COMMENTS ON FIXED POINT AND COINCIDENCE THEOREMS ON MULTIMAPS WITH NONCONVEX OR NONCOMPACT DOMAINS

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Dedicated to Professor S. P. Singh on his 70th birthday

Abstract. Some basic fixed point and coincidence theorems in [15, Sections 2 and 3] and [14] are generalized and improved by removing redundant restrictions.

1. Introduction

Most of fixed point theorems in topological vector spaces are for multimaps having compact convex domains or for compact multimaps having convex domains.

However, Tian [14] first considered the existence of fixed points of multimaps having noncompact and nonconvex domains, and obtained certain consequences of corresponding known results on multimaps having compact convex domains.

In a recent work [15], its authors obtained some existence results on fixed points of expansive multimaps and inner multimaps on not necessarily convex or compact sets in Hausdorff topological vector spaces. As a consequence, they proved a new intersection theorem concerning not necessarily convex or compact sets and its applications. They also gave new coincidence and section theorems for multimaps defined on not necessarily convex sets in Hausdorff topological vector spaces. Actually, using a technique based on the investigation of the ranges of multimaps,

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the authors of [15] obtained a number of new fixed point, coincidence, intersection, and section theorems of Fan-Browder type.

On the other hand, there have also appeared new versions of the Fan-Browder fixed point theorem due to the present author [8,9]. More recently, some general forms of the Kakutani fixed point theorem are also known in [6,10,11].

Applying those new results, in this paper, we show that basic fixed point and coincidence theorems of [15, Sections 2 and 3] and [14, Theorem 2] can be generalized or improved by removing certain redundant restrictions.

2. Preliminaries

A t.v.s. means a topological vector space E and $\mathcal{V}$ denotes a neighborhood system of the origin 0 of E..

A *multimap* (simply, a *map*) $T : X \multimap Y$ is a function from X into the power set 2^Y of Y. $T(x)$ is called the *value* of T at $x \in X$ and $T^-(y) := \{x \in X : y \in T(x)\}$ the *fiber* of T at $y \in Y$. Let $T(A) := \bigcup\{T(x) : x \in A\}$ for $A \subset X$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X, and *compact* if its range $T(X)$ is contained in a compact subset of Y.

For a set D, let $\langle D \rangle$ denote the set of non-empty finite subsets of D.

Let X be a subset of a vector space and D a nonempty subset of X. We call (X, D) a *convex space* if $\operatorname{co} D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D \rangle$; see Lassonde [5] and Park [10]. If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [4].

The following version of the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem for convex spaces is known in [5,9]:

Theorem 2.1. *Let (X, D) be a convex space and $F : D \multimap X$ a multimap such that*

(1) *$F(z)$ is open [resp. closed] for each $z \in D$; and*
(2) *F is a KKM map (that is, $\operatorname{co} N \subset F(N)$ for each $N \in \langle D \rangle$).*

Then $\{F(z)\}_{z\in D}$ *has the finite intersection property.* (*More precisely, for any* $N \in \langle D\rangle$, *we have* $\operatorname{co} N \cap [\bigcap_{z\in N} F(z)] \neq \emptyset$.)

From Theorem 2.1, we deduced the following new Fan-Browder fixed point result [8,9,13]:

Theorem 2.2. *Let* (X, D) *be a convex space and* $P : X \multimap D$ *a multimap. If there exist* $z_1, z_2, \cdots, z_n \in D$ *and non-empty open* [*resp. closed*] *subsets* $G_i \subset P^-(z_i)$ *for each* $i = 1, 2, \cdots, n$ *such that* $\operatorname{co}\{z_1, z_2, \cdots, z_n\} \subset \bigcup_{i=1}^n G_i$, *then the map* $\operatorname{co} P : X \multimap X$ *has a fixed point* $x_0 \in X$ (*that is,* $x_0 \in \operatorname{co} P(x_0)$).

Usually, a multimap having nonempty convex values and open fibers is called a *Browder map.* Note that Theorem 2.2 can be applied to Browder maps and then the well-known Fan-Brwder fixed point theorem [1, Theorem 1] follows. In fact, in our previous work (Sy and Park [13]), Theorem 2.2 is applied to obtain several forms of the Fan-Browder fixed point theorem, other (approximate) fixed point theorems, and so on.

