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THE KKM, MATCHING, AND FIXED POINT THEOREMS IN GENERALIZED CONVEX SPACES

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Abstract. From our version of the KKM theorem for G-convex spaces, we deduce generalizations of the intersection theorems of Sperner [40] and Alexandroff-Pasynkoff [1], matching theorems of Lassonde [17] and Klee [14], and various forms of the Fan-Browder type fixed point theorems. Applications to the existence of maximal elements and approximate fixed points are added.

1. Introduction

The KKM theory is the study of applications of various equivalent formulations of the classical Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM principle). It was initiated by the work [15] and developed first by Ky Fan [5-7]. Since then, there have appeared numerous generalizations of known results and new applications. For the literature, see Granas [8, 9] and Park [18-21].

At the beginning, the theory was mainly concerned with convex subsets of topological vector spaces. Later, it was extended to convex spaces by Lassonde [16, 17], and to spaces having certain families of contractible subsets (simply, C-spaces or H-spaces) by Horvath [10, 11]. This line of generalizations of earlier works is followed by many authors. Especially, in [22-36], the author introduced generalized convex (simply, G-convex) spaces and basic properties of KKM maps for such spaces, which seem to be more adequate for various purposes. Actually, our new concept of G-convex spaces is a common generalization of the usual convexity in a topological vector space and many of abstract convexities which have been developed in connection mainly with the fixed point theory and the KKM theory.

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In this paper, from our version of the KKM theorem for G-convex spaces, we deduce generalizations of the intersection theorems of Sperner [40] and Alexandroff-Pasynkoff [1], matching theorems of Lassonde [17] and Klee [14], and various forms of the Fan-Browder type fixed point theorems. For some of such results, we have to recover the isotonicity requirement for the definition of G-convex spaces, which was discarded once by ourselves. Finally, applications to the existence of maximal elements and approximate fixed points are added.

In Section 2, as a preliminary, we list basic known theorems equivalent to the Brouwer fixed point theorem and the KKM principle. Section 3 deals with G-convex space versions of the intersection theorems of Sperner [40] and Alexandroff-Pasynkoff [1], which are shown to be equivalent to the Brouwer theorem. In Section 4, matching theorems of Lassonde [17] and Klee [14] are generalized to G-convex spaces. Section 5 deals with various forms of the Fan-Browder type fixed point theorems for G-convex spaces. As applications of the Fan-Browder theorem, we give a maximal element theorem in Section 6 and an approximate fixed point theorem in Section 7.

2. Some equivalent forms of the Brouwer theorem

It is well-known that the Brouwer fixed point theorem, the Sperner (combinatorial) lemma, and the Knaster–Kuratowski–Mazurkiewicz (simply, KKM) theorem are mutually equivalent. For the literature, see [4, 20, 31].

Let $\Delta_n = v_0v_1 \cdots v_n$ be an n-simplex and $\partial\Delta_n = \bigcup_{i=0}^{n} v_0v_1 \cdots \widehat{v_i} \cdots v_n$ its boundary, that is, the union of $(n-1)$-faces of Δ_n.

The Brouwer fixed point theorem [3] as follows is one of the most well-known and useful theorems in topology:

Theorem B (Brouwer)**.** *A continuous map $f : \Delta_n \to \Delta_n$ has a fixed point $x_0 = f(x_0) \in \Delta_n$.*

The following is the combinatorial lemma of Sperner [40]:

Lemma S (Sperner)**.** *Let K be a simplicial subdivision of an n-simplex $v_0v_1 \cdots v_n$. To each vertex of K, let an integer be assigned in such a way that whenever a vertex u of K lies on a face $v_{i_0}v_{i_1} \cdots v_{i_k}$ $(0 \le k \le n$, $0 \le i_0 < i_1 < \cdots < i_k \le n)$, the number assigned to u is one of the integers $i_0, i_1, \cdots, i_k$. Then the total number of those n-simplexes of K, whose vertices receive all $n+1$ integers $0, 1, \cdots, n$, is odd. In particular, there is at least one such n-simplex.*

Recently, the author and Jeong [32] gave a new proof of Lemma S from the Brouwer fixed point theorem.

Recall that the Sperner lemma [40] was applied to obtain the closed version of the following in [15]:

Theorem (KKM)**.** *Let $F_i\,(0 \le i \le n)$ be $n+1$ closed [open] subsets of an n-simplex $v_0v_1\cdots v_n$. If the inclusion relation*

$$v_{i_0}v_{i_1}\cdots v_{i_k} \subset F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

holds for all faces $v_{i_0}v_{i_1}\cdots v_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, then $\bigcap_{i=0}^n F_i \neq \emptyset$.

