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On solutions of generalized complementarity and eigenvector problems

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Abstract

From a new Fan–Browder type fixed point theorem due to the second author, we deduce an existence theorem for a solution of an equilibrium problem in Section 3. This theorem is applied to generalized complementarity problems in Section 4 and to eigenvector problems in Section 5.

Keywords: KKM mappings; Fixed point; Equilibrium problem; Generalized complementarity problem; Eigenvector problem

1. Introduction

The complementarity theory has been applied in many fields of mathematics such as optimization, game theory, economics, and engineering [3,8–10]. The study of solvability of complementarity problems is the most important issue in the complementarity theory, and has been investigated by a number of authors [1–3,8–12,16]. We know that the existence of solutions of a complementarity problem depends on the coercivity (or compactness) conditions of the functional and its domain in the given problem. Karamardian's condition introduced in [12] has been applied by many authors. In 1990, Harker and Pang introduced another condition in [7]. In 2001, Isac and Li introduced a more general condition in [11]. By these conditions those authors obtained some existence theorems.

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It is known that in some cases, finding a fixed point of a mapping, solving a variational inequality, and solving a complementarity problem are equivalent. One of the useful tools to solve an equilibrium problem is the Fan–KKM theorem. Therefore fixed point theory, KKM mappings, and a Fan–KKM theorem play key roles in studying of the existence of solutions of a complementarity problem. In Section 2, we prove a Fan–Browder-type fixed point theorem and recall a generalized Fan–KKM theorem due to the second author. In Section 3, we introduce a new coercivity condition and prove some existence theorems that are more general than the conditions mentioned above. In Section 4, we use the equilibrium theorems obtained in Section 3 to study the eigenvector problems investigated by Li [15].

2. Basic existence theorems

A *multimap* (or simply, a *map*) $F: X \multimap Y$ is a function from a set X into the power set 2^Y of a set Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X : y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) : x \in A\}$.

For a set D, let $\langle D\rangle$ denote the set of nonempty finite subsets of D.

Let X be a subset of a vector space and D a nonempty subset of X. We call (X, D) a *convex space* if $\operatorname{co} D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D\rangle$ (see Park [17]). If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [13]. If X is compact, then the convex space (X, D) is said to be compact. Every nonempty convex subset X of a topological vector space is a convex space with respect to any nonempty subset D of X, and the converse is known to be not true.

The following version of the Knaster–Kuratowski–Mazurkiewicz (simply, KKM) principle for convex spaces is known [17,19,20].

Lemma 1. *Let (X, D) be a convex space and $F : D \multimap X$ a multimap such that*

- *$F(z)$ is open [resp. closed] for each $z \in D$; and*
- *F is a KKM map (that is, $\operatorname{co} N \subset F(N)$ for each $N \in \langle D\rangle$).*

Then $\{F(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for any $N \in \langle D\rangle$, we have $\operatorname{co} N \cap [\bigcap_{z\in N} F(z)] \neq \emptyset$.)

For a convex space (X, D), a subset Y of X is called a *convex subspace* of (X, D) if $(Y, Y \cap D)$ itself is a convex space, that is, $\operatorname{co} A \subset Y$ for any $A \in \langle Y \cap D\rangle$.

The following form of the KKM theorem is due to Park [20–22].

Lemma 2. *Let (X, D) be a convex space, K a nonempty compact subset of X, and $F : D \multimap X$ a multimap such that*

- $\bigcap_{z\in D} F(z) = \bigcap_{z\in D} \overline{F(z)}$;
- *F is a KKM map; and*

- *for each* $N \in \langle D \rangle$, *there exists a compact convex subspace* L_N *of* (X, D) *containing* N *such that*

$$L_N \cap \bigcap \{\overline{F(z)} : z \in L_N \cap D\} \subset K.$$

Then $K \cap \bigcap \{F(z) : z \in D\} \neq \emptyset$.

From Lemma 1, we deduced the following form of the Fan–Browder fixed point theorem; (see Park [21,22]).

