COINCIDENCE THEOREMS ON ω-CONNECTED SPACES

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Abstract. We obtain general coincidence theorems and related results for multimaps in very large classes defined on ω-connected spaces. Our typical consequence is as follows: Let X be a compact ω-connected topological space, and $F : X \multimap X$ a multimap with nonempty values and open fibers such that, for each open subset $O \subset X$, $\bigcap_{x \in O} Fx$ is empty or ω-connected. Then F has a fixed point.

1. Introduction

Let us consider the following well-known fixed point theorem due to Felix Browder [4] in 1968: *Let X be a compact convex subset of a topological vector space, and $F : X \multimap X$ a multimap with nonempty convex values Fx for $x \in X$ and open fibers $F^{-}y := \{x \in X : y \in Fx\}$ for $y \in X$. Then F has a fixed point $x_0 \in X$, that is, $x_0 \in Fx_0$.* This is usually called the Fan-Browder fixed point theorem.

Our principal aim in this paper is to obtain far-reaching generalizations of the theorem to ω-connected topological spaces; for example, we have the following: *Let X be a compact ω-connected topological space, and $F : X \multimap X$ a multimap with nonempty values and open fibers such that, for each open subset $O \subset X$, $\bigcap_{x \in O} Fx$ is empty or ω-connected. Then F has a fixed point.* Some related coincidence theorems and other results are also obtained.

2000 *Mathematics Subject Classification.* Primary 54H25, 55M20, Secondary 47H10.

Key words and phrases. Multimap (map), admissible class of maps, generalized convex (or G-convex) space, H-space, C-space, hyperconvex metric space.

Recall that, in a sequence of papers [8-12], Charles Horvath initiated the study of his C-spaces, which have a very useful abstract convexity. More precisely, replacing convexity by contractibility (or more generally, by ω-connectedness), he obtained generalizations of many results in convex analysis including KKM theory and fixed point theory. Especially, Horvath obtained a selection theorem, coincidence theorems, and fixed point theorems in contractible spaces. This line of study is followed by Tarafdar and Yuan [35, 37], Park and Jeong [27], Ding [5-7], and others.

In fact, Tarafdar and Yuan [35] obtained an interesting coincidence theorem for compact upper semicontinuous multimaps with contractible values defined on contractible spaces. This generalizes results on convex-valued multimaps defined on convex spaces. Park and Jeong [27] showed that this result holds for non-compact multimaps in very general classes, and their new theorem includes a large number of known results.

On the other hand, the present author generalized the concept of C-spaces to that of generalized convex spaces (or simply, G-convex spaces) and established the foundations of the KKM theory on such new spaces; see the references in [22, 31].

Based on such new developments on the abstract convexity, in this paper, we are mainly concerned with the coincidence theory for multimaps in very general classes $\mathfrak{A}_c^\kappa$ defined on ω-connected topological spaces. Consequently, we obtain far-reaching generalizations of fixed point or coincidence theorems on convex sets originated from Browder [4] to the new ones for ω-connected spaces, and we can show that results on contractible spaces in [5-12, 27, 35-37] are consequences of corresponding ones for G-convex spaces.

In Sections 2 and 3, preliminaries on the admissible classes $\mathfrak{A}_c^\kappa$ of multimaps and generalized convex spaces are given. Section 4 concerns with new coincidence theorems for ω-connected spaces, which are deduced from corresponding ones for generalized convex spaces due to the present author in [21, 26, 32]. In Section 5, we obtain fixed point or maximal element theorems as consequences of results in Section 4. Finally, in Section 6, results in previous sections are applied to hyperconvex metric spaces.

2. Admissible classes of multimaps

For topological spaces X and Y, a *multimap* or a *map* $T : X \multimap Y$ is a function from X into the set of nonempty subsets of Y. Recall that $T^{-}y = \{x \in X : y \in Tx\}$ for $y \in Y$, and hence $x \in T^{-}y$ if and only if $y \in Tx$. A map $T : X \multimap Y$ is *upper semicontinuous* (*u.s.c.*) if for each open subset G of Y, the set $\{x \in X : Tx \subset G\}$ is open in X; and *compact* if the range $T(X) = \{y \in Y : y \in Tx$ for some $x \in X\}$ is contained in a compact subset of Y. A polytope is a finite dimensional compact convex subset of a topological vector space.