The open-valued version of the KKM theorem was due to Kim [13] and Shih–Tan [39]. For the history of generalizations and applications of the open-valued version, see Park et al. [20, 31, 36].

In [31], we deduced the following intersection theorem from the KKM theorem:

Theorem S (Sperner)**.** *Let F_i $(0 \le i \le n)$ be $n+1$ nonempty closed [resp. open] sets covering an n-simplex $\Delta_n = v_0v_1\cdots v_n$. If, for each i, F_i is disjoint from the $(n-1)$-face $v_0v_1\cdots\widehat{v_i}\cdots v_n$, then $\bigcap_{i=0}^n F_i \neq \emptyset$.*

The closed version of Theorem S is due to Sperner [40] and applied to prove the invariance of dimension, and the open version is due to Stromquist; see [31] for references.

Theorem AP (Alexandroff–Pasynkoff)**.** *Let X_i $(0 \le i \le n)$ be $n+1$ closed [resp. open] sets covering an n-simplex $\Delta_n = v_0v_1\cdots v_n$ such that $v_0\cdots\widehat{v_i}\cdots v_n \subset X_i$ for each i. Then $\bigcap_{i=0}^n X_i \neq \emptyset$.*

The closed version of Theorem AP is due to Alexandroff and Pasynkoff [1] and applied to the essentiality of the identity map of the boundary of a simplex, and the open version is noted by Lassonde [17]; see also [31].

3. The KKM type theorems in G-convex spaces

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of a set Y; that is, a function with the values $F(x) \subset Y$ for

$x \in X$ and the *fibers* $F^-(y) := \{x \in X : y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) : x \in A\}$.

It was known that the KKM theorem holds for topological spaces with abstract convexity without any linear structure. The following concept is due to the first author:

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J := \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n an n-simplex with vertices $v_0, v_1, \ldots, v_n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$. For details on G-convex spaces, see [22-27, 32-36], where basic theory was extensively developed.

In certain cases, for example, when $D = A$ is finite, we may assume $\Gamma_J = \phi_A(\Delta_J)$.

There exist a large number of examples of G-convex spaces; see [20, 22, 25, 27, 33-36]. We give a few typical examples as follows:

If $X = D$ is a convex subset of a vector space and each Γ_A is the convex hull of $A \in \langle X \rangle$ equipped with the Euclidean topology, then $(X; \Gamma)$ becomes a *convex space* in the sense of Lassonde [16]. Note that any convex subset of a topological vector space is a convex space, but not conversely.

If $X = D$ and each Γ_A is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geq 0$) and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then (X, Γ) becomes a *C-space* (or an *H-space*) due to Horvath [10, 11].

If X is compact, then the G-convex space $(X, D; \Gamma)$ is said to be compact.

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$. For a G-convex space, we obtained the following KKM theorem:

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a multimap with nonempty closed* [*resp. open*] *values. Suppose that F is a KKM map. Then $\{F(a)\}_{a \in D}$ has the finite intersection property. More precisely, for any $A \in \langle D \rangle$, we have*

$$\Gamma_A \cap \bigcap_{a \in A} F(a) \neq \emptyset.$$

Remarks. 1. For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \mathrm{co}$, the convex hull, Theorem 1 reduces to the KKM theorem [15]. If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 1 reduces to Fan's KKM lemma [6]. Note that Lassonde [17] obtained particular forms of Theorem 1 as his Principe KKM_f and KKM_o for convex sets X with the finite topology.

2. Theorem 1 is due to Park [24, 25, 27, 31] where a number of applications in various fields were given.

Recall that, at first, a G-convex space was defined under the additional isotonicity condition:

$$(*) \qquad \text{if } M, N \in \langle D \rangle \text{ and } M \subset N, \text{ then } \Gamma_M \subset \Gamma_N;$$

see [34-36]. Condition $(*)$ holds for convex spaces or C-spaces in the sense of Horvath [10, 11], but not for G-convex spaces in general. Later, it is known that this restriction was superfluous (see [22-27]) in most applications.

However, when $D = A$ is finite, by putting $\Gamma_J = \phi_A(\Delta_J)$, we may assume the isotonicity condition $(*)$.

