any $A \in \langle D \rangle$ with $|A| = n + 1$, the set $\overline{F(\phi_A(\Delta_n))}$ in its induced topology is a normal space, belongs to $\mathfrak{KC}(X, Z)$.*

The following improves Lemma 3.2(1):

Corollary 4.7. *Let $(X, D; \Gamma)$ be a G-convex space and Z a normal space. Then $\mathfrak{B}(X, Z) \subset \mathfrak{KC}(X, Z)$.*

Corollary 4.8. *Let $(X, D; \Gamma)$ be a G-convex space and Z a Hausdorff space. Then*

(1) *every compact closed map $F \in \mathfrak{B}(X, Z)$ belongs to $\mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$; and,*

(2) *in the class of compact closed maps defined on a G-convex space $(X \supset D; \Gamma)$ satisfying condition $(*)$ into Z, we have $\mathfrak{B}(X, Z) = \mathfrak{KC}(X, Z) = \mathfrak{KO}(X, Z)$.*

Proof. (1) Since a compact closed map F is u.s.c. and compact-valued, F sends compact sets to compact sets. Therefore the set $F(\phi_A(\Delta_n))$ is compact in the Hausdorff space Z. Hence, it is normal. Now, by Theorem 4.5, the conclusion follows.

(2) By (1), $\mathfrak{B}(X, Z) \subset \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$. By Lemma 3.2(3), $\mathfrak{KC}(X, Z) \subset \mathfrak{B}(X, Z)$. Moreover, by Theorem 4.4, $\mathfrak{KC}(X, Z) = \mathfrak{KO}(X, Z)$. Therefore, the conclusion follows.

Recall that a convex space $(X \supset D; \Gamma)$ satisfies the condition $(*)$.

Corollary 4.8. *Let $(X \supset D; \Gamma)$ be a convex space and Z a Hausdorff space. Then, in the class of compact closed maps $F : X \multimap Z$, we have $\mathfrak{B}(X, Z) = \mathfrak{KC}(X, Z) = \mathfrak{KO}(X, Z)$.*

5. Remarks on examples of $\mathfrak{KC}$

In Sections 3 and 4, we obtained mutual relations among multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ when those maps are closed or compact or the ones having normal ranges. Without such restrictions, it seems to be hard to establish any other conclusion.

In 2002, Jeng, Huang, and Zhang [6] obtained some interesting examples of $\mathfrak{KC}$-maps. This section is concerned with comments on their results.

(I) [6, Theorem 2.2] *Let X and Y be two convex spaces and $F : X \multimap Y$ satisfy that $\overline{F(C)}$ is convex for any convex subset C of X. Then $F \in \mathfrak{KC}(X, Y)$.*

Closely examining the proof of the above, we can see the following:

(I)′ *Let X and Y be two convex spaces and $F : X \multimap Y$ satisfy that $F(C)$ is convex for any convex subset C of X. Then $F \in \mathfrak{KC}(X, Y) \cap \mathfrak{KO}(X, Y)$.*

(II) [6, Theorem 2.3] *Let X be a nonempty interval of $\mathbb{R}$ and Y a topological space. If $F : X \multimap Y$ satisfies that $\overline{F([a, b])}$ is connected for any $a, b \in X$ with $a < b$, then $F \in \mathfrak{KC}(X, Y)$.*

Similarly, we have

(II)′ *Let X be a nonempty interval of $\mathbb{R}$ and Y a topological space. If $F : X \multimap Y$ satisfies that $F([a, b])$ is connected for any $a, b \in X$ with $a < b$, then $F \in \mathfrak{KC}(X, Y) \cap \mathfrak{KO}(X, Y)$.*

Example 5.1 [2]**.** Let $X = [0, 1]$ with the usual topology, and let $F : X \multimap X$ be defined as

$$F(x) := \begin{cases} \{|\sin 1/x|\}, & \text{for } x \in (0, 1]; \\ \{0\}, & \text{for } x = 0. \end{cases}$$

This example was originally given in [2] to show that $\mathfrak{KC}(X, X) \supsetneq \mathfrak{A}_c^\kappa$. Now it is also an example of (II) and (II)′.

Note that the converses of (II) and (II)′ are not true:

Example 5.2 [6]**.** Let $F : [0,1] \to [0,1]$ be defined by $F(0) := \{0,1\}$ and $F(x) := \{1\}$ for $x \in (0,1]$. Since F has a continuous selection f such that $f(x) = 1$ for all $x \in [0,1]$.

