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FIXED POINT THEOREMS ON $\mathfrak{KC}$-MAPS IN ABSTRACT CONVEX SPACES

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Abstract. We introduce basic results in the KKM theory on abstract convex spaces and the map classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$. We study the nature of Kakutani type maps, $\mathfrak{B}$-maps, and $\mathfrak{KC}$-maps in G-convex spaces; and show that generalizations of the key results in [2-4, 12] are consequences of the G-convex space theory and the new abstract convex space theory.

1. Introduction

Recently, in [8], we introduced a new concept of abstract convex spaces and a map class $\mathfrak{K}$ having certain KKM property which are adequate to establish the KKM theory. With this new concept, in [10], we generalized and simplified known results of the theory on convex spaces, H-spaces, G-convex spaces, and others. In fact, from a basic KKM type theorem for a $\mathfrak{K}$-map defined on an absract convex space without any topology, we deduced ten equivalent formulations of the theorem. As applications of the equivalents, in the frame of abstract convex spaces, we obtained Fan-Browder type fixed point theorems, almost fixed point theorems for multimaps, mutual relations between the map classes $\mathfrak{K}$ and $\mathfrak{B}$, variational inequalities, the von Neumann type minimax theorems, and the Nash equilibrium theorems.

More early, in [1-4, 12], their authors investigated fixed point theorems on Kakutani type maps or the map classes $\mathfrak{K}$ and $\mathfrak{B}$ on particular types of our generalized convex (G-convex) spaces. Especially, Fakhar and Zafarani [3] obtained new theorems on those map classes and applications. Kuo, Jeng, and Huang [4] introduced the concept of the strict KKM property and investigated the fixed point problem for multimaps having this property on almost Γ-convex subsets of locally G-convex uniform spaces. Their new theorem generalizes the Fan-Glicksberg theorem and partially extends the Himmelberg theorem. Its application to a minimax theorem is also added in [4].

In the present paper, we introduce basic results in the KKM theory on abstract convex spaces and the map classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$. We study the nature of

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Kakutani type maps, $\mathfrak{B}$-maps, and $\mathfrak{KC}$-maps in G-convex spaces; and show that generalizations of key results in [2-4, 12] are consequences of known ones. Consequently, a number of key results in [2-4, 12] are improved by applying our G-convex space theory and the new abstract convex space theory.

Section 2 is for preliminaries for abstract convex spaces taken from our previous works in [8, 10]. In Section 3, some necessary definitions and facts on generalized convex spaces in [9, 10] are introduced. Section 4 concerns with generalizations of results in [3] and comments to them. Finally, in Section 5, main results in [2, 4, 12] are generalized or improved by applying our G-convex space theory or the new abstract convex space theory.

2. Abstract convex spaces

This section concerns with preliminaries for abstract convex spaces introduced in [8, 10]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values, where $\langle D \rangle$ denotes the set of nonempty finite subsets of D. We sometimes denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if, for any $A \in \langle X \cap D \rangle$, we have $\Gamma_A \subset X$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

In [8], we gave a lot of examples of abstract convex spaces and generalizations of the KKM principle. We have abstract convex subspaces as follows:

Proposition 2.1. *For an abstract convex space $(E \supset D; \Gamma)$, let X be a Γ-convex subset of E, and D' a nonempty subset of $X \cap D$. Let $\Gamma' : \langle D' \rangle \multimap X$ be a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \;\; for \;\; A \in \langle D' \rangle.$$

Then $(X \supset D'; \Gamma')$ itself is an abstract convex space called a subspace.

Definitions. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. A multimap $F : E \multimap Z$ with nonempty values is called *a $\mathfrak{K}$-map* if, for any map $G : D \multimap Z$ satisfying

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \;\; \text{is a} \;\; \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation KKM(E, Z) for $\mathfrak{KC}(E, Z)$.