In this section, from the KKM Theorem 1, we deduce the following generalizations of Theorems S and AP for G-convex spaces, resp.:

Theorem 2. *Let $(X, D; \Gamma)$ be a G-convex space with $D = \{a_0, a_1, \cdots, a_n\}$ and $F : D \multimap X$ a multimap with nonempty closed [resp. open] values such that*

(i) *$X = F(D)$ and*

(ii) *for each i, $0 \leq i \leq n$, $F(a_i)$ is disjoint from $\Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_n\})$.*

Then $\bigcap_{i=0}^n F(a_i) \neq \emptyset$.

Proof. It suffices to show that F is a KKM map. Let $N \in \langle D \rangle$. If $N = D$, then $\Gamma_N \subset X = F(N)$ by (i). Suppose that $N \subsetneq D$. Then there exists an index j, $0 \leq j \leq n$, such that $a_j \notin N$. By (ii) and condition $(*)$, we have

$$F(a_j) \cap \Gamma_N \subset F(a_j) \cap \Gamma(\{a_0, \cdots, \widehat{a_j}, \cdots, a_n\}) = \emptyset.$$

However, we have $\Gamma_N \subset X = F(D) = \bigcup_{i=0}^n F(a_i)$ and hence

$$\Gamma_N \subset \bigcup \{F(a_i) : a_i \in N\} = F(N).$$

Now the conclusion follows from Theorem 1. ■

For $D = \{a_0, a_1, \cdots, a_n\}$, we denote as follows:

$$D_0 := \{a_0, \cdots, a_{n-1}\},$$
$$D_i := \{a_0, \cdots, \widehat{a_{i-1}}, \cdots, a_n\}$$

for $1 \leq i \leq n$.

Theorem 3. *Let $(X, D; \Gamma)$ be a G-convex space with $D = \{a_0, a_1, \cdots, a_n\}$ and $T : D \multimap X$ a multimap with nonempty closed [resp. open] values such that*

(i) $X = T(D)$ *and*
(ii) $\Gamma_{D_i} \subset T(a_i)$ *for* $0 \leq i \leq n$.

Then $\bigcap_{i=0}^n T(a_i) \neq \emptyset$.

Proof. We show that T is a KKM map. Let $N \in \langle D \rangle$. If $N = D$, then $\Gamma_N \subset X = T(N)$ by (i). Suppose that $N \subsetneq D$. Then, by (ii) and condition $(*)$,

$$\Gamma_N \subset \Gamma_{D_i} \subset T(a_i) \quad \text{for some} \quad a_i \in N,$$

and hence

$$\Gamma_N \subset \bigcup \{T(a_i) : a_i \in N\} = T(N).$$

Now the conclusion follows. ■

Ky Fan [5, 7] noted that each of Theorems S and AP can be easily derived from the other. We show that this can be done for Theorems 2 and 3 as follows:

Theorem 2 $\Longrightarrow$ Theorem 3. Suppose that $\bigcap_{i=0}^n T(a_i) \neq \emptyset$. Let $F : D \multimap X$ be a map with open [resp. closed] values defined by

$$F(a_i) := X \backslash T(a_{i+1}) = T(a_{i+1})^c \quad \text{for } 0 \leq i < n,$$
$$F(a_n) := X \backslash T(a_0) = T(a_0)^c.$$

Then we have

$$\bigcup_{i=0}^n F(a_i) = \bigcup_{i=0}^n T(a_i)^c = [\bigcap_{i=0}^n T(a_i)]^c = X$$

and hence condition (i) of Theorem 2 holds. Moreover, since

$$\Gamma_{D_{i+1}} = \Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_n\}) \subset T(a_{i+1}) \quad \text{for } 0 \le i < n$$

and

$$\Gamma_{D_0} = \Gamma(\{a_0, \cdots, a_{n-1}, \widehat{a_n}\}) \subset T(a_0),$$

we have

$$F(a_i) \cap \Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_n\}) = \emptyset \quad \text{for } 0 \le i \le n,$$

and hence condition (ii) of Theorem 2 holds. Therefore,

$$\bigcap_{i=0}^{n} F(a_i) = \bigcap_{i=0}^{n} T(a_i)^c = [\bigcup_{i=0}^{n} T(a_i)]^c = [T(D)]^c = \emptyset$$

by (i). This contradicts Theorem 2.

Theorem 3 $\Longrightarrow$ *Theorem 2.* Similarly we can prove.

As usual, $\mathbb{B}^n$ denotes the unit ball in the Euclidean space $\mathbb{R}^n$ and $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ denotes the $(n-1)$-sphere.

Let $h : \Delta_n \to \mathbb{B}^n$ be a homeomorphism and $D = \{v_i\}_{i=0}^n$. Then $(\mathbb{B}^n, D; \Gamma)$ becomes a G-convex space where

$$\Gamma(\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}) := h(\mathrm{co}\,\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\})$$

for each $\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\} \in \langle D \rangle$.