Theorem 1. *Let* (X, D) *be a convex space and* $P : X \multimap D$ *a multimap. If there exist* $z_1, z_2, \ldots, z_n \in D$ *and nonempty open* [*resp. closed*] *subsets* $G_i \subset P^-(z_i)$ *for each* $i = 1, 2, \ldots, n$ *such that* $\mathrm{co}\{z_1, z_2, \ldots, z_n\} \subset \bigcup_{i=1}^n G_i$, *then the map* $\mathrm{co}\, P : X \multimap X$ *has a fixed point* $x_0 \in X$ (*more precisely*, *there exists an* $N \in \langle P(x_0) \rangle$ *such that* $x_0 \in \mathrm{co}\, N \subset \mathrm{co}\, P(x_0)$).

Theorem 2. *Let* (X, D) *be a convex space and* $P : X \multimap D$ *a multimap. Suppose that*

(i) *there exist a finite subset* $N = \{y_1, y_2, \ldots, y_n\} \subset D$ *and nonempty open* [*resp. closed*] *subsets* $A_i \subset P^-(y_i)$ *for each* $i = 1, 2, \ldots, n$; *and*
(ii) *there exists a convex subspace* L_N *of* (X, D) *containing* N, *a finite subset* $M = \{z_1, z_2, \ldots, z_m\} \subset L_N \cap D$, *and nonempty open* [*resp. closed*] *subsets* $B_j \subset P^-(z_j)$ *for each* $j = 1, 2, \ldots, m$, *such that*

$$L_N \backslash K \subset \bigcup_{j=1}^{m} B_j \quad \text{where } K := \bigcup_{i=1}^{n} A_i.$$

Then the map $\mathrm{co}\, P : X \multimap X$ *has a fixed point*.

Proof. Note that

$$\mathrm{co}(M \cup N) \subset L_N = (L_N \backslash K) \cup (L_N \cap K) \subset \bigcup_{j=1}^{m} B_j \cup \bigcup_{i=1}^{n} A_i.$$

Therefore, by Theorem 1, $\mathrm{co}\, P : X \multimap X$ has a fixed point in L_N. □

The following is a practical form of Theorem 2.

Corollary 2.1. *Let* (X, D) *be a convex space*, K *a nonempty subset of* X, *and* $S : X \multimap D$, $T : X \multimap X$ *multimaps*. *Suppose that*

(i) *for each* $x \in X$, $\mathrm{co}\, S(x) \subset T(x)$;
(ii) $K \subset \bigcup \{\mathrm{Int}\, S^-(z) : z \in N\}$ *for some* $N \in \langle D \rangle$; *and*

(iii) *there exists a convex subspace* L_N *of* (X, D) *containing* N *such that*

$$L_N \backslash K \subset \bigcup_{z \in M} \operatorname{Int} S^-(z) \quad \textit{for some } M \in \langle L_N \cap D \rangle.$$

Then T *has a fixed point in* L_N.

From now on we consider convex spaces $X = (X, X)$ for the case $X = D$ for simplicity.

Let X be a convex space. According to Lassonde [13], a nonempty set $L \subset X$ is called a *c-compact* set if for each finite subset $N \subset X$ there is a compact convex set $L_N \subset X$ such that $L \cup N \subset L_N$.

The following are trivial examples of c-compact sets:

1. In any convex space, every finite set and every convex hull of a finite set are c-compact.
2. Let X be any convex set in a Hausdorff topological vector space. Every nonempty compact convex set in X is c-compact.
3. Let X be any quasi-complete convex set in a Hausdorff locally convex topological vector space. Every nonempty precompact set in X is c-compact.

Corollary 2.2. *Let* X *be a convex space and* $S : X \multimap X$ *a multimap such that*

(i) *for each* $x \in X$, $S(x)$ *is open in* X;
(ii) *for each* $y \in X$, $S^-(y)$ *is nonempty and convex*; *and*
(iii) *for some c-compact set* $L \subset X$, *the set* $X \backslash \bigcup_{x \in L} S(x)$ *is compact*.

Then S *has a fixed point*.

Corollary 2.2 is a particular form of Lassonde [13, Theorem 1.1] and extends the Fan–Browder fixed point theorem.