An *admissible class* $\mathfrak{A}_c^{\kappa}(X,Y)$ of maps $T : X \multimap Y$ is one such that, for each compact subset K of X, there exists a map $S \in \mathfrak{A}_c(K,Y)$ satisfying $S(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite compositions of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(iii) for each polytope P, each $F \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$, the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of uniform spaces), admissible maps of Górniewicz, σ-selectional maps of Haddad and Lasry, permissible maps of Dzedzej, and others. Further, $\mathbb{K}_c^+$ due to Lassonde, $\mathbb{V}_c^+$ due to Park et al., and approximable maps $\mathbb{A}^{\kappa}$ due to Ben-El-Mechaiekh and Idzik are examples of $\mathfrak{A}_c^{\kappa}$. For the literature, see [18, 19, 23]. Many other careless authors or printers mistook $\mathfrak{A}$ for $\mathcal{U}$.

3. Generalized convex spaces

An *ω-connected space* X is a topological space which is n-connected for all $n \geq 0$ (or *infinitely connected*; that is, any continuous function defined on the boundary of a finite dimensional ball with values in X can be extended to a continuous function on the ball with values in X).

We give some examples of ω-connected spaces as follows: (1) convex or star-shaped subsets of topological vector spaces; (2) convex spaces; (3) hyperconvex metric spaces; (4) contractible spaces (for examples, see Horvath [10, 11]); (5) the union of two comb spaces with identifying particular points in each space; see Spanier [33]; and (6) a path-connected topological semilattice X with an element $\overline{x} \in X$ such that $x \leq \overline{x}$ for all $x \in X$ [13, Lemma 1].

In our earlier works [21-26, 28, 30, 31], we introduced the following unified generalization of various abstract convexities without any linear structure:

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each finite $A = \{a_0, a_1, \cdots, a_n\} \subset D$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subset \{0, 1, \cdots, n\}$ implies $\phi_A(\Delta_J) \subset \Gamma(\{a_j : j \in J\})$, where Δ_n is an n-simplex with vertices $v_0, v_1, \cdots, v_n$ and Δ_J=co$\{v_j : j \in J\}$ the face of Δ_n corresponding to J. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$, where $\langle D \rangle$ denotes the set of all nonempty finite subsets of D. If $X = D$, then we denote $(X; \Gamma) = (X, X; \Gamma)$.

For a G-convex space $(X, D; \Gamma)$ with $X \supset D$, a subset C of X is said to be *Γ-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. For details on G-convex spaces, see [21-26], where basic theory was extensively developed.

There are a lot of examples of G-convex spaces:

If $X = D$ is a convex subset of a vector space and each Γ_A is the convex hull of $A \in \langle X \rangle$ equipped with the Euclidean topology, then $(X; \Gamma)$ becomes a *convex space* in the sense of Lassonde [17]. Note that any convex subset of a topological vector space is a convex space, but not conversely.

A G-convex space $(X, D; \Gamma)$ reduces to an H-*space* [29] if each Γ_A for $A \in \langle D \rangle$ is contractible (or more generally, ω-connected) in X and, for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$. If $X = D$, then the H-space $(X; \Gamma)$ becomes a C-*space* due to Horvath [11, 12].

The other major examples of G-convex spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, L-spaces of Ben-El-Mechaiekh et al. [2], continuous images of G-convex spaces, Verma's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, generalized H-spaces of Stachó, and mc-spaces of Llinares. Moreover, Ben-El-Mechaiekh et al. [2] gave examples of G-convex spaces $(X; \Gamma)$ as follows: B'-simplicial convexity, hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces.

Futhermore, any hyperbolic space X in the sense of Kirk and Reich-Shafrir is G-convex space, since the closed convex hull of any $A \in \langle X \rangle$ is contractible. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic. For the literature, see [19, 21-26].

We need the following coincidence theorem [26, 32]:

Theorem 3.1. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$ two maps, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that*

(1) *for each $z \in D$, Sz is open [resp. closed] in Y;*
(2) *for each $y \in F(X)$, $M \in \langle S^- y \rangle$ implies $\Gamma_M \subset T^- y$ [or, $\phi_M(\Delta_{|M|-1}) \subset T^- y$]; and*
(3) *$\overline{F(X)} \subset S(N)$ [resp. $Y = S(N)$] for some $N \in \langle D \rangle$.*

Then there exists an $\overline{x} \in X$ such that $F\overline{x} \cap T\overline{x} \neq \emptyset$.

Note that if F is single-valued, we do not need the Hausdorffness of Y.