From Theorem 3, we give a simple proof of the non-retract theorem due to Bohl [2].

Theorem 4 (Bohl)**.** *For* $n \ge 1$, $\mathbb{S}^{n-1}$ *is not a retract of* $\mathbb{B}^n$*.*

Proof. Suppose, on the contrary, that there exists a continuous retraction $g : \mathbb{B}^n \to \mathbb{S}^{n-1}$ such that $g|_{\mathbb{S}^{n-1}} = \mathrm{id}_{\mathbb{S}^{n-1}}$ Note that

$$\Gamma_{D_0} = h(\mathrm{co}\{v_0, v_1, \cdots, v_{n-1}\})$$

and

$$\Gamma_{D_i} = h(\mathrm{co}\{v_0, \cdots, \widehat{v_{i-1}}, \cdots, v_n\}) \quad \text{for } 1 \le i \le n$$

are closed sets in $\mathbb{S}^{n-1}$, and that $\mathbb{S}^{n-1} = h(\partial\Delta_n) = \bigcup_{i=0}^n \Gamma_{D_i}$ and $\bigcap_{i=0}^n \Gamma_{D_i} = \emptyset$. Let $T : D \multimap \mathbb{B}^n$ be a map defined by $T(v_i) := g^{-1}(\Gamma_{D_i})$ for $0 \leq i \leq n$. Then T is closed-valued and $\mathbb{B}^n = T(D)$. Moreover,

$$T(v_i) = g^{-1}(\Gamma_{D_i}) = g^{-1}g(\Gamma_{D_i}) \supset \Gamma_{D_i} \quad \text{for } 0 \leq i \leq n.$$

However, we have

$$\bigcap_{i=0}^{n} T(v_i) = \bigcap_{i=0}^{n} g^{-1}(\Gamma_{D_i}) = g^{-1}(\bigcap_{i=0}^{n} \Gamma_{D_i}) = g^{-1}(\emptyset) = \emptyset.$$

This contradicts the conclusion of Theorem 3. ■

Remark. It is well-known that Theorem 4 is equivalent to the Brouwer fixed point theorem. Since Theorems 1 - 4 are consequences of the KKM principle, each of them is also equivalent to the Brouwer theorem.

4. Matching theorems in G-convex spaces

From the KKM Theorem 1, we can deduce the following generalization of Lassonde [17]:

Theorem 5. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a closed [resp. open] valued multimap. Suppose that there exists an $A \in \langle D \rangle$ such that $\Gamma_A \subset F(A)$. Then there exists a $B \in \langle D \rangle$ such that $\Gamma_B \cap \bigcap_{b \in B} F(b) \neq \emptyset$.*

Proof. Suppose that $\Gamma_B \cap \bigcap_{b \in B} F(b) = \emptyset$ for all $B \in \langle D \rangle$. Then

$$\Gamma_B \subset \bigcup_{b \in B} (X \backslash F(b)) = F^c(B),$$

where $F^c : D \multimap X$ is a multimap defined by $F^c(b) := X \backslash F(b)$ for $b \in D$. Then F^c is an open [resp. closed] valued KKM map. Hence, by the KKM Theorem 1, for any $A \in \langle D \rangle$, we have

$$\Gamma_A \cap \bigcap_{a \in A} F^c(a) \neq \emptyset \iff \Gamma_A \cap (X \backslash F(A)) \neq \emptyset \iff \Gamma_A \not\subset F(A),$$

which violates our assumption. ■

For a convex space X, Theorem 5 reduces to the "matching" theorem of Lassonde [17].

From Theorem 5, we deduce the following:

Theorem 6. *Let $(X, D; \Gamma)$ be a G-convex space, $\{A_i\}_{i=1}^n$ a family of closed [resp. open] subsets covering X, and $\{z_i\}_{i=1}^n$ a family of n elements in D. Then there exists a subset $J \subset I = \{1, 2, \cdots, n\}$ such that $\Gamma(\{z_j : j \in J\}) \cap \bigcap_{j \in J} A_j \neq \emptyset$.*

Proof. Let $D' := \{z_1, z_2, \cdots, z_n\} \subset D$. Consider the G-convex space $(X, D'; \Gamma)$ where $\Gamma : \langle D' \rangle \multimap X$ is the restriction of the original $\Gamma : \langle D \rangle \multimap X$. Define a map $F : D' \multimap X$ by $F(z_i) := A_i$ for each $z_i \in D'$. Since

$$\Gamma_{D'} \subset X = \bigcup_{i=1}^{n} A_i = \bigcup_{z \in D'} F(z) = F(D'),$$

by Theorem 5, we have a $J \subset I$ such that $\Gamma(\{z_j : j \in J\}) \cap \bigcap_{j \in J} F(z_j) \neq \emptyset$. This completes our proof. ■

For a convex space X, Theorem 6 reduces to the "matching" theorem of Ky Fan due to Lassonde [17].