3. Equilibrium theorems

The following main result of this section is an existence theorem on an equilibrium problem:

Theorem 3. *Let* X *be a convex space and* $g, h : X \times X \to \mathbb{R}$ *real functions such that*

(i) *for any* $x, y \in X$, $g(x, y) \leqslant h(x, y)$ *and* $h(x, x) \leqslant 0$;
(ii) *for any* $x \in X$, $\{y \in X : g(x, y) > 0\}$ *is open in* X;
(iii) *for any* $y \in X$, $\{x \in X : h(x, y) > 0\}$ *is convex*; *and*
(iv) *there is a nonempty compact subset* K *of* X *such that, for each* $N \in \langle X \rangle$, *we have a compact convex subset* L_N *of* X *containing* N *such that, for each* $y \in L_N \backslash K$, *there exists a* $z \in L_N$ *satisfying* $g(z, y) > 0$.

Then there exists a $y_0 \in K$ *such that*

$$g(x, y_0) \leqslant 0 \quad \textit{for all } x \in X.$$

Proof. Suppose on the contrary that for each $y \in K$ there exists a $z \in X$ such that $g(z, y) > 0$. Define a map $F : X \multimap X$ by

$$F(y) := \{z \in X : h(z, y) > 0\} \quad \text{for } y \in X.$$

Then each $F(y)$ is convex by (iii). Moreover, for each $z \in X$,

$$F^{-}(z) = \{y \in X : h(z, y) > 0\} \supset \text{Int } F^{-}(z) \supset \{y \in X : g(z, y) > 0\}$$

by (i) and (ii). By our assumption, for each $y \in K$, there exists a $z \in X$ such that $y \in \text{Int } F^{-}(z)$. Since K is compact, there exists an $N \in \langle X \rangle$ such that $K \subset \bigcup_{z \in N} \text{Int } F^{-}(z)$. Then, by (iv), there exists a compact convex subset L_N of X containing N such that $L_N \backslash K \subset \bigcup_{z \in L_N} \text{Int } F^{-}(z)$. Since L_N is compact and $L_N \subset (L_N \backslash K) \cup K$, we have an $N' \in \langle L_N \rangle$ such that

$$L_N \subset \left[\bigcup_{z \in N'} \text{Int } F^{-}(z)\right] \cup \left[\bigcup_{z \in N} \text{Int } F^{-}(z)\right] = \bigcup_{z \in M} \text{Int } F^{-}(z),$$

where $M := N' \cup N \in \langle L_N \rangle$. Then, by Corollary 2.1 with $X = D$ and $S = T = F$, F has a fixed point $x_0 \in L_N$; that is, $x_0 \in F(x_0)$ or $h(x_0, x_0) > 0$, which contradicts (i). This completes our proof. □

For a convex space X, a real function $f : X \to \mathbb{R}$ is said to be *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is convex for each $r \in \mathbb{R}$.

Recall that a real function $f : X \to \mathbb{R}$, where X is a topological space, is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is open for each $r \in \mathbb{R}$.

Remarks. 1. In Theorem 3, conditions (ii) and (iii) can be replaced by the following:

(ii)′ for any $x \in X$, $y \mapsto g(x, y)$ is l.s.c.;
(iii)′ for any $y \in X$, $x \mapsto h(x, y)$ is quasiconcave.

2. In Theorem 3, K is not necessarily compact. Instead, if $K \subset F^{-}(N) = \bigcup_{z \in N} \{y \in X : h(z, y) > 0\}$ for some $N \in \langle X \rangle$ and if, for this N, we have a subset L_N (for example, co N) satisfying the requirement as in (iv), then we have the conclusion.

3. Some authors might prefer to say that the set in condition (ii) is compactly open. This is not practical and, by choosing the compactly generated extension of the original topology on X, Theorem 3 still works (see Park [20]).

Corollary 3.1. *In Theorem* 3, *condition* (iv) *can be replaced by the following without affecting its conclusion*:

(iv)′ *X has a nonempty c-compact subset L such that* $K := \{y \in X : g(x, y) \leqslant 0$ *for all* $x \in L\}$ *is compact.*

Then there exists a point $y_0 \in K$ *such that* $g(x, y_0) \leqslant 0$ *for all* $x \in X$.

Remark. Corollary 3.1 includes results of Fan [6], Allen [1], and many others.