From Theorem 3.1 with $F := 1_X$, the identity map on X, we have the following in [24]:

Theorem 3.2. *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ two maps satisfying*

(1) *for each $z \in D$, Sz is open [resp. closed];*

(2) *for each $y \in X$, $M \in \langle S^- y\rangle$ implies $\Gamma_M \subset T^- y$ [or, $\phi_M(\Delta_{|M|-1}) \subset T^- y$]; and*

(3) *$X = S(N)$ for some $N \in \langle D\rangle$.*

Then T has a fixed point $x_0 \in X$; that is, $x_0 \in Tx_0$.

If X is a convex subset of a topological vector space and $\Gamma_N = \operatorname{co} N$ for $N \in \langle X\rangle$, then Theorem 3.2 is a generalization of the Fan-Browder fixed point theorem; see [4, 24] and references therein. Theorem 3.2 is obtained in [24] and applied to various forms of the Fan–Browder theorem, the Ky Fan intersection theorem, and the Nash equilibrium theorem for G-convex spaces.

From Theorem 3.2, we deduced the following in [26]:

Theorem 3.3. *Let $(X \supset D; \Gamma)$ be a G-convex space and $A : X \multimap X$ a multimap such that Ax is Γ-convex for each $x \in X$. If there exist $z_1, z_2, \cdots, z_n \in D$ and nonempty open [resp. closed] subsets $G_i \subset A^- z_i$ for $i = 1, 2, \cdots, n$ such that $X = \bigcup_{i=1}^n G_i$, then A has a fixed point.*

The following selection theorem is given in the proof of [30, Theorem 1] implicitly or [21, Theorem 1(i)] explicitly.

Theorem 3.4. *Let X be a Hausdorff space, $(Y, D; \Gamma)$ a G-convex space, and $S : X \multimap D$, $T : X \multimap Y$ maps satisfying*

(1) *for each $x \in X$, $M \in \langle Sx\rangle$ implies $\Gamma_M \subset Tx$; and*

(2) *$X = \bigcup\{\operatorname{Int} S^- y : y \in D\}$.*

Then $T \in \mathbb{C}^\kappa(X,Y) \subset \mathfrak{A}_c^\kappa(X,Y)$*. More precisely, for any nonempty compact subset* K *of* X*,* $T|_K$ *has a continuous selection* $f : K \to Y$*; that is,* $fx \in Tx$ *for all* $x \in K$*, such that* $f(K) \subset \Gamma_A$ *for some* $A \in \langle D \rangle$*.*

We will show that all of the results in this paper are consequences of Theorem 3.1 with or without the aid of Theorem 3.4, and so are results on contractible spaces in [5-12, 27, 35-37].

4. Coincidence theorems for ω-connected spaces

We begin, in this section, with the following coincidence theorem:

Theorem 4.1. *Let* X *be an* ω*-connected space,* Y *a Hausdorff space, and* $F \in \mathfrak{A}_c^\kappa(X,Y)$*. Suppose that a map* $G : Y \multimap X$ *satisfies the following:*

(1) *for each open subset* O *of* Y*, the set* $\bigcap_{y \in O} Gy$ *is empty or* ω*-connected; and*

(2) $\overline{F(X)} \subset \bigcup\{\operatorname{Int} G^- x : x \in D\}$ *for some* $D \in \langle X \rangle$*.*

Then there exist an $x_0 \in X$ *and a* $y_0 \in Y$ *such that* $y_0 \in Fx_0$ *and* $x_0 \in Gy_0$*.*

Proof. Let us define an H-space $(X, D; \Gamma)$ as follows: For any $J \in \langle D \rangle$, let

$$\Gamma_J := \begin{cases} \bigcap\{Gy : y \in \bigcap_{x \in J} \operatorname{Int} G^- x\} & \text{if } \bigcap_{x \in J} \operatorname{Int} G^- x \neq \emptyset, \\ X & \text{otherwise.} \end{cases}$$

Note that if $y \in \bigcap_{x \in J} \operatorname{Int} G^- x$, then $J \in \langle Gy \rangle$. Therefore, if $O = \bigcap_{x \in J} \operatorname{Int} G^- x \neq \emptyset$, then $\Gamma_J = \bigcap_{y \in O} Gy$ is an ω-connected set by (1). Moreover, it is clear that $\Gamma_J \subset \Gamma_{J'}$ whenever $J \subset J' \in \langle D \rangle$. Now $(X, D; \Gamma)$ is a G-convex space.