We apply Theorem 6 to obtain the Sperner intersection theorem and the Alexan- droff–Pasynkoff theorem. In fact, from Theorem 6, we can deduce Theorems 2 and 3 if $D = \{a_0, a_1, \cdots, a_n\}$ is finite:

Theorem 6 $\Longrightarrow$ Theorem 3. Let $A_i := T(a_{i+1})$ for each $a_i \in D$, $0 \leq i \leq n-1$, and $A_n := T(a_0)$. Then by Theorem 6, there exists a subset $J \subset I = \{0, 1, \cdots, n\}$ such that

$$\Gamma(\{a_j : j \in J\}) \cap \bigcap_{j \in J} A_j \neq \emptyset.$$

However, the isotonicity $(*)$ and condition (iii) of Theorem 3 implies

$$\Gamma(\{a_j : j \in J\}) \subset \Gamma(\{a_0, \cdots, \widehat{a_i}, \cdots, a_n\}) \subset T(a_{i+1}) = A_i$$

for each $i \in I \backslash J$. Therefore,

$$\Gamma(\{a_j : j \in J\}) \subset \bigcap_{i \in I \backslash J} A_i$$

and hence

$$\Gamma(\{a_j : j \in J\}) \subset \bigcap_{i \in I} A_i \neq \emptyset.$$

Thus, we have $\bigcap_{i\in I} T(a_i) = \bigcap_{i\in I} A_i \neq \emptyset$. ■

Theorem 6 $\Longrightarrow$ *Theorem 2.* This is clear since Theorems 2 and 3 are equivalent under the condition $(*)$.

Remark. Now, each of Theorems 1 – 6 is equivalent to the Brouwer fixed point theorem.

In case to emphasize $X \supset D$, a G-convex space $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$.

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D\rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$; and for any subset $Y \subset X$, the Γ-*convex hull* of Y is defined as follows:

$$\Gamma\text{-co}\, Y := \bigcap\{Z \subset X : Z \text{ is a } \Gamma\text{-convex subset of } X \text{ containing } Y\}.$$

It is easily seen that $\Gamma\text{-co}\, Y = \bigcup\{\Gamma\text{-co}\, N : N \in \langle Y\rangle\}$.

As an another application of Theorem 6, we deduce the following Klee type theorem:

Theorem 7. *Let $(X; \Gamma)$ be a G-convex space satisfying*

$$(**) \qquad N \subset \Gamma_N \quad \textit{for each } N \in \langle X\rangle.$$

Let $\{A_i\}_{i=1}^n$ be a family of n closed [resp. open] subsets covering X such that any $n-1$ subsets in the family has nonempty intersection. Then there exists a $J \subset I = \{1, 2, \cdots, n\}$ such that

$$\Gamma\text{-co}(\bigcap\{A_i : i \in I\backslash J\}) \cap \bigcap_{j\in J} A_j \neq \emptyset.$$

Proof. For each $i \in I$, choose a $z_i \in \bigcap\{A_j : j \in I$ and $j \neq i\}$. Then by Theorem 6, there exists a subset $J \subset I$ such that $\Gamma(\{z_j : j \in J\}) \cap \bigcap_{j\in J} A_j \neq \emptyset$. Since $\{z_j : j \in J\} \subset \bigcap\{A_i : I \in I\backslash J\}$, by $(**)$, we have

$$\{z_j : j \in J\} \subset \Gamma(\{z_j : j \in J\}) \subset \Gamma\text{-co}(\bigcap\{A_i : i \in I\backslash J\}).$$

Hence, we have the conclusion. ■

For any convex space X, Theorem 7 reduces to a result of Lassonde [17], which is a generalization of a lemma of Klee [14].

5. The Fan–Browder type fixed point theorems

The KKM theorem has been a rich source of fixed point results from the very beginning; see [18-21, 30, 31]. Recent works on fixed point theorems on G-convex spaces can be seen in [23-29, 35].

In this section, from Theorem 1, we deduce some new forms of the Fan–Browder type fixed point theorems.