Corollary 3.2. *Let X be a convex space and* $g : X \times X \to \mathbb{R}$ *a real function such that*

(i) *for any* $x \in X$, $\{y \in X : g(x, y) < g(y, y)\}$ *is open in X*;
(ii) *for any* $y \in X$, $\{x \in X : g(x, y) < g(y, y)\}$ *is convex*; *and*
(iii) *there is a nonempty compact subset K of X such that*, *for each* $N \in \langle X \rangle$, *we have a compact convex subset* L_N *of X containing N such that*, *for each* $y \in L_N \backslash K$, *there exists a* $z \in L_N$ *satisfying* $g(z, y) < g(y, y)$.

Then there exists a $y_0 \in K$ *such that*

$$g(x, y_0) \geqslant g(y_0, y_0) \quad \text{for all } x \in X.$$

Proof. In Theorem 3, put $g = h$ and replace $g(x, y)$ by $g(y, y) - g(x, y)$. □

If $X = K$, then Corollary 3.2 reduces to the following:

Corollary 3.3. *Let X be a compact convex space and* $g : X \times X \to \mathbb{R}$ *a real function satisfying conditions* (i) *and* (ii) *of Corollary* 3.2. *Then there exists a point* $y_0 \in X$ *such that*

$$g(x, y_0) \geqslant g(y_0, y_0) \quad \text{for all } x \in X.$$

Remark. Conditions (i) and (ii) can be replaced by the following, respectively:

(i)′ for any $x \in X$, $y \mapsto g(x, y) - g(y, y)$ is u.s.c. on X; and
(ii)′ for any $y \in X$, $x \mapsto g(x, y)$ is quasiconvex on X.

Then Corollary 3.3 improves to that by Fan [6, Corollary 1], which is actually equivalent to Fan minimax inequality [6].

4. Generalized complementarity problems

Generalized complementarity problems are introduced by Karamardian [14] as follows:

Let E be a topological vector space, F a vector space (for example, the dual space of E), and $\langle\ ,\ \rangle : E \times F \to \mathbb{R}$ a bilinear form. A nonempty subset X of E is called a *cone* iff

$$\alpha x + \beta y \in X \quad \text{for all } \alpha, \beta \geqslant 0 \text{ and } x, y \in X.$$

The *polar* X^* of a cone X is the cone in F defined by

$$X^* := \{p \in F : \langle x, p\rangle \geqslant 0, \forall x \in X\}.$$

Then, the *generalized complementary problem* (GCP) for a function $f : X \to F$ is to find an $x_0 \in X$ satisfying

$$f(x_0) \in X^*, \quad \langle x_0, f(x_0)\rangle = 0. \tag{0}$$

From Theorem 2, we have the following theorem.

Theorem 4. *Let E, F, X, and f be given as above. The GCP, as given by* (0), *has a solution x_0 whenever*:

(i) *for any $x \in X$, $y \mapsto \langle x - y, f(y)\rangle$ is u.s.c. on X*;
(ii) *there exists a nonempty compact subset K of X such that, for each nonempty finite subset N of X, we have a compact convex subset L_N of X containing N such that, for each $x \in L_N \backslash K$, there is a $u \in L_N$ satisfying*

$$\langle x - u, f(x)\rangle > 0.$$

Moreover, $x_0 \in K$.

Proof. Put $g(x, y) = \langle x - y, f(y)\rangle$ in Corollary 3.2. Then, all of the requirements are satisfied. Therefore, there exists an $x_0 \in K$ such that

$$\langle x, f(x_0)\rangle \geqslant \langle x_0, f(x_0)\rangle, \quad \forall x \in X.$$

This implies that

$$\langle \alpha x, f(x_0)\rangle \geqslant \langle x_0, f(x_0)\rangle, \quad \forall \alpha > 1 \text{ and } \forall x \in X.$$

By considering $x = 2x_0$, we have

$$\langle x_0, f(x_0)\rangle \geqslant 0.$$

On the other hand, by considering $x = 0$, we have

$$\langle x_0, f(x_0)\rangle \leqslant 0.$$

Therefore,

$$\langle x_0, f(x_0)\rangle = 0$$

and

$$\langle x, f(x_0)\rangle \geqslant 0, \quad \forall x \in X,$$

whence we have $f(x_0) \in X^*$. This completes our proof. □

Remark. Theorem 4 is first obtained by Park [16, Corollary 2.2] for a continuous map $f : X \to E^*$, and later [18, Theorem 3.1] whenever the function $(x, y) \mapsto \langle x, f(y)\rangle$ is continuous on $X \times X$. In [21], different methods were adopted.