The following is originated from Schauder in 1935:

Conjecture 1. *Let* E *be a Hausdorff t.v.s.,* C *a convex subset of* E*, and* f *a continuous function from* C *into* C*. If* f *is compact, then* f *has a fixed point* $x_0 \in C$*, that is,* $x_0 = f(x_0)$.

In 2001, Cauty [2] claimed the affirmativity of Conjecture 1. Later, it is known that his proof has a gap.

Recall that a *Kakutani map* is an upper semicontinuous (u.s.c.) map having nonempty compact convex values. The following is also related to Conjecture 1:

Conjecture 2. *Let* X *be a compact convex subset of a Hausdorff t.v.s.* E*. Then every Kakutani map* $T : X \multimap X$ *has a fixed point* $x_0 \in X$*, that is,* $x_0 \in T(x_0)$.

In order to give some most general resolutions of Conjectures 1 and 2, we need the following concepts.

Let X be a subset of a t.v.s. E. A subset K of X is said to be *Klee approximable in* X [10] if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E. X is said to be *admissible* (in the sense of Klee) if every compact subset K of X is Klee approximable in X.

Let X be a nonempty closed convex subset of a t.v.s. E. We say that X is *weakly admissible* (in the sense of Nhu and Aranđelović) [6] if for every $V \in \mathcal{V}$ there exist closed convex subsets $X_1, X_2, \cdots, X_n$ of X with $X = \operatorname{co}(\bigcup_{i=1}^n X_i)$ and continuous functions $f_i : X_i \to X \cap L, i = 1, 2, \cdots, n$, where L is a finite dimensional subspace of E, such that $\sum_{i=1}^n (f_i(x_i) - x_i) \in V$ for every $x_i \in X_i$ and $i = 1, 2, \cdots, n$.

A subset B of a t.v.s. E is said to be *convexly totally bounded* (simply, c.t.b.) [3] if for every $V \in \mathcal{V}$, there exist a finite subset $\{x_i\}_{i=1}^n \subset B$ and a finite family of convex subsets $\{C_i\}_{i=1}^n$ of V such that $B \subset \bigcup_{i=1}^n (x_i + C_i)$.

The following is the most general known resolution of Conjectures 1 and 2:

Theorem 2.3. [3,6,10] *Let X be a convex subset of a t.v.s. E. Then a compact Kakutani map $T : X \multimap X$ has a fixed point if one of the following conditions holds:*

(1) (Idzik) $\overline{T(X)}$ *is convexly totally bounded (c.t.b.).*

(2) (Okon) X *is compact and weakly admissible.*

(3) (Park) $\overline{T(X)}$ *is Klee approximable.*

Note that (3) holds whenever $\overline{T(X)}$ is locally convex or X is admissible (in the sense of Klee).

Let us say that a topological space X has the *fixed point property* (simply, f.p.p.) if any continuous selfmap $f : X \to X$ has a fixed point $x_0 \in X$. A subset X of a t.v.s. is said to have the *Kakutani fixed point property* if every Kakutani map $\Phi : X \multimap X$ has a fixed point.

In view of Theorem 2.3, the most general known examples of compact convex sets having the Kakutani fixed point property are weakly admissible ones or c.t.b. ones.

It is well-known that, for each locally finite open cover of a normal topological space, there is a partition of unity subordinated to it. Form this fact, we deduce the following selection theorem, where Δ_n is the standard n-simplex.

Theorem 2.5. *Let K be a normal topological space, X a convex subset of a t.v.s.*

E, and $S : K \multimap X$ a multimap such that

(1) *for each $x \in K$, $S(x)$ is convex; and*

(2) *$K = \bigcup_{i=1}^{n+1} \operatorname{Int} S^-(y_i)$ for some $\{y_i\}_{i=1}^{n+1} \subset X$.*

Then there exist continuous maps $g : \Delta_n \to \operatorname{co}\{y_i\}_{i=1}^{n+1}$ and $\phi : K \to \Delta_n$ such that $f = g\phi : K \to \operatorname{co}\{y_i\}_{i=1}^{n+1}$ is a continuous selection of S, that is, $f(x) \in S(x)$ for all $x \in K$.