Theorem 8. *Let $(X, D; \Gamma)$ be a G-convex space and $S : X \multimap D$, $T : X \multimap X$ maps such that*

(1) *for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$; and*
(2) *there exist $D' := \{z_1, z_2, \cdots, z_n\} \in \langle D \rangle$ and nonempty open [resp. closed] subsets $\{G_i\}_{i=1}^n$ of X such that*

$$X = \bigcup_{i=1}^{n} G_i \quad \text{and} \quad G_i \subset S^-(z_i) \quad \text{for each } i.$$

Then T has a fixed point $x_* \in X$; that is, $x_* \in T(x_*)$.

Proof. Consider the G-convex space $(X, D'; \Gamma)$ where $\Gamma : \langle D' \rangle \multimap X$ is actually the restriction $\Gamma|_{\langle D' \rangle}$ of the original Γ. Define a map $F : D' \multimap X$ by $F(z_i) = X \backslash G_i$ for each $z_i \in D'$. Then each $F(z_i)$ is closed [resp. open] in X, and

$$\bigcap_{i=1}^{n} F(z_i) = X \backslash \bigcup_{i=1}^{n} G_i = X \backslash X = \emptyset.$$

Therefore, the family $\{F(z)\}_{z \in D'}$ does not have the finite intersection property, and hence, F is not a KKM map by Theorem 1. Thus there exists an $N \in \langle D' \rangle$ such that $\Gamma_N \not\subset F(N) = \bigcup\{X \backslash G_i : z_i \in N\}$. Hence there exists an $x_* \in \Gamma_N$ such that $x_* \in G_i \subset S^-(z_i)$ for each $z_i \in N$; that is, $N \in \langle S(x_*) \rangle$. Therefore $x_* \in \Gamma_N \subset T(x_*)$ by (1). This completes our proof. ■

Remark. In Theorem 8, let $X_T := \{x \in X : x \notin T(x)\}$. Then condition $X = \bigcup_{i=1}^n G_i$ in (2) can be replaced by $X_T = \bigcup_{i=1}^n G_i$ without affecting the conclusion of Theorem 8.

In fact, suppose that T has no fixed point, that is, $X = X_T$. Then by Theorem 8, T has a fixed point, a contradiction.

For a G-convex space $(X \supset D; \Gamma)$, we have the following:

Corollary 8.1. *Let* $(X \supset D;\Gamma)$ *be a* G*-convex space and* $P : X \multimap D$ *a multimap. If there exist* $D' := \{z_1, z_2, \cdots, z_n\} \in \langle D\rangle$ *and nonempty open [resp. closed] subsets* $\{G_i\}_{i=1}^n$ *such that* $G_i \subset P^-(z_i)$ *for each* i *and* $\Gamma\text{-co}\, D' \subset \bigcup_{i=1}^n G_i$*. Then the map* $\Gamma\text{-co}\, P : X \multimap X$ *has a fixed point. More precisely, there exist a point* $x_0 \in X$ *and a subset* $N \in \langle P(x_0)\rangle$ *such that* $x_0 \in \Gamma_N \subset \Gamma\text{-co}\, P(x_0)$*.*

A convex space version of Theorem 8 is obtained by Park [28], Sy and Park [41] and applied to obtain the Fan–Browder fixed point theorem, approximate fixed point theorems for convex-valued upper [or lower] semicontinuous multimaps, and the Himmelberg fixed point theorem.

From Theorem 8 and Corollary 8.1, we can deduce various forms of the Fan–Browder type fixed point theorems as follows:

Corollary 8.2. *Let* $(X, D;\Gamma)$ *be a* G*-convex space and* $S : X \multimap D$, $T : X \multimap X$ *maps such that*

(1) *for each* $x \in X$, $M \in \langle S(x)\rangle$ *implies* $\Gamma_M \subset T(x)$;
(2) $S^-(z)$ *is open [resp. closed] for each* $z \in D$; *and*
(3) $X = \bigcup\{S^-(z) : z \in N\}$ *for some* $N \in \langle D\rangle$.

Then T *has a fixed point* $x_* \in X$*; that is,* $x_* \in T(x_*)$*.*

Corollary 8.3. *Let* $(X, D;\Gamma)$ *be a* G*-convex space and* $S : X \multimap D$, $T : X \multimap X$ *maps such that*

(1)′ *for each* $x \in X$, $M \in \langle S(x)\rangle$ *implies* $\Gamma_M \subset T(x)$; *and*
(3)′ $X = \bigcup\{\text{Int}\, S^-(z) : z \in N\}$ *for some* $N \in \langle D\rangle$.