We apply the open case of Theorem 3.1 with $S = (\operatorname{Int} G^-)|_D$ and $T = G^-$. Then conditions (1) and (3) of Theorem 3.1 are readily satisfied. We show that condition (2) of Theorem 3.1 holds. In fact, for each $y \in Y$ and $M \in \langle S^- y \rangle \subset \langle D \cap Gy \rangle$, we have $y \in \bigcap_{x \in M} Sx = \bigcap_{x \in M} \operatorname{Int} G^- x \neq \emptyset$. Hence $\Gamma_M = \bigcap\{Gz : z \in \bigcap_{x \in M} \operatorname{Int} G^- x\} \subset Gy = T^- y$. Therefore, by Theorem 3.1, the conclusion follows.

Remarks. 1. Theorem 4.1 is a restatement of [24, Theorem 5.1] and contains a number of particular cases by replacing ω-connectedness by its particular forms—convexity, star-shapeness, contractibility, and others. If F is single-valued, then the Hausdorffness of Y is redundant.

2. If X is a convex space, then Theorem 4.1 reduces to a result equivalent to [18, Theorem 5], which extends many known theorems and has numerous applications in the KKM theory.

3. If ω-connected sets are replaced by contractible sets, then Theorem 4.1 reduces to Park and Jeong [27, Theorem 2].

Similarly, from the closed case of Theorem 3.1, we have the following:

Theorem 4.1′. *Let X be an ω-connected space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X,Y)$. Suppose that a map $G : Y \multimap X$ satisfies the following:*

(1) *for each closed subset C of Y, the set $\bigcap_{y\in C} Gy$ is empty or ω-connected; and*

(2) *there exists a $D \in \langle X\rangle$ such that, for each $z \in D$, $G^- z$ is closed and $Y = G^-(D)$.*

Then there exist an $x_0 \in X$ and a $y_0 \in Y$ such that $y_0 \in Fx_0$ and $x_0 \in Gy_0$.

If F is a compact map in Theorem 4.1, we have the following:

Corollary 4.2. *Let X be an ω-connected space, Y a Hausdorff space, and $A \in \mathfrak{A}_c^\kappa(X,Y)$ a compact map. Suppose that $B : Y \multimap X$ is a map such that*

(1) *for each open subset O in Y, the set $\bigcap_{y\in O} By$ is empty or ω-connected; and*

(2) $\overline{A(X)} \subset \bigcup_{x\in X} \operatorname{Int} B^- x$.

Then there exist an $x_0 \in X$ and a $y_0 \in Y$ such that $y_0 \in Ax_0$ and $x_0 \in By_0$.

Proof. Let $F := A$ and $G := B$ in Theorem 4.1. Since $\overline{A(X)}$ is compact, condition (2) implies (2) of Theorem 4.1. Then the conclusion follows.

Remarks. 1. Corollary 4.2 originates from Browder [4, Theorem 7].

2. Replacing ω-connected sets by contractible sets, Tarafdar and Yuan [35, Theorem 1] obtained a particular form of Corollary 4.2 for an u.s.c. map A with compact contractible values and, later, gave some applications in [37].

3. Replacing ω-connected sets by contractible sets, Theorem 4.2 reduces to Park and Jeong [27, Theorem 3].

4. If $X = Y$ and $A = F = 1_X$, the identity map on X, each of Theorem 4.1 and Corollary 4.2 reduces to a fixed point theorem.

5. The main result of Ding [5, Theorem 1] is a particular form of Theorem 4.2 by replacing ω-connectedness by contractibility for an acyclic map A. Moreover, he should assume the Hausdorffness of Y.

In many cases, X itself is assumed to be compact in Corollary 4.2. For a long period, many authors tried to weaken the compactness in such situation. The following is the prototype of such intention:

Theorem 4.3. *Let X be a topological space, Y a Hausdorff space, K a nonempty subset of Y, $F : X \multimap Y$, and $G : Y \multimap X$. Suppose that*

(1) $K \cap \overline{F(X)} \subset \bigcup_{x \in N} \operatorname{Int} G^{-}x$ *for some* $N \in \langle X \rangle$;

(2) *there exists an ω-connected subset L_N of X containing N such that $F|_{L_N} \in \mathfrak{A}_c^{\kappa}(L_N, Y)$;*

(3) *for each open subset O of Y, the set $L_N \cap \bigcap_{y \in O} Gy$ is empty or ω-connected; and*

(4) $\overline{F(L_N)} \backslash K \subset \bigcup_{x \in M} \operatorname{Int} G^{-}x$ *for some* $M \in \langle L_N \rangle$.

Then there exist an $x_0 \in X$ and a $y_0 \in K$ such that $y_0 \in Fx_0$ and $x_0 \in Gy_0$.