Proof. Let $\{\alpha_i\}_{i=1}^{n+1}$ be a partition of unity subordinated to the finite cover $\{\text{Int } S^-(y_i)\}_{i=1}^{n+1}$ of the normal space K; that is,

(3) for each i, $\alpha_i : K \to [0,1]$ is continuous;

(4) Supp $\alpha_i \subset \operatorname{Int} S^-(y_i)$ for each i ; and

(5) for each $x \in K$, $\sum_{i=1}^{n+1} \alpha_i(x) = 1$.

Define a continuous function $\phi : K \to \Delta_n$ by

$$\phi(x) = \sum_{i=1}^{n+1} \alpha_i(x) e_i = \sum_{y_j \in A_x} \alpha_j(x) e_j \quad \text{for } x \in K,$$

where e_i are vertices of Δ_n and

$$y_j \in A_x \subset \{y_i\}_{i=1}^{n+1} \Longleftrightarrow \alpha_j(x) \neq 0 \Longrightarrow x \in \text{Int } S^-(y_j) \Longrightarrow y_j \in S(x)$$

and hence $A_x \subset \{y_i\}_{i=1}^{n+1} \cap S(x)$ and $\phi(x) \in \operatorname{co}\{e_j \mid y_j \in A_x\}$. Let $g : \Delta_n \to \operatorname{co}\{y_i\}_{i=1}^{n+1}$ be the map given by $g(\sum_{i=1}^{n+1} \lambda_i e_i) = \sum_{i=1}^{n+1} \lambda_i y_i$ for each $\{\lambda_i\}_{i=1}^{n+1}$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$. Let $f := g\phi : K \to \operatorname{co}\{y_i\}_{i=1}^{n+1}$. Then f is continuous and, for each $x \in K$, we have

$$\begin{aligned} f(x) = g(\phi(x)) \in g(\operatorname{co}\{e_j \mid y_j \in A_x\}) &= \operatorname{co}\{y_j \mid y_j \in A_x\} \\ &\subset \operatorname{co}(\{y_i\}_{i=1}^{n+1} \cap S(x)) \subset \operatorname{co}(S(x)) = S(x). \end{aligned}$$

This completes our proof.

Theorem 2.5 is motivated from Browder [1] and Park [7].

In a t.v.s., a convex hull of a finite subset is called a *polytope.*

3. Fixed points of expansive multimaps on not necessarily convex or compact sets

In this section, we derive new versions of results of Section 2 of [15].

Theorem 3.1. *Let C be a nonempty subset of a t.v.s. E, $F : C \multimap E$, and K a convex subset of E. Assume that the following conditions hold:*

(1) $C \subset K \subset F(C)$;

(2) *for each* $c \in C$, $F(c)$ *is open [resp. closed] in* $F(C)$;

(3) $F(C) = \bigcup_{i=1}^{n} F(c_i)$ *for some* $c_1, c_2, \cdots, c_n \in C$;

(4) *for each* $y \in K$, $F^{-1}(y)$ *is convex.*

Then there exists $u \in C$ such that $u \in F(u)$.

Proof. Let $D := \{c_1, c_2, \cdots, c_n\} \subset C$. Then (K, D) is a convex space. Let $T : D \multimap K$ be a map defined by $T(c_i) = K \backslash F(c_i)$ for each i. Then T has closed [resp. open] values. Moreover, $\bigcap_{i=1}^{n} T(c_i) = K \backslash \bigcup_{i=1}^{n} F(c_i) \subset F(C) \backslash \bigcup_{i=1}^{n} F(c_i) = \emptyset$. Therefore, by the KKM Theorem 2.1, T can not be a KKM map. Hence, co $N \not\subset T(N)$ for some $N \subset D$, that is, there exists a $u \in$ co $N \subset K$ such that $u \notin T(c_j) = K \backslash F(c_j)$ for each $c_j \in N$. Therefore $u \in F(c_j) \cap K$ or $c_j \in F^{-}(u)$ for each $c_j \in N$. Since $F^{-}(u)$ is convex in C, we have $u \in$ co $N \subset F^{-}(u)$. Therefore, $u \in F(u)$. This completes our proof.