Then T *has a fixed point.*

Corollary 8.4. *Let* $(X \supset D;\Gamma)$ *be a* G*-convex space, and* $S : X \multimap D$, $T : X \multimap X$ *maps. Suppose that*

(1) *for each* $x \in X$, $M \in \langle S(x)\rangle$ *implies* $\Gamma_M \subset T(x)$;
(2) *there exist a nonempty subset* K *of* X*, a subset* $N := \{z_1, z_2, \cdots, z_n\} \in \langle D\rangle$*, and nonempty open [resp. closed] subsets* $\{G_i\}_{i=1}^n$ *of* X *such that*

$$K \subset \bigcup_{i=1}^n G_i \quad \text{and} \quad G_i \subset S^-(z_i) \quad \text{for each } \; i,\ 1 \le i \le n; \ \text{and}$$

(3) *there exist a* Γ*-convex subset* L_N *of* X *containing* N*, a subset* $M := \{z_{n+1}, z_{n+2}, \cdots, z_{n+m}\} \in \langle L_N \cap D\rangle$*, and nonempty open [resp. closed]*

subsets $\{G_i\}_{i=n+1}^{n+m}$ *of* X *such that*

$$L_N \backslash K \subset \bigcup_{i=n+1}^{n+m} G_i \quad \text{and} \quad G_i \subset S^{-}(z_i) \quad \text{for each} \;\; i, \; n+1 \leq i \leq n+m.$$

Then T *has a fixed point in* L_N*.*

Corollary 8.5. *Let* $(X \supset D; \Gamma)$ *be a* G*-convex space,* K *a nonempty subset of* X*, and* $S : X \multimap D$*,* $T : X \multimap X$ *multimaps. Suppose that*

(1) *for each* $x \in X$*,* $M \in \langle S(x) \rangle$ *implies* $\Gamma_M \subset T(x)$*;*
(2) $K \subset \bigcup\{\operatorname{Int} S^{-}(z) : z \in N\}$ *for some* $N \in \langle D \rangle$*; and*
(3) *there exists a* Γ*-convex subset* L_N *of* X *containing* N *such that*

$$L_N \backslash K \subset \bigcup\{\operatorname{Int} S^{-}(z) : z \in M\}$$

for some $M \in \langle L_N \cap D \rangle$*.*

Then T *has a fixed point in* L_N*.*

Corollary 8.6. *Let* $(X \supset D; \Gamma)$ *be a* G*-convex space,* K *a nonempty compact subset of* X*, and* $S : X \multimap D$*,* $T : X \multimap X$ *multimaps. Suppose that*

(1) *for each* $x \in X$*,* $M \in \langle S(x) \rangle$ *implies* $\Gamma_M \subset T(x)$*;*
(2) $K \subset \bigcup\{\operatorname{Int} S^{-}(z) : z \in D\}$*; and*
(3) *for each* $N \in \langle D \rangle$*, there exists a compact* Γ*-convex subset* L_N *of* X *containing* N *such that*

$$L_N \backslash K \subset \bigcup\{\operatorname{Int} S^{-}(z) : z \in L_N \cap D\}.$$

Then T *has a fixed point in* L_N*.*

Particular forms of results in this section or their applications have appeared in [18, 24, 26-30].

6. Maximal elements

From Theorem 8, we have the following:

Theorem 9. *Let $(X;\Gamma)$ be a G-convex space and $P : X \multimap X$ a map having open [resp. closed] fibers. If $P(X)$ is covered by a finite number of fibers of P, then either the map $\Gamma\text{-co}\, P : X \multimap X$ has a fixed point or $P^-(y) = \emptyset$ for some $y \in X$.*

Proof. Suppose $P^-(y) \neq \emptyset$ for all $y \in X$. Then there exists an $x \in P^-(y)$ or $y \in P(x)$. Therefore, $X = P(X)$ and X is covered by a finite number of open [resp. closed] fibers of P. Now, by Corollary 8.1, $\Gamma\text{-co}\, P$ has a fixed point. ■

Theorem 10. *Let $(X \supset D;\Gamma)$ be a compact G-convex space and $P : X \multimap D$ a map such that*

(1) *$x \notin \Gamma\text{-co}\, P(x)$ for all $x \in X$; and*

(2) *$P^-(y)$ is open for all $y \in D$.*

Then there exists an $\overline{x} \in X$ such that $P(\overline{x}) = \emptyset$.