We give some particular forms of the coercivity condition (ii) in Theorem 4 as follows:

- "(I)" (Karamardian) *E is Hausdorff and there exists a nonempty compact convex subset D of X with the property that, for every* $x \in X\backslash D$, *there exists* $z \in D$ *such that*

$$\langle x - z, f(x)\rangle > 0.$$

We show (I) $\Longrightarrow$ (ii). Put $K := D$ and $L_N := \overline{\mathrm{co}}(D \cup N)$ for any $N \in \langle X\rangle$. Since D is a c-compact subset of the convex set X and E is Hausdorff, we have $L_N \subset X$. Then for each $x \in L_N\backslash K \subset X\backslash D$, there exists $z \in D \subset L_N$ such that $\langle x - z, f(x)\rangle > 0$. Karamardian's condition (I) was applied to obtain particular forms of Theorem 4 by Karamardian [12] and Isac [9].

- "(II)" (Harker–Pang) *There exists an* $x_* \in X$ *such that the set*

$$X(x_*) := \{x \in X : \langle x - x_*, f(x)\rangle \leqslant 0\}$$

is relatively compact (*or empty*) *in X.*

We show (II) $\Longrightarrow$ (ii). Put $K := \overline{X(x_*)}$, the closure in X, and $L_N := \mathrm{co}(N \cup \{x_*\})$ for each $N \in \langle X\rangle$. Then for each $x \in L_N\backslash K \subset X\backslash X(x_*)$, there exists $x_* \in L_N$ such that $\langle x - x_*, f(x)\rangle > 0$. Condition (II) was used by Harker and Pang [7] to study the solvability of variational inequality in $\mathbb{R}^n$.

- "(III)" (Ding–Tan) *E is Hausdorff and there exist a nonempty compact convex subset* $D_0 \subset X$ *and a nonempty compact subset* $D_* \subset X$ *such that for each* $x \in X\backslash D_*$, *there is a* $y \in \mathrm{co}(D_0 \cup \{x\})$ *verifying* $\langle x - y, f(x)\rangle > 0$.

We show (III) $\Longrightarrow$ (iii). Put $K := D_*$ and $L_N := \overline{\mathrm{co}}(D_0 \cup N) \subset X$ for any $N \in \langle X\rangle$. Then for each $x \in L_N\backslash K \subset X\backslash D_*$, there exists a $y \in \mathrm{co}(D_0 \cup \{x\}) \subset L_N$ such that $\langle x - y, f(x)\rangle > 0$. Condition (III) appeared in [4,9], and in many other papers, and is more general than Karamardian's condition (I).

- "(IV)" (Isac–Li) *X is a quasi-complete cone in a Hausdorff locally convex topological vector space E*; *and there exists a precompact subset D of X such that the set*

$$X(D) := \{x \in X : \langle x - y, f(x)\rangle \leqslant 0 \text{ for all } y \in D\}$$

is compact.

We show (IV) $\Longrightarrow$ (ii). In fact, put $K := X(D)$ and $L_N := \overline{\mathrm{co}}(D \cup N)$ for each $N \in \langle X\rangle$. Then L_N is a compact convex set since X is quasi-complete and E is locally convex. Now, for each $x \in L_N\backslash K$, we have $x \in X\backslash X(D)$. Hence, by (IV), there exists a $y \in D \subset L_N$ such that $\langle x - y, f(x)\rangle > 0$. Condition (IV) was applied to obtain particular forms of Theorem 4 by Isac and Li [11, Theorems 4, 6, and 7]. Note that the solution $x_0 \in X(D)$.

- "(V)" (Isac–Li) *$E = F$ is an inner-product space and there exists a nonempty compact convex subset D of X such that*

$$M := \bigcup_{y \in D} \{x \in X : \|f(x)\| \leqslant \|y - (x - f(x))\|\}$$

is relatively compact.

We show (V) $\Longrightarrow$ (ii). Let K be the closure of M and let $L_N := \overline{\mathrm{co}}(D \cup N)$ for each $N \in \langle X\rangle$. Then, for each $x \in L_N \backslash K \subset X \backslash M$, there exists a $y \in D \subset L_N$ such that

$$\|f(x)\| > \|y - (x - f(x))\|,$$

which can be shown to imply $\langle x - y, f(x)\rangle > 0$. Condition (V) was applied to obtain a particular form of Theorem 4 by Isac and Li [11, Theorem 3]. Note that, in this case, the solution $x_0 \in K = \overline{M}$.