Proof. Let $A := F|_{L_N} : L_N \multimap Y$ and let $B : Y \multimap L_N$ be defined by $By := Gy \cap L_N$ for $y \in Y$. Note that $A \in \mathfrak{A}_c^{\kappa}(L_N, Y)$. We apply Corollary 4.2 with $X = L_N$.

For each open subset O of Y, the set $\bigcap_{y\in O} By = L_N \cap \bigcap_{y\in O} Gy$ is empty or ω-connected by (3). Moreover,

$$\begin{aligned}\overline{A(L_N)} = \overline{F(L_N)} &\subset (\overline{F(L_N)} \setminus K) \cup (\overline{F(X)} \cap K)\\ &\subset [\bigcup_{x\in M} \operatorname{Int} G^- x] \cup [\bigcup_{x\in N} \operatorname{Int} G^- x]\\ &\subset \bigcup_{x\in M\cup N} \operatorname{Int} G^- x = \bigcup_{x\in M\cup N} \operatorname{Int} B^- x,\end{aligned}$$

where $M \cup N \in \langle L_N \rangle$.

Therefore, by Corollary 4.2, we have an $x_0 \in L_N$ and a $y_0 \in Y$ such that $y_0 \in Ax_0 = Fx_0$ and $x_0 \in By_0 = Gy_0$.

Remarks. 1. If L_N is compact, then we may assume $\overline{F(L_N)} = F(L_N)$ since Y is Hausdorff.

2. Particular forms of Theorem 4.3 appear in Ding [6, Theorems 1.1 and 1.2].

3. If F is a compact map, then by putting $Y = K = \overline{F(X)}$, condition (4) holds trivially.

From Theorem 3.4 we have the following particular case:

Lemma 4.4. *Let X be a Hausdorff compact space, Y an ω-connected space, and $S, T : X \multimap Y$ maps satisfying*

(i) *for each $y \in Y$, $S^- y$ is open;*
(ii) *for each $x \in X$, $Sx \subset Tx$; and*
(iii) *for any open set O of X, $\bigcap_{x\in O} T(x)$ is empty or ω-connected.*

Then T has a continuous selection.

Proof. Since our maps are nonempty-valued by definition, for any $x \in X$, there exists a $y \in Sx$ or $x \in S^- y$. Since X is compact, there exists a subset $D \in \langle Y \rangle$ such that $X = \bigcup\{S^- y : y \in D\}$.

We can make an H-space $(Y, D; \Gamma)$ as follows: For any $J \in \langle D \rangle$, let

$$\Gamma_J := \begin{cases} \bigcap\{Tx : x \in \bigcap_{y \in J} S^- y\} & \text{if } \bigcap_{y \in J} S^- y \neq \emptyset, \\ Y & \text{otherwise.} \end{cases}$$

Note that if $x \in \bigcap_{y \in J} S^- y$, then $J \in \langle Sx \rangle$. Therefore, if $O = \bigcap_{y \in J} S^- y \neq \emptyset$, then $\Gamma_J = \bigcap_{x \in O} Tx$ is not empty (since $O \neq \emptyset$ and $J \subset Sx \subset Tx$ for each $x \in O$) and hence ω-connected by (iii). Moreover, it is clear that $\Gamma_J \subset \Gamma_{J'}$ whenever $J \subset J' \in \langle D \rangle$. Therefore $(Y, D; \Gamma)$ is a G-convex space.

Define $S' : X \multimap D$ by

$$S'x := \{y \in D : y \in Sx\} = D \cap Sx.$$

For each $x \in X$ and $M \in \langle S'x \rangle = \langle D \cap Sx \rangle$, we have $x \in \bigcap_{y \in M} S^- y \neq \emptyset$. Hence $\Gamma_M = \bigcap\{Tx : \bigcap_{y \in M} S^- y\} \subset Tx$. Therefore condition (1) of Theorem 3.4 for S' is satisfied.

Recall that $X = \bigcup\{S^- y : y \in D\} = \bigcup\{(S')^- y : y \in D\}$, which shows that condition (2) of Theorem 3.4 is also satisfied. Therefore, by Theorem 3.4 for S', T has a continuous selection.

Remarks. 1. If we replace ω-connectedness by convexity, then Lemma 4.4 was given implicitly in [4].

2. If we replace ω-connectedness by contractibility, Lemma 4.4 reduces to Horvath [9, Theorem 3] and includes [8, Lemma 2].