Proof. Suppose $P(x) \neq \emptyset$ for all $x \in X$. Then $X = \bigcup_{y\in D} P^-(y)$. Since X is compact, $X = \bigcup_{y\in N} P^-(y)$ for some $N \in \langle D\rangle$. Then by Theorem 8, $\Gamma\text{-co}\, P$ has a fixed point, which contradicts (1). ■

For a convex space $X = D$, each of Theorems 9 and 10 reduces to the existence theorem of maximal elements due to Yannelis–Prabhakar [42] and others.

7. Approximate fixed points

In this section, we introduce a general approximate fixed point theorem which subsumes many known generalizations of the Brouwer fixed point theorem.

We need the following notion due to Himmelberg [12]: A nonempty subset X of a topological vector space E is said to be almost convex if for any neighborhood V of the origin 0 in E and for any finite set $\{x_1, \cdots, x_n\} \subset X$, there exists a finite set $\{z_1, \cdots, z_n\} \subset X$ such that for each $i \in \{1, \cdots, n\}$, $z_i - x_i \in V$ and $\text{co}\{z_1, \cdots, z_n\} \subset X$. Clearly, each convex set is almost convex, but the converse is not true.

Proposition. *Let X be a subset of a topological vector space E. If X has an almost convex subset Y, then for any $D \in \langle Y \rangle$, (X, D) has a G-convex structure.*

Proof. Choose a neighborhood V of the origin 0 of E. For any $D = \{y_1, y_2, \cdots, y_n\} \in \langle Y \rangle$, there exists a $D' = \{z_1, z_2, \cdots, z_n\} \in \langle Y \rangle$ such that $z_i - y_i \in V$ for all $i = 1, 2, \cdots, n$ and $\operatorname{co} D' \subset Y \subset X$. Define $\Gamma : \langle D \rangle \multimap X$ by $\Gamma(\{y_{i_1}, \cdots, y_{i_k}\}) = \operatorname{co}\{z_{i_1}, \cdots, z_{i_k}\}$ for each $\{y_{i_1}, \cdots, y_{i_k}\} \subset D$. Then $(X, D; \Gamma)$ becomes a G-convex space. ■

In a previous work of Park [27, Lemma 1], Proposition is incorrectly proved.

From Theorem 8, we have the following:

Theorem 11. *Let X be a subset of a topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(y)$ is convex for all $y \in Y$. If there is a totally bounded subset K of X such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$, then for any open [resp. closed] convex neighborhood U of the origin 0 of E, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Proof. There exists a symmetric open neighborhood V of 0 such that $\overline{V} + \overline{V} \subset U$. Since K is totally bounded in E, there exists a finite subset $\{x_1, \cdots, x_n\} \subset K$ such that $K \subset \bigcup_{i=1}^n (x_i + V)$. Moreover, since Y is almost convex and dense in X, there exists a finite subset $\{y_1, \cdots, y_n\}$ of Y such that $y_i - x_i \in V$ for each $i = 1, \cdots, n$, and $Z := \operatorname{co}\{y_1, \cdots, y_n\} \subset Y$.

Now we apply Theorem 8 with $(Z, \{y_i\}_{i=1}^n; \operatorname{co})$ instead of $(X, D; \Gamma)$. Since $T(y) \cap K \neq \emptyset$ for each $y \in Z \subset Y$ and $K \subset \bigcup_{i=1}^n (x_i + V) \subset \bigcup_{i=1}^n (y_i + U)$, we have $(T(y) - U) \cap \{y_i\}_{i=1}^n \neq \emptyset$.

Let us define a map $F : Z \multimap Z$ by $F(y) = (T(y) - U) \cap Z$ for $y \in Z$. Then F has nonempty convex values.

Case 1. T is lower semicontinuous and U is open: Then for each $z \in Z$, $F^-(z) = \{y \in Z : z \in T(y) - U\} = \{y \in Z : T(y) \cap (z + U) \neq \emptyset\}$ is open.

Case 2. T is upper semicontinuous and U is closed: Then for each $z \in Z$, $F^-(z)$ is closed.

Moreover, $Z = \bigcup_{i=1}^n F^-(y_i)$ since, for each $z \in Z$, we have a $y_i \in Z$ such that $y_i \in (T(z) - U) \cap Z = F(z)$.

Therefore, by Theorem 8, we have a fixed point $y_U \in Z \subset Y$ of F, that is, $y_U \in F(y_U) = (T(y_U) - U) \cap Z$ and hence $T(y_U) \cap (y_U - U) \neq \emptyset$. ■

Particular forms of Theorem 11 and their applications have appeared in [21, 37, 38] with different proofs.

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