Remark. Note that [15, Theorem 2.1] is a particular form of the 'open' version of Theorem 3.1 with the additional requirement that (a) E is Hausdorff, (b) $F(C)$ is a compact subset of E, and (c) $F^{-}(y)$ is nonempty. The authors of [15] adopted the partition of unity argument.

Similarly, [15, Theorems 2.2 and 2.3] can be improved as follows.

Theorem 3.2. *Let C be a nonempty subset of a t.v.s. E and $F : C \multimap E$. Suppose that*

(1) F *is expansive, that is,* $C \subset F(C)$;

(2) $F(C)$ *is convex;*

(3) *for each* $c \in C$, $F(c)$ *is open [resp. closed] in* $F(C)$;

(4) $F(C) = \bigcup_{i=1}^{n} F(c_i)$ *for some* $c_1, c_2, \cdots, c_n \in C$;

(5) *for each* $y \in F(C)$, $F^{-}(y)$ *is convex.*

Then there exists $u \in C$ such that $u \in F(u)$.

Proof. Put $K = F(C)$ in Theorem 3.1.

Theorem 3.3. *Let C be a nonempty subset of a t.v.s. E and $F : C \multimap E$. Suppose that*

(1) *F is expansive, that is, $C \subset F(C)$;*
(2) *C is convex;*
(3) *for each $c \in C$, $F(c)$ is open [resp. closed] in $F(C)$;*
(4) *$F(C) = \bigcup_{i=1}^n F(c_i)$ for some $c_1, c_2, \cdots, c_n \in C$;*
(5) *for each $y \in C$, $F^-(y)$ is convex.*

Then there exists $u \in C$ such that $u \in F(u)$.

Proof. Put $K = C$ in Theorem 3.1.

Remark. The expansiveness is called surjectivity by some other authors. In view of Theorems 3.2 and 3.3, [15, Theorems 2.2 and 2.3] hold without assuming the Hausdorffness of E and the compactness of $F(C)$.

Moreover, we have the following:

Theorem 3.4. *Let C be a nonempty convex subset of a Hausdorff t.v.s. E, $F : C \multimap E$ a multimap, and $f : C \to E$ a continuous function such that $f(C) \subset F(C)$. Suppose that*

(1) *$F(C)$ is a compact subset of E;*
(2) *for each $c \in C$, $F(c)$ is open in $F(C)$;*
(3) *for each $y \in F(C)$, $F^-(y)$ is nonempty and convex.*

Then there exists $u \in C$ such that $f(u) \in F(u)$.

Proof. Since $F^-|_{F(C)} : F(C) \multimap C$ is a Browder map, and $F(C)$ is compact, by Theorem 2.5, $F^-|_{F(C)}$ has a continuous selection $s : F(C) \to C$. Note that $s(F(C)) \subset P$ for some polytope $P \subset C$. Since $f(C) \subset F(C)$, $sf(P) \subset P$. Now by the Brouwer fixed point theorem, sf has a fixed point $u \in P \subset C$, that is, $(sf)(u) = u$ or $f(u) \in s^-(u) \subset (F^-)^-(u) = F(u)$. This completes our proof.

Remark. Note that [15, Theorem 2.6] is a particular form of Theorem 3.4 with additional assumption that F is expansive. Note that for $f = 1_C$, the identity map of C, Theorem 3.4 reduces to [15, Theorem 2.3].

4. Fixed points of inner multimaps on not necessarily convex or compact sets

In this section, we derive new versions of results of Section 3 of [15], where a map $F : C \multimap E$ is said to be *inner* if $F(C) \subset C$.

Theorem 4.1. *Let C be a nonempty subset of a t.v.s. E and $F : C \multimap C$ a multimap such that $F(C)$ is convex. Suppose that*

(1) *for each $c \in C$, $F(c)$ is convex;*
(2) *for each $y \in F(C)$, $F^-(y)$ is open [resp. closed];*
(3) *C is covered by a finite number of $F^-(y)$'s.*

Then there exists $u \in C$ such that $u \in F(u)$.