(III) [6, Theorem 2.4] *Let X be any convex space and $F \in \mathfrak{KC}(X,Y)$ such that $\overline{F(x)}$ is connected for any $x \in X$. Then $\overline{F(C)}$ is connected for any convex subset C of X.*

Closely examining the proof of the above, we can see the following:

(III)$'$ *Let X be any convex space and $F \in \mathfrak{KC}(X,Y)$ or $F \in \mathfrak{KO}(X,Y)$ such that $F(x)$ is connected for any $x \in X$. Then $F(C)$ is connected for any convex subset C of X.*

In [6], the following characterization of single-valued functions in the $\mathfrak{KC}$ is obtained as an immediate consequence of (II) and (III):

(IV) [6, Theorem 2.5] *Let X be a nonempty interval of $\mathbb{R}$, Y a topological space and $f : X \to Y$. Then $f \in \mathfrak{KC}(X,Y)$ if and only if $\overline{f([a,b])}$ is connected for any $a, b \in X$ with $a < b$.*

As an immediate consequence of (II)$'$ and (III)$'$, we have

(IV)$'$ *Let X be a nonempty interval of $\mathbb{R}$, Y a topological space and $f : X \to Y$. Then $f \in \mathfrak{KC}(X,Y) \cup \mathfrak{KO}(X,Y)$ if and only if $f([a,b])$ is connected for any $a, b \in X$ with $a < b$.*

6. Historical remarks on $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$

1. At the Halifax conference in 1991, the author [10] first named the KKM theory as the research area on applications of various equivalent formulations of the classical KKM theorem.

2. In 1993, the author [11] introduced the admissible class $\mathfrak{A}_c^\kappa(X,Y)$ of multimaps $X \multimap Y$ between topological spaces and gave examples of multimaps belonging to the class.

3. In the same year, the author initiated the study of generalized convex spaces and various subclasses of the admissible class. Consequently, we could generalize the KKM theory to G-convex spaces; see [22-24] for earlier developments.

4. In 1994, the author [12] showed that fundamental results in the KKM theory can be obtained in far-reaching generalized forms related to $\mathfrak{A}_c^\kappa(X,Y)$ and that this class has the KKM property when X is a convex space and Y is a Hausdorff space as follows:

Corollary 2 [12]**.** *Let (X,D) be a convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X,Y)$, and $H : D \to 2^Y$ such that, for any $N \in \langle D \rangle$, $F(\mathrm{co}\, N) \subset H(N)$. Then the family $\{\overline{Hx} : x \in D\}$ has the finite intersection property.*

5. Motivated by this, Chang and Yen [2] in 1996 defined the KKM class of maps on convex subsets of topological vector spaces. Naturally, their $\mathfrak{KC}$ class contains $\mathfrak{A}_c^\kappa$-class under the hypothesis of the above corollary, but no significant proper example

of the former not in the latter was found. Note that Example 5.1 was the only one in their class not belonging to $\mathfrak{A}_c^\kappa$.

6. In the same year, Chang and Yen [2] introduced the concept of s-KKM map. A large number of authors followed their way and tried to rewrite many known results in the KKM theory using this concept or its modifications like S-KKM class.

7. On the other hand, the author [13] extended the $\mathfrak{A}_c^\kappa$-class to the 'better' admissible $\mathfrak{B}$-class on convex spaces, supplied a large number of examples, and showed that, in the class of compact closed multimaps from convex spaces into Hausdorff spaces, two subclasses $\mathfrak{B}$ and $\mathfrak{KC}$ coincide.

8. For G-convex spaces, such multimap classes are extended and investigated by a number of authors.

9. In 2003, the author [16] introduced the classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ of multimaps from a G-convex spaces into a topological spaces. This is followed by H. Kim and the author [8].

10. In 2004, the author [17] showed that a compact closed s-KKM map from a convex subset of a t.v.s. into itself belongs to $\mathfrak{B}$. Note that the convexity is redundant.

11. In 2005, H. Kim [7] showed that two classes KKM and s-KKM of multimaps from a convex space into a topological space are identical whenever s is surjective [this is the only case S-KKM is meaningful].

12. Recently, in [18], we observed that any S-KKM class on abstract convex spaces is included in the $\mathfrak{KC}$ class.

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