- "(VI)" (Li) *X is a nonempty, closed, and convex subset of a Banach space B with dual space B^*. Let $T : X \to B^*$ be a continuous function. There exists an element $y_0 \in K$ such that the subset of X*

$$\{x \in X : 2\langle Jx - \alpha(Tx - \xi), y_0 - x\rangle + \|x\|^2 \leqslant \|y_0\|^2\}$$

is compact, where $\xi \in B^*$ and α is an arbitrary positive constant.

We show that (VI) $\Longrightarrow$ (ii) for the case that $\xi = 0$.

Let $K := \{x \in X : 2\langle Jx - \alpha(Tx - \xi), y_0 - x\rangle + \|x\|^2 \leqslant \|y_0\|^2\}$. Since $y_0 \in K$, K is a nonempty, bounded subset of X. For any $N \in \langle X\rangle$ we choose $L_N = K \cup N$. It is clear that L_N is a closed, bounded subset of X. For any $x \in L_N \backslash K$, we take $y_0 \in K \subset L_N$. Since $x \notin K$, from the definition of K, we have

$$2\langle Jx - \alpha(Tx - \xi), y_0 - x\rangle + \|x\|^2 > \|y_0\|^2.$$

Noting $\langle Jx, x\rangle = \|x\|^2$, we get

$$2\langle Jx, y_0\rangle - \|x\|^2 - 2\alpha\langle Tx, y_0 - x\rangle > \|y_0\|^2.$$

Thus,

$$\begin{aligned} 2\alpha\langle Tx, x - y_0\rangle &> \|y_0\|^2 + \|x\|^2 - 2\langle Jx, y_0\rangle \\ &\geqslant \|y_0\|^2 + \|x\|^2 - 2\|x\|\,\|y_0\|^2 \\ &= (\|y_0\| - \|x\|)^2 \geqslant 0. \end{aligned}$$

This shows (VI) $\Longrightarrow$ (ii).

In [15], Li used condition (VI) to obtain some results of the existence of solutions of the variational inequality. We note that in (VIII), it is not necessary for X to be a convex cone.

In case X is a closed cone, condition (VI) reduces to the following:

- "(VII)" (Li)

$$\left\{x \in X : \langle Tx - \xi, x\rangle \leqslant \frac{1}{2\alpha}\|x\|^2\right\}$$

is compact.

Similar to the proof of (VI) $\Longrightarrow$ (ii), we can prove that (VII) $\Longrightarrow$ (ii).

Corollary 4.1. *Let X be a cone in a reflexive Banach space E* (*for example*, *a Hilbert space*), $\langle\ ,ran : E \times E \to \mathbb{R}$ *a bilinear form*, *and* $f : X \to E$ *a function*. *Then the GCP*, *as given by* (0), *has a solution* x_0 *whenever*

- "(i)" *for any* $x \in X$, $y \mapsto \langle x - y, f(y)\rangle$ *is weakly u.s.c. on X*;
- "(ii)" *there exists a nonempty bounded subset K of X such that*, *for each nonempty finite subset N of X*, *we have a closed bounded subset* L_N *of X containing N such that*, *for each* $x \in L_N \backslash K$, *there is a* $u \in L_N$ *satisfying* $\langle x - u, f(x)\rangle > 0$.

Proof. We apply Theorem 4 with the weak topology on E. Let K' be the weak closure of K in X. Then Theorem 4 holds for K' instead of K. In fact, for each $x \in L_N\backslash K' \subset L_N\backslash K$, there exists a $u \in L_N$ satisfying $\langle x - u, f(x)\rangle > 0$.

We give some particular forms of condition (ii) in Corollary 4.1 as follows:

- "(VIII)" (Isac–Li) *X is a closed cone and there exists a bounded set D of X such that the set*

$$X(D) = \{x \in X : \langle x - y, f(x)\rangle \leqslant 0 \text{ for all } y \in D\}$$

is bounded.