Proof. Let $X := F(C)$ and $T := F|_X$. Then $T : X \multimap X$ is convex-valued. Suppose that $C = \bigcup_{i=1}^n F^-(y_i)$ for some $D := \{y_1, y_2, \cdots, y_n\} \subset F(C) \subset C$. Then $T^-(y_i) = X \cap F^-(y_i) = F(C) \cap F^-(y_i)$ and hence $X = F(C) = \bigcup_{i=1}^n T^-(y_i)$. Then, by Theorem 2.2, T has a fixed point $u \in X \subset C$, that is, $u \in T(u) = F(u)$. This completes our proof.

Remark. Note that [15, Theorem 3.1] is a particular form of the 'open' version of Theorem 4.1 with the additional requirement that (a) E is Hausdorff, (b) C is compact, and (c) $F(c)$ is nonempty. As for Theorem 3.2, we can prove Theorem 4.1 by applying the KKM theorem directly.

The following is a coincidence theorem for a Browder map and a Kakutani map:

Theorem 4.2. *Let C be a nonempty subset of a Hausdorff t.v.s. E, $F : C \multimap C$ a Browder map, and $G : C \multimap C$ a Kakutani map such that $G(C) \subset F(C)$. If $F(C)$ is a compact convex subset of C and has the Kakutani fixed point property, then there exists $(u, v) \in C \times F(C)$ such that $u \in F(v)$ and $v \in G(u)$.*

Proof. Since $F|_{F(C)} : F(C) \multimap F(C)$ is a Browder map and $F(C)$ is compact, $F|_{F(C)}$ has a continuous selection $f : F(C) \to P$ into a polytope $P \subset F(C) \subset C$. Since $G|_P : P \multimap G(C)$ is a Kakutani map and $G(C) \subset F(C)$, the composition $Gf : F(C) \multimap F(C)$ is a Kakutani map. Since $F(C)$ is a compact convex subset of a Hausdorff t.v.s. E and has the Kakutani fixed point property, Gf has a fixed point $v \in F(C)$, that is, $v \in (Gf)(v)$. By letting $u := f(v) \in P \subset C$, we have $u = f(v) \in F(v)$ and $V \in G(u)$. This completes our proof.

Remarks. 1. Note that [15, Theorem 3.2] is a variant of Theorem 4.2 for the case where (a) C is compact, (b) E is locally convex, and (c) G is upper demicontinuous (instead of upper semicontinuous).

2. In view of [12, Proposition 2], the upper semicontinity, the upper demicontinuity, and the upper hemicontinuity of G in [15, Theorem 3.2] are all equivalent to each other. Therefore, Theorem 4.2 generalizes [15, Theorem 3.2].

3. In view of Theorem 2.3, we may assume that $F(C)$ is weakly admissible or c.t.b.

In the later part (Sections 4-7) of [15], further interesting results on intersection theorems, coincidence theorems and section theorems on various types of multimaps or spaces were obtained. However, there should be rooms for some minor improvements of those results, for example, the Hausdorffness in Theorems 4.1 and 4.2 of [15] are redundant.

5. A necessary and sufficient condition for the existence of fixed points

In this section, we improve a result in [14] by removing redundant restrictions as follows:

Theorem 5.1. *Let X be a nonempty subset of a t.v.s. E and $F : X \multimap E$ a multimap with open fibers and convex values. Then F has a fixed point if and only if there exists a nonempty compact convex subset K of X such that*

$$F(x) \cap K \neq \emptyset \quad for\ all\ \ x \in K.$$

Proof. Necessity. Suppose F has a fixed point $x^* \in F(x^*)$. Let $K := \{x^*\}$. Then the singleton K clearly satisfies the required condition.

Sufficiency. Define a multimap $G : K \multimap K$ by $G(x) := F(x) \cap K$ for each $x \in K$. Then G has nonempty convex values. Moreover, for each $y \in K$, $G^-(y) = \{X \in K : y \in F(x) \cap K\} = F^-(y) \cap K$ is relatively open in K. Therefore, by Theorem 2.2, G has a fixed point $x^* \in G(x^*) = F(x^*) \cap K$, that is, $x^* \in F(x^*)$.

Remarks. 1. Note that [14, Theorem 2] is a particular form of Theorem 5.1 for the case where (a) X is paracompact, (b) E is Hausdorff and locally convex, and (c) F has nonempty closed values.

2. It is known that [14, Theorem 1] is incorrectly stated.

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