Theorem 4.5. *Let X be a Hausdorff compact ω-connected space, and $R, S : X \multimap X$ maps satisfying*

(1) *for each $x \in X$, $S^- x \neq \emptyset$;*
(2) *for each $x \in X$, $R^- x$ and Sx are open; and*
(3) *for any open subset O of X, $\bigcap_{x \in O} Rx$ and $\bigcap_{x \in O} S^- x$ are empty or ω-connected.*

Then there exists an $x_0 \in X$ *such that* $Rx_0 \cap Sx_0 \neq \emptyset$.

Proof. By Lemma 4.4, the map R has a continuous selection $f : X \to X$ and $f \in \mathfrak{A}_c^\kappa(X, X)$. Consider $B := S^- : X \multimap X$. By (1), for each $x \in X$, there exists a $y \in Bx \neq \emptyset$ or $x \in B^- y$. Since $B^- y = Sy$ is open, $X = \bigcup\{B^- y : y \in X\}$. Therefore, by Corollary 4.2 with $A := f$, there exist an $x_0 \in X$ and a $y_0 \in X$ such that $y_0 = fx_0$ and $x_0 \in By_0$. Hence $y_0 \in Rx_0 \cap B^- x_0 = Rx_0 \cap Sx_0$. This completes our proof.

Remark. Theorem 4.5 generalizes Horvath [8, Theorem 3], which is originated from Ky Fan.

5. Fixed point theorems for ω-connected spaces

The following simple consequence of Theorem 4.1 with $F := 1_X$ or Corollary 4.2 with $A := 1_X$ generalizes the Fan-Browder theorem:

Theorem 5.1. *Let* X *be an* ω*-connected space and* $G : X \multimap X$ *a map satisfying*

(1) *for each open subset* O *of* X*, the set* $\bigcap_{x \in O} Gx$ *is empty or* ω*-connected; and*

(2) $X = \bigcup_{x \in D} \operatorname{Int} G^- x$ *for some* $D \in \langle X \rangle$.

Then G *has a fixed point* $x_0 \in X$.

Remarks. 1. If ω-connected sets are replaced by convex sets, Theorem 5.1 reduces to the Fan-Browder fixed point theorem.

2. If ω-connected sets are replaced by contractible sets, Theorem 5.1 contains results due to Horvath [8-12]; see also [27, Theorem 4].

3. Theorem 5.1 sharpens Bielawski [3, Corollary 4.10].

4. Note that Theorem 5.1 follows also from Theorems 3.2 or 3.3.

From Theorem 4.1′ with $F := 1_X$, we have the following:

Theorem 5.1′. *Let X be an ω-connected space and $G : X \multimap X$ a map satisfying*

(1) *for each closed subset C of X, the set $\bigcap_{y\in C} Gy$ is empty or ω-connected; and*

(2) *there exists a $D \in \langle X\rangle$ such that, for each $z \in D$, $G^{-}z$ is closed and $X = G^{-}(D)$.*

Then G has a fixed point $x_0 \in X$.

Remark. Theorem 5.1′ follows also from Theorems 3.2 or 3.3.

From Theorem 5.1, we deduce the following maximal element theorem (where a map may have empty values):

Theorem 5.2. *Let X be a compact ω-connected space and $G : X \multimap X$ a map satisfying*

(1) *for each open subset O of X, the set $\bigcap_{x\in O} Gx$ is empty or ω-connected; and*

(2) *for each $x \in X$, $G^{-}x$ is open.*

Then either G has a fixed point $x_0 \in X$ or a maximal element $y_0 \in X$ (that is, $Gy_0 = \emptyset$).

Proof. Suppose $Gy \neq \emptyset$ for all $y \in X$. Then $y \in G^{-}x$ for some $x \in X$, and hence $X = \bigcup_{x\in X} G^{-}x$. Since each $G^{-}x$ is open we have $X = \bigcup_{x\in X} \operatorname{Int} G^{-}x$. Therefore by Theorem 5.1, G has a fixed point $x_0 \in X$.

Remarks. 1. If we replace ω-connectedness by convexity, then Theorem 5.2 reduces to the classical result of Yannelis and Prabhakar [36].

2. If we replace ω-connectedness by contractibility, Theorem 5.2 reduces to Horvath [8, Theorem 2].

Corollary 5.3. *Let X and G be the same as in Theorem 5.2. If $Gx \neq \emptyset$ for all $x \in X$, then for every continuous function $f : X \to X$ there exists $x_0 \in X$ such that $x_0 \in G(fx_0)$.*

Proof. The map $G' : X \multimap X$ defined by $G'x = G(fx)$ for $x \in X$ satisfies the hypothesis of Theorem 5.2.