We show (VIII) $\Longrightarrow$ (ii). Let $K := X(D)$ and $L_N := \overline{\text{co}}(D \cup N)$ for each $N \in \langle X\rangle$. Then, for each $x \in L_N\backslash K \subset X\backslash X(D)$, there is a $y \in D \subset L_N$ satisfying $\langle x - y, f(x)\rangle > 0$. Condition (VIII) was used to obtain a particular form of Corollary 4.1 by Isac and Li [11, Theorem 6].

- "(IX)" *There exists* $\rho > 0$ *such that for all* $x \in X$ *with* $\|x\| > \rho$, *there exists* $y \in X$ *with* $\|y\| < \|x\|$ *such that* $\langle x - y, f(x)\rangle > 0$.

We show that (IX) $\Longrightarrow$ (ii). Let $\overline{B_\rho} := \{x \in E : \|x\| \leqslant \rho\}$ for $\rho > 0$. Let $K := \overline{B_\rho} \cap X$. For each $N \in \langle X\rangle$, let

$$\nu = \max\{\|x\| : x \in N\}, \quad \mu = \max\{\nu, \rho\}, \quad L_N := \overline{B_\mu} \cap X.$$

Then L_N is a closed bounded convex subset of X containing N. Now for each $x \in L_N\backslash K$, we have $\mu \geqslant \|x\| > \rho$, and hence, by (IX), there exists a $y \in X$ with $\|y\| < \|x\|$ such that $\langle x - y, f(x)\rangle > 0$. Since $\mu \geqslant \|x\| > \|y\|$, we have $y \in \overline{B_\mu}$ and hence $y \in L_N$. This shows (IX) $\Longrightarrow$ (ii).

Remark. Condition (IX) is a little stronger than condition (θ) of Isac [10], which implies several existence theorems for particular cases of GCPs.

5. Eigenvector problems

In this section, we apply our equilibrium theorem to the eigenvector problems investigated by Li [14].

Let $(E, \|\cdot\|)$ be a normed linear space with origin O. Let $f : X \to E$ be a function from a nonempty subset X of E into E. A point $x \in X$, $x \neq O$, satisfying $f(x) \neq O$, is called an *eigenvector* of f if $x \in \mathbb{R}f(x) := \{rf(x) : r \in \mathbb{R}\}$, that is,

$$d(x, \mathbb{R}f(x)) := \inf\{\|x - y\| : y \in \mathbb{R}f(x)\} = 0.$$

In this section, we obtain results that, if X and f satisfy certain conditions, then there exists a point $x_0 \in X$ such that x_0 is one of the closest points in X to the line $\mathbb{R}f(x_0)$; that is, there exists an $x_0 \in X$ satisfying

$$d(x_0, \mathbb{R}f(x_0)) = \min\{d(x, \mathbb{R}f(x_0)) : x \in X\} = d(X, \mathbb{R}f(x_0)).$$

Theorem 5. *Let E be a normed vector space and X a convex subset of E. Let $f : X \to E$ be a continuous map satisfying $O \notin f(X)$. Suppose that*

- "(∗)" *there is a nonempty compact subset K of X such that, for each $N \in \langle X\rangle$, we have a compact convex subset L_N of X containing N such that, for each $y \in L_N \backslash K$, there exists a $z \in L_N$ satisfying*

  $$d(z, \mathbb{R}f(y)) < d(y, \mathbb{R}f(y)).$$

Then there exists a $y_0 \in K$ such that

$$d(x, \mathbb{R}f(y_0)) \geqslant d(y_0, \mathbb{R}f(y_0)) \quad \textit{for all } x \in X.$$