We need the following in [8]:

Theorem 2.2. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, $S : D \multimap Z$, $T : E \multimap Z$ maps, and $F \in \mathfrak{K}(E, Z)$. Suppose that*

(1) for each $z \in F(E)$, $M \in \langle S^-(z) \rangle$ implies $\Gamma_M \subset T^-(z)$; and

(2) $F(E) \subset S(N)$ for some $N \in \langle D \rangle$. Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Definition. For a given abstract convex space $(E, D; \Gamma)$ and a topological space X, a map $H : X \multimap E$ is called a Φ-*map* (or a *Fan-Browder map*) if there exists a map $G : X \multimap D$ such that

(i) for each $x \in X$, $M \in \langle G(x) \rangle$ implies $\Gamma_M \subset H(x)$; and

(ii) $X = \bigcup \{\text{Int}\, G^-(y) \mid y \in D\}$.

Definitions. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is an abstract convex space with a base $\mathcal{U}$ of a uniform structure of E.

In $(E, D; \Gamma; \mathcal{U})$, a subset Z of E is called a Φ-*set* if, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : Z \multimap E$ such that $\text{Gr}(H) \subset U$. If E itself is a Φ-set, then it is called a Φ-*space*.

For any $U \in \mathcal{U}$, a point $x \in E$ is called a *U-fixed point* of a map $F : E \multimap E$ if $F(x) \cap U[x] \neq \emptyset$. The map F is said to have the *almost fixed point property* whenever it has a U-fixed point for any $U \in \mathcal{U}$.

The following almost fixed point theorem and its corollaries are given in [10]:

Theorem 2.3. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space, and $F \in \mathfrak{KC}(E, E)$ be a compact map. If $\overline{F(E)}$ is a Φ-set, then F has the almost fixed point property.*

Corollary 2.4. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex Hausdorff uniform space, and $F \in \mathfrak{KC}(E, E)$ a compact closed map such that $\overline{F(E)}$ is a Φ-set. Then F has a fixed point.*

Corollary 2.5. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex Hausdorff uniform Φ-space such that the identity map $1_E \in \mathfrak{KC}(E, E)$. If $f : E \to E$ is a compact continuous function, then f has a fixed point.*

3. Generalized convex spaces

This section concerns with preliminaries on generalized convex spaces which are typical examples of abstract convex spaces:

Definition. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. It is possible to assume $\Gamma(A) = \phi_A(\Delta_n)$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$. In case $X \supset D$, the G-convex space is denoted by $(X \supset D; \Gamma)$.

We have established a large amount of literature on G-convex spaces. In the remainder of this section, we follow mainly [9, 10]:

Definition. A *G-convex uniform space* $(X, D; \Gamma; \mathcal{U})$ is a G-convex space such that $(X, \mathcal{U})$ is a uniform space with a base $\mathcal{U}$ of the uniformity consisting of symmetric entourages. For each $U \in \mathcal{U}$, let

$$U[x] := \{x' \in X \mid (x, x') \in U\}$$

be the *U-ball* around a given element $x \in X$.

Definition. A G-convex uniform space $(X \supset D; \Gamma; \mathcal{U})$ is called an *LG-space* if D is dense in X and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] := \{x \in X \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset X$ is Γ-convex.

Note that a singleton is not necessarily Γ-convex in an LG-space.

We give a general definition of Kakutani maps as follows:

Definition. Let Y be a topological space and $(X \supset D; \Gamma)$ a G-convex space. A map $F : Y \multimap X$ is called a *Kakutani map* if it is u.s.c. and has nonempty compact Γ-convex values.

The following is the main result of [6]:

Theorem 3.1. *Let $(X \supset D; \Gamma; \mathcal{U})$ be a Hausdorff LG-space and $T : X \multimap X$ a compact Kakutani map. Then T has a fixed point.*

Definition. A G-convex uniform space $(X \supset D;\ \Gamma;\ \mathcal{U})$ is said to be *locally G-convex* if D is dense in X and, for each $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \subset U$ and, for each $x \in X$,

$$N \in \langle V[x] \cap D\rangle \Rightarrow \Gamma_N \subset U[x].$$

Remark. In particular, if the U-ball $U[x]$ itself is Γ-convex for each $x \in X$, then $(X \supset D;\ \Gamma;\ \mathcal{U})$ is locally G-convex. In such case, if X is Hausdorff, every singleton is Γ-convex since $\{x\} = \bigcap_{U \in \mathcal{U}} U[x]$ and the intersection of Γ-convex subsets is Γ-convex.

Definition. For a G-convex uniform space $(X, D; \Gamma; \mathcal{U})$, a subset Y of X is called a *Φ-set* if for each entourage $U \in \mathcal{U}$, there exists a Φ-map $T : Y \multimap X$ such that $\mathrm{Gr}(T) \subset U$ (that is, $T(y) \subset U[y]$ for all $y \in Y$). If X itself is a Φ-set, then it is called a *Φ-space*.

Note that every subset Y of a Φ-space is a Φ-set.

Theorem 3.2. *For a locally G-convex space $(X \supset D;\ \Gamma;\ \mathcal{U})$, any nonempty subset Y of X is a Φ-set.*

Definition. Let $(X, D; \Gamma)$ be a G-convex space and Z a topological space. In 1996, we defined *the better admissible class* $\mathfrak{B}$ of maps from X into Z as follows:

$F \in \mathfrak{B}(X, Z) \iff F : X \multimap Z$ is a map such that for any $N \in \langle D\rangle$ with the cardinality $|N| = n + 1$ and any continuous function $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n)$.

Theorem 3.3. *Let $(X, D; \Gamma)$ be a G-convex space and Z a topological space.*
(1) If Z is a Hausdorff space, then every compact map $F \in \mathfrak{B}(X, Z)$ belongs to $\mathfrak{KC}(X, Z)$.
(2) If $F : X \multimap Z$ is a closed map such that $F\phi_N \in \mathfrak{KC}(\Delta_n, Z)$ for any $N \in \langle D \rangle$ with the cardinality $|N| = n+1$, then $F \in \mathfrak{B}(X, Z)$.

Theorem 3.4. *Let $(X, D; \Gamma; \mathcal{U})$ be a G-convex uniform space, Y a subset of X, and $F \in \mathfrak{B}(Y, Y)$ a map such that $\overline{F(Y)}$ is a compact Φ-subset of Y. Then F has the almost fixed point property.*

Further if Y is Hausdorff and F is closed, then F has a fixed point $x_0 \in Y$.

Theorem 3.5. *Let $(X, D; \Gamma; \mathcal{U})$ be a Hausdorff Φ-space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

4. Fixed points of $\mathfrak{KC}$-maps in abstract convex spaces

In this section, we give comments on the results on $\mathfrak{B}$-maps and $\mathfrak{KC}$-maps appeared in [3].

We begin with the following continuous selection theorem for multimaps with noncompact domain given as [7, Theorem J]:

Lemma 4.1. *Let Y be a normal space, $(X, D; \Gamma)$ a G-convex space, and $S : Y \multimap D$ a map such that $Y = \bigcup\{\text{Int } S^-(y) \mid y \in A\}$ for some $A \in \langle D \rangle$. Then there exists a continuous function $s : Y \to \Gamma_A$ such that $s(y) \in \Gamma(A \cap S(y))$ for all $y \in Y$. In fact, if $|A| = n+1$, then $s = \phi_A \circ p$, where $\phi_A : \Delta_n \to \Gamma_A$ and $p : Y \to \Delta_n$ are continuous functions.*

The following simplifies and improves [3, Theorem 2.2]:

Theorem 4.2. *Let $(X, D; \Gamma)$ be a G-convex space, Z a normal space, and $F \in \mathfrak{B}(X, Z)$. Let $S : Z \multimap D$ and $T : Z \multimap X$ satisfy the following:*
(1) for each $y \in D$, $S^-(y)$ is open in Z;
(2) for each $z \in F(X)$, $M \in \langle S(z) \rangle$ implies $\Gamma_M \subset T(z)$; and
(3) $\overline{F(X)} \subset S^-(A)$ for some $A \in \langle D \rangle$. Then there exist a point $x_0 \in X$ and a point $y_0 \in Z$ such that $x_0 \in T(y_0)$ and $y_0 \in F(x_0)$.

Proof. Since $Y := \overline{F(X)}$ is normal and included by $S^-(A)$, by Lemma 4.1, there exists a continuous function $s : Y \to \Gamma_A$ such that $s(y) \in \Gamma(A \cap S(y))$ for all $y \in Y$ and, if $|A| = n+1$, $s = \phi_A \circ p$, where $\phi_A : \Delta_n \to \Gamma_A$ and $p : Y \to \Delta_n$ are continuous functions. Then, as in the proof of [10, Theorem 14], $z \in F\phi_A(\Delta_n)$ implies $z \in (T^-\phi_A p)(z)$. Since $F \in \mathfrak{B}(X, Z)$, $pF\phi_A$ has a fixed point $a_0 \in \Delta_n$; that is, $a_0 \in (pF\phi_A)(a_0)$. Put $x_0 := \phi_A(a_0)$. Since $p^{-1}(a_0) \subset (F\phi_A)(a_0) = F(x_0)$, for any $z \in p^{-1}(a_0)$, we have $z \in F\phi_A(\Delta_n)$, $(\phi_A p)(z) = \phi_A(a_0) = x_0$, and $z \in (T^-\phi_A p)(z) = T^-(x_0)$. Therefore, $p^{-1}(a_0) \subset T^-(x_0)$ and hence $p^{-1}(a_0) \subset F(x_0) \cap T^-(x_0)$. This completes our proof.

Remarks. 1. Theorem 4.2 was noted in Remarks for Theorem 14 in [10].

2. Note that Theorem 4.2 simplifies [3, Theorem 2.2]. Note that [3, Corollary 2.4] holds without the normality of X; see [10, Corollary 2.4].

Note that the following improves our Theorem 3.3(1):

Lemma 4.3 [3, Lemma 2.5]. *Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. Then any map $F \in \mathfrak{B}(X, Y)$ such that for any $A \in \langle D \rangle$ with $|A| = n+1$, the set $F(\phi_A(\Delta_n))$ in its induced topology is a normal space, belongs to $\mathfrak{KC}(X, Y)$.*

Remark. Ours is for the particular case when Y is Hausdorff and F is compact.

Recall that Ben-El-Mechaiekh initiated the study of approachable or approximable maps and obtained a lot of results. In [1], the following is given:

Lemma 4.4 [1, Proposition 3.9]. *Let X be a compact space and $(Y \supset D; \Gamma)$ be an LG-space. Then any u.s.c. map $T : X \multimap Y$ having nonempty Γ-convex values is approachable.*

It is known by the author that any u.s.c. approachable map having nonempty compact values belongs to the class $\mathfrak{B}$. Hence, by Theorem 3.3(1), we have the following. Here we give another proof.

Theorem 4.5. *Let $(X, D; \Gamma)$ be a G-convex space and $(Y \supset D'; \Sigma)$ a Hausdorff LG-space. If $F : X \multimap Y$ is a Kakutani map, then $F \in \mathfrak{B}(X, Y) \cap \mathfrak{KC}(X, Y)$.*

Proof. Let $N \in \langle D \rangle$ with $|N| = n + 1$. Then $F \circ \phi_A : \Delta_n \multimap Y$ is u.s.c. and has nonempty Γ-convex values, and hence, by Lemma 4.4, $F \circ \phi_A$ is an approachable map. Then for any continuous function $p : F\phi_A(\Delta_n) \to \Delta_n$, the composition $p \circ F\phi_A$ is approachable by [1, Lemma 2.4]. Then $p \circ F\phi_A : \Delta_n \multimap \Delta_n$ has a fixed point by [1, Lemma 4.1]. Therefore, $F \in \mathfrak{B}(X, Y)$.

Moreover, for any $A \in \langle D \rangle$ with $|A| = n + 1$, the set $F(\phi_A(\Delta_n))$ is compact in a Hausdorff space Y, and hence, normal. Therefore, $F \in \mathfrak{KC}(X, Y)$ by Lemma 4.3. This completes our proof.

Remarks. 1. Note that Theorem 4.5 for $X = Y$ reduces to [9, Lemma 8.9] and [3, Lemma 2.6].

2. Recall that, early in 1997 [11], for a G-convex space $(X, D; \Gamma)$ and a Hausdorff space Y, we showed that an acyclic map $F : X \multimap Y$ or, more generally, a map $F \in \mathfrak{A}_c^\kappa(X, Y)$ belongs to the class $\mathfrak{KC}$. This is reclaimed in [3, Lemma 2.7]. More early in 1994 [5], the result was obtained for convex spaces and this is the origin of the study of the so-called KKM-class of multimaps.

3. Note that [3, Theorem 2.8] is a particular form of our Theorem 2.3. From Theorem 2.3, we can easily obtain the following generalization of [3, Theorem 2.9] and our Corollary 2.4.

Theorem 4.6. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex Hausdorff uniform space, and $F \in \mathfrak{KC}(E, E)$ a map such that $\overline{F(E)}$ is a Φ-set. Suppose that $G : E \multimap E$ is a compact closed map such that $F(x) \subset G(x)$ for each $x \in E$, then G has a fixed point.*

Proof. By Theorem 2.3, for each $U \in \mathcal{U}$, there is an $x_U \in E$ such that $F(x_U) \cap U[x_U] \neq \emptyset$ and so $G(x_U) \cap U[x_U] \neq \emptyset$. Choose $y_U \in G(x_U) \cap U[x_U] \neq \emptyset$. Since $\overline{G(X)}$ is compact and Hausdorff, we may assume x_U and y_U converge to $\hat{x} \in \overline{G(X)}$. Since the graph of G is closed, we have $\hat{x} \in G(\hat{x})$. This completes our proof.

Remarks. 1. For $F = G$, Theorem 4.6 reduces to Corollary 2.4.

2. Since every LG-space whose singletons are Γ-convex is a Φ-space, Corollary 2.4 generalizes [3, Corollary 2.10].

Similarly, we have the following generalization of [3, Theorem 2.12]:

Theorem 4.7. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex Hausdorff uniform space and Z a compact space. Suppose that a map $T : Z \multimap E$ has a continuous selection f such that $f(Z)$ is a Φ-subset of E. If $F \in \mathfrak{KC}(E, Z)$ is a closed map, then there exist a point $z_0 \in Z$ and a point $x_0 \in E$ such that $z_0 \in F(x_0)$ and $x_0 \in T(z_0)$.*

Proof. Since $F \in \mathfrak{KC}(E, Z)$ and $f : Z \to E$ is continuous, we have $fF \in \mathfrak{KC}(E, E)$. Note that fF is closed and $\overline{fF(E)} \subset f(Z)$ is a Φ-subset of E. Therefore, by Corollary 2.4, fF has a fixed point $x_0 \in E$. This completes our proof.

Remarks. 1. For $Z \subset E$ and $T = 1_E$, Theorem 4.7 reduces to Corollary 2.4.

2. The authors of [3] incorrectly claimed that our Theorem 3.1, the main theorem in [6], is [3, Corollary 2.13] of [3, Theorem 2.12].

From our Corollary 2.4 or Theorem 4.7, we obtain the following generalization of our Theorem 3.5.

Corollary 4.8. *Let $(X, D; \Gamma; \mathcal{U})$ be an G-convex Hausdorff uniform space, and $F \in \mathfrak{KC}(X, X)$ a compact closed map such that $\overline{F(X)}$ is a Φ-set. Then F has a fixed point.*

Remarks. 1. In view of Theorem 3.3(1), $\mathfrak{KC}$ can be replaced by $\mathfrak{B}$, and hence, by the admissible class $\mathfrak{A}_c^\kappa$.

2. Since every LG-space whose singletons are Γ-convex is a Φ-space, our Corollary 4.8 generalizes [3, Corollary 2.14] for acyclic maps, which has been known for a long time in more general forms.

3. [3, Corollary 2.15] is redundant.

5. Fixed points of $\mathfrak{KC}$-maps in almost Γ-convex sets

In this section, for simplicity, all topological spaces are assumed to be Hausdorff otherwise explicitly stated. Our aim in this section is to examine results on $\mathfrak{KC}$-maps appeared in [2, 4, 12]. Note that all topological spaces are assumed to be Hausdorff in [2] and [11] implicitly and in [4] explicitly. Moreover, the authors of [2, 4, 12] adopted particular forms of our G-convex spaces, LG-spaces, and locally G-convex spaces for the case $X = D$. Furthermore, in [2, 4, 12], a G-convex uniform space $(E; \Gamma; \mathcal{U})$ is said to be locally G-convex whenever $U[x]$ is Γ-convex for each $x \in E$ and each $U \in \mathcal{U}$.

Checking the proof of [12, Lemma 1], the authors of [4] noticed the following:

Lemma 5.1 [4, Lemma 2.4]**.** *Suppose X is a nonempty compact subset of a locally G-convex space $(E; \Gamma; \mathcal{U})$, $p : X \to \Delta_n$ is continuous and $T : \Delta_n \multimap X$ is u.s.c. with nonempty closed Γ-convex values. Then $p \circ T : \Delta_n \to \Delta_n$ has a fixed point.*

From Lemmas 4.1 and 5.1, we can deduce the following improved version of [12, Theorem 3.1]:

Corollary 5.2. *Let $(X;\Gamma)$ be a locally G-convex space and $(Y;\Sigma)$ an arbitrary G-convex space. Suppose $F : X \to 2^Y$ such that*

1. *$F(x)$ is Σ-convex for all $x \in X$;*
2. *$F^-(y)$ contains an open set O_y (which may be empty for some y);*
3. *$\bigcup_{y\in Y} O_y = X$. Then for each compact Kakutani map $G : Y \to 2^X$ there exists a coincidence point; that is, a point $x_0 \in X$ such that $F(x_0) \cap G^-(x_0) \neq \emptyset$.*

Proof. Note that $X' := \overline{G(Y)}$ is a Hausdorff compact subset of Y. Then, by Lemma 4.1, $F|_{X'}$ has a continuous selection $s : X' \to \Sigma_N$ for some $N \in \langle Y\rangle$, $|N| = n + 1$ such that $s = \phi_n \circ p$ where $\phi_N : \Delta_n \to \Sigma_N$ and $p : X' \to \Delta_n$. Note that $G \circ \phi_N : \Delta_n \multimap X'$ is a compact Kakutani map. Hence, by Lemma 5.1, $p \circ G \circ \phi_N : \Delta_n \multimap \Delta_n$ has a fixed $a_0 \in \Delta_n$, that is, $a_0 \in p \circ G \circ \phi_N(a_0)$. Then $y_0 := \phi_n(a_0) \in (\phi_N \circ p)G(y_0) = (sG)(y_0) \subset (FG)(y_0)$. Then $y_0 \in F(x_0)$ and $x_0 \in G(y_0)$ for some $x_0 \in X$. This completes our proof.

From Lemma 5.1, we have the following:

Proposition 5.3. *Let $(X;\Gamma;\mathcal{U})$ be a locally G-convex space. Then any Kakutani map $F : X \multimap X$ belongs to $\mathfrak{B}(X,X)$.*

Proof. For any $N \in \langle X\rangle$, let $\phi_N : \Delta_n \to \Gamma_N \subset X$. Then $F \circ \phi_N : \Delta_n \multimap X$ is a compact Kakutani map. Then, for any continuous function $p : F\phi_N(\Delta_n) \to \Delta_n$, the map $p\circ F\circ\phi_N : \Delta_n \multimap \Delta_n$ has a fixed point by Lemma 5.1. Therefore, $F \in \mathfrak{B}(X,X)$.

Definition [4, Definition 2.3]**.** Suppose X is a nonempty subset of a G-convex space $(E;\Gamma)$ and Z a topological space, and $T, F : X \multimap Z$. We say that F is a generalized KKM mapping with respect to T if, for any $A = \{x_1,\dots,x_n\} \in \langle X\rangle$, there is $B = \{y_1,\dots,y_n\} \in \langle X\rangle$ satisfying

(a) $\Gamma_B \subset X$, and

(b) $T(\Gamma_{\{y_i:i\in I\}}) \subset \bigcup_{i\in I} F(x_i)$ for any nonempty subset I of $\{1,\dots,n\}$.

If a multimap $T : X \multimap Z$ satisfies that, for any generalized KKM mapping $F : X \multimap Z$ with respect to T the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the strict KKM property. (We corrected [4, Definition 2.3].)

Remark. In [2, Definition 2], a generalized KKM mapping is called a generalized Γ^*-mapping and the strict KKM property is called the Γ^*-KKM property. We will not use such terminology.

For each $A \in \langle X\rangle$, choose $B \in \langle X\rangle$ as in Definition and define $\Gamma'_A := \Gamma_B$. Then we have the following:

Proposition 5.4. *Under the situation of the above definition, $(X;\Gamma')$ is a G-convex space. Moreover, the map T having the strict KKM property is a $\mathfrak{KC}$-map in our sense and $T \in \mathfrak{KC}(X,Z)$.*

Note that [4, Proposition 2.5] can be generalized as follows:

Proposition 5.5. *Under the situation of the above definition, let X be a nonempty subset of a G-convex space $(E;\Gamma)$ and $(E';\Sigma;\mathcal{U})$ a locally G-convex space. Then any compact Kakutani map $T : X \multimap E'$ belongs to $\mathfrak{B}(X,E') \cap \mathfrak{KC}(X,E')$.*

This can be shown by our theory more easily than the proof in [4].

Proof. For the G-convex space $(X;\Gamma')$ as in Proposition 5.4, consider

$$\Delta_n \xrightarrow{\phi_N} \Gamma'_N \overset{T|_{\Gamma'_N}}{\multimap} T(\Gamma'_N) \xrightarrow{p} \Delta_n.$$

Since $T \circ \phi_N$ is also a compact Kakutani map, by Lemma 5.1, $p \circ T \circ \phi_N$ has a fixed point. Therefore $T \in \mathfrak{B}(X,Y)$. Since E is Hausdorff and T is compact, by our Theorem 3.3(1), we have $T \in \mathfrak{KC}(X,Y)$. This completes our proof.

Note that Proposition 5.5 with $E = E'$ reduces to [4, Proposition 2.5].

The following concept is well-known for topological vector spaces:

Definition [3, 4, Definition 3.1]**.** A nonempty subset X of a G-convex uniform space $(E,D;\Gamma;\mathcal{U})$ is said to be almost Γ-convex if, for any $\{x_1,\ldots,x_n\} \in \langle X\rangle$ and for any entourage $U \in \mathcal{U}$, there is $\{y_1,\ldots,y_n\} \in \langle X\rangle$ such that $y_i \in U[x_i]$ for each $i \in \{1,\ldots,n\}$ and Γ-co $\{y_1,\ldots,y_n\} \subset X$, where Γ-co B denotes the intersection of all Γ-convex sets containing B.

We show the following:

Proposition 5.6. *An almost Γ-convex subset X of a G-convex uniform space $(E;\Gamma;\mathcal{U})$ can be made into a G-convex uniform space $(X;\Gamma';\mathcal{U}')$.*

Proof. We choose $\mathcal{U}'$ the relative uniformity on X and define $\Gamma' : \langle X\rangle \multimap X$ as follows: for each $A := \{x_1,\ldots,x_n\} \in \langle X\rangle$, choose a $B := \{y_1,\ldots,y_n\} \in \langle X\rangle$, and define $\Gamma'(A) := \Gamma_B$. Then $(X;\Gamma';\mathcal{U}')$ is an abstract convex uniform space.

In the remainder of this section, we show that Theorems 3.3, 3.4 and Corollary 3.5 in [4] follow easily from our results in Sections 2 and 3:

Theorem 5.7 [4, Theorem 3.3]**.** *Let X be an almost Γ-convex subset of a locally G-convex space $(E;\Gamma;\mathcal{U})$. If $T \in \mathfrak{KC}(X,X)$ is compact, then for any $U