Corollary 5.4. *Let X be a compact ω-connected space. If the diagonal $\Delta \subset X \times X$ has a basis $\{V_i\}_{i \in I}$ of open neighborhoods such that, for each $i \in I$ and each open subset $U \subset X$, the intersection $\bigcap_{x \in U} V_i[x]$, where $V_i[x] := \{y \in X : (x, y) \in V_i\}$, is empty or ω-connected, then X has the fixed point property.*

Proof. If $f : X \to X$ is a continuous function without fixed point, then

$$\{(fx, x) : x \in X\} \cap \Delta = \emptyset.$$

Hence there is an $i_0 \in I$ such that $\{(fx, x) : x \in X\} \cap V_{i_0} = \emptyset$, which contradicts Corollary 5.3.

Corollary 5.5. *Let X be a compact ω-connected metric space. If there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ and every open subset $U \subset X$, the intersection $\bigcap_{x \in U} B(x, \varepsilon)$ of open balls is empty or ω-connected, then X has the fixed point property.*

Remarks. 1. Corollaries 5.3–5.5 are essentially due to Horvath [8] and can be extended to the G-convex space cases.

2. Horvath [8] noted that Corollary 5.4 generalizes Ky Fan's extension of the Tychnoff fixed point theorem for topological vector spaces having sufficiently many linear functionls; for references, see [19].

From Theorem 5.1, we have the following:

Theorem 5.6. *Let X be a Hausdorff compact topological space, Y an ω-connected subspace of X, and $A, B : Y \multimap X$ maps satisfying*

(1) *for each $x \in X$, $B^- x \neq \emptyset$ and for each $y \in Y$, By is open;*
(2) *for each $y \in Y$, $By \subset Ay$; and*
(3) *for each open subset O of X, the set $\bigcap_{x \in O} A^- x$ is empty or ω-connected.*

Then A has a fixed point.

Remark. Replacing ω-connectedness by contractibility, Theorem 5.6 reduces to Horvath [9, Theorem 4].

For $X = Y$ and $F = 1_X$, Theorem 4.3 reduces to the following:

Theorem 5.7. *Let X be a topological space, K a nonempty subset of X, and $G : X \multimap X$. Suppose that*

(1) *$K \subset \bigcup_{x \in N} \operatorname{Int} G^- x$ for some $N \in \langle X \rangle$;*
(2) *there exists an ω-connected subset L_N of X containing N;*
(3) *for each open subset O of X, the set $L_N \cap \bigcap_{x \in O} Gx$ is empty or ω-connected; and*
(4) *$L_N \backslash K \subset \bigcup_{x \in M} \operatorname{Int} G^- x$ for some $M \in \langle L_N \rangle$.*

Then there exists an $x_0 \in X$ such that $x_0 \in Gx_0$.

Remarks. 1. Ding [6, Corollary 1.1] is a particular form of Theorem 5.7 and was applied to obtain an abstract variational inequality [6, Theorem 2.1].

2. Ding [7, Lemma 1.1] is an another particular form of Theorem 5.7 and applied to the existence of solutions of a general quasi-equilibrium problem [7, Theorem 2.1] and other problems [7, Theorem 3.1].

3. Since Theorem 5.7 is a variant of Theorem 5.1, this can be applied to the problems considered by Ding for more simple, but still equivalent, situations. In such a way, any interested reader can simplify the main results of [6, 7] in more essential forms.

6. Hyperconvex metric spaces

A metric space (H, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

It is known that the space $\mathbb{C}(E)$ of all continuous real functions on a Stonian space E (that is, an extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space $\mathbb{C}(E)$ for some Stonian space E. Therefore, $(\mathbf{R}^n, \|\cdot\|_\infty)$, l^∞, and L^∞ are concrete examples of hyperconvex spaces.

Results of Aronszajn and Panitchpakti [1, Theorem 1$'$] and Isbell [14, Theorem 1.1] are combined in the following:

Lemma 6.1. *A hyperconvex space is complete and* (*freely*) *contractible.*

The concepts of C-spaces, LC-spaces, and LC-metric spaces were introduced and extensively studied by Horvath in a sequences of papers [8-12]:

A C-space $(X; \Gamma)$ is called an *LC-space* (or a *locally H-convex space*) if X is a Hausdorff uniform space and there exists a basis $\{V_\lambda\}_{\lambda \in I}$ for the uniform structure such that for each $\lambda \in I$, $\{x \in X : E \cap V_\lambda[x] \neq \emptyset\}$ is Γ-convex whenever $E \subset X$ is Γ-convex, where

$$V_\lambda[x] = \{x' \in X : (x, x') \in V_\lambda\}.$$

For example, any nonempty convex subset X of a locally convex Hausdorff topological vector space is an LC-space with $\Gamma_A = \operatorname{co} A$, the convex hull of $A \in \langle X \rangle$.