Proof. We apply Corollary 3.2 by putting $g(x, y) := d(x, \mathbb{R}f(y))$ for $(x, y) \in X \times X$. Define a multimap $F : X \multimap X$ by $F(x) := \{y \in X : d(x, \mathbb{R}f(y)) < d(y, \mathbb{R}f(y))\} = \{y \in X : g(x, y) < g(y, y)\}$ for each $x \in X$. Then $F(x)$ is shown to be open (see the argument in [17, pp. 740–743]). Moreover, for each $y \in X$, $F^-(y) = \{x \in X : d(x, \mathbb{R}f(y)) < d(y, \mathbb{R}f(y))\} = \{x \in X : g(x, y) < g(y, y)\}$ is convex. In fact, let $r := d(y, \mathbb{R}f(y))$ and we have to show that $d(x_1, \mathbb{R}f(y)) < r$ and $d(x_2, \mathbb{R}f(y)) < r$ imply $d(tx_1 + (1 - t)x_2, \mathbb{R}f(y)) < r$ for $t \in (0, 1)$. Since $\|x_1 - z_1\| < r$ and $\|x_2 - z_2\| < r$ for some $z_1, z_2 \in \mathbb{R}f(y)$, we have $\|tx_1 + (1 - t)x_2 - (tz_1 + (1 - t)z_2\| \leqslant t\|x_1 - z_1\| + (1 - t)\|x_2 - z_2\| < r$ and $tz_1 + (1 - t)z_2 \in \mathbb{R}f(y)$. This shows the convexity of $F^-(y)$. Further, condition (∗) is the same to (iii) of Corollary 3.2. Therefore, we have the conclusion.

Corollary 5.1. *In Theorem* 5, *the coercivity condition* $(*)$ *can be replaced by one of the following without affecting its conclusion*:

- "$(**)$" *There exists a compact convex subset L of X such that*

$$K := \bigcap_{z\in L} \{y \in X : d(z, \mathbb{R}f(y)) \geqslant d(y, \mathbb{R}f(y))\}$$

  is contained in a compact subset of X.
- "$(^{***})$" *X itself is compact.*

Remark. Li [14, Theorem 1] obtained Corollary 5.1 under the restriction that X itself is closed in E from a form of the KKM theorem due to Fan [5, Lemma 1] which was generalized by Park [19–23] in several ways, from which Theorem 5 follows by following Li's method.

From Theorem 5, we obtain an existence theorem of eigenvectors as follows:

For a subset X of a normed vector space E, the *vertical cone* $\mathrm{VC}(X)$ of X is defined by

$$\mathrm{VC}(X) := \bigcup_{u\in X} \mathbb{R}u;$$

see [17].

Theorem 6. *Let X be a convex subset of a normed vector space E, and* $f : X \to E$ *a continuous function satisfying* $O \notin f(X) \subset VC(X)$. *Suppose that condition* $(*)$ *of Theorem* 5 *holds. Then there exists an* $x_0 \in X$ *such that* $x_0 \in \mathbb{R}f(x_0)$. *Further, if* $O \notin X$, *then f has an eigenvector.*

Proof. Since f is continuous, it satisfies condition $(^*)$ of Theorem 5. From the conclusion of Theorem 5, there exists $x_0 \in K$ such that

$$d(x_0, \mathbb{R}f(x_0)) \leqslant d(x, \mathbb{R}f(x_0)) \quad \text{for all } x \in K.$$

From the condition $f(X) \subset VC(X)$, there exists $x_1 \in X$ and $r_1 \in \mathbb{R}$ such that $f(x_0) = r_1 x_1$.

Since $O \notin f(x)$, $r_1 \neq 0$ and $x_1 \neq O$. We get $x_1 = (1/r_1) f(x)$. So, $x_1 \in \mathbb{R}f(x)$, which yields $d(x_1, \mathbb{R}f(x_0)) = 0$. From $d(x_0, \mathbb{R}f(x_0)) \leqslant d(x_1, \mathbb{R}f(x_0)) = 0$, we obtain $d(x_0, \mathbb{R}f(x_0)) = 0$. So $x_0 \in \mathbb{R}f(x_0)$. This proves the first part of this theorem.

From $x_0 \in \mathbb{R}f(x_0)$, there exists $r_0 \in \mathbb{R}$, such that $x_0 = r_0 f(x_0)$. If $O \notin X$, then $r_0 \neq 0$. So $f(x_0) = (1/r_0)x_0$. Hence, x_0 is an eigenvector of f with the eigenvalue $1/r_0$. Thus this theorem is proved. □

Corollary 6.1. *In Theorem* 6, *the coercivity condition* $(*)$ *can be replaced by condition* $(**)$ *or condition* $(***)$ *of Corollary* 5.1 *without affecting its conclusion.*

Remark. Li [14, Theorem 2 and Corollary 2] obtained Corollary 6.1 under the restriction that X itself is closed in E. He applied Corollary 6.1 for case $(^{***})$ to existence theorems of eigenvectors in finite-dimensional normed vector spaces (Euclidean spaces).

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