A triple $(X, d; \Gamma)$ is called an *LC-metric space* whenever (X, d) is a metric space and $(X; \Gamma)$ is a C-space such that open balls are Γ-convex, and any neighborhood $\{x \in X : d(x, Y) < r\}$ of a Γ-convex set $Y \subset X$ is also Γ-convex.

Horvath [12, Theorem 9] obtained the following:

Lemma 6.2. *Any hyperconvex metric space H is a complete LC-metric space with*

$$\Gamma(A) = \Gamma_A := \bigcap\{B : B \text{ is a closed ball containing } A\}$$

for each $A \in \langle H \rangle$.

From now on, a hyperconvex metric space $(H, d; \Gamma)$ is simply denoted by H, and $\mathrm{BI}(H)$ denotes the set of nonempty closed ball intersections in H. Elements of $\mathrm{BI}(H)$ are sometimes called *admissible subsets* of H; see [15]. It is well-known that any admissible subset of a hyperconvex metric space is hyperconvex and hence contractible.

From Theorem 3.1 for the open case, we have the following:

Theorem 6.3. *Let X be a hyperconvex metric space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that a map $G : Y \multimap X$ satisfies the following:*

(1) *for each $y \in Y$, Gy is Γ-convex [e.g., $Gy \in \mathrm{BI}(X)$]; and*
(2) *$\overline{F(X)} \subset \bigcup\{\mathrm{Int}\, G^- x : x \in D\}$ for some $D \in \langle X \rangle$.*

Then there exist an $x_0 \in X$ and a $y_0 \in Y$ such that $y_0 \in Fx_0$ and $x_0 \in Gy_0$.

From the closed case of Theorem 3.1, we have the following:

Theorem 6.3′. *Let X be a hyperconvex metric space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Suppose that a map $G : Y \multimap X$ satisfies the following:*

(1) *for each $y \in Y$, Gy is Γ-convex [e.g., $Gy \in \mathrm{BI}(X)$]; and*
(2) *there exists a $D \in \langle X \rangle$ such that, for each $z \in D$, $G^- z$ is closed and $Y = G^-(D)$.*

Then there exist an $x_0 \in X$ and a $y_0 \in Y$ such that $y_0 \in Fx_0$ and $x_0 \in Gy_0$.

If F is a compact map in Theorem 6.3, we have the following:

Corollary 6.4. *Let X be a hyperconvex metric space, Y a Hausdorff space, and $A \in \mathfrak{A}_c^{\kappa}(X,Y)$ a compact map. Suppose that $B : Y \multimap X$ is a map such that*

(1) *for each $y \in Y$, By is Γ-convex* [*e.g.*, $By \in \mathrm{BI}(X)$]; *and*
(2) $\overline{A(X)} \subset \bigcup\{\mathrm{Int}\, B^{-}x : x \in X\}$.

Then there exist an $x_0 \in X$ and a $y_0 \in Y$ such that $y_0 \in Ax_0$ and $x_0 \in By_0$.

From Theorem 6.3, we have the following:

Corollary 6.5. *Let X be a hyperconvex metric space and $T : X \multimap X$ a map such that*

(1) *for each $x \in X$, Tx is Γ-convex*; *and*
(2) $X \subset \bigcup\{\mathrm{Int}\, T^{-}x : x \in D\}$ *for some $D \in \langle X \rangle$.*

Then there exist an $x_0 \in X$ such that $x_0 \in Tx_0$.

A particular form of Corollary 6.5 was obtained by Park [20, Theorem 3].

From Theorem 4.5, we have the following:

Theorem 6.6. *Let X be a compact hyperconvex metric space, and $R, S : X \multimap X$ maps satisfying*

(1) *for each $x \in X$, $S^{-}y \neq \emptyset$*;
(2) *for each $x \in X$, $R^{-}x$ and Sx are open*; *and*
(3) *for each $x \in X$, Rx and $S^{-}x$ are closed ball-intersections.*

Then there exists an $x_0 \in X$ such that $Rx_0 \cap Sx_0 \neq \emptyset$.

Finally, for other forms of fixed point theorems on hyperconvex metric spaces, the reader may refer to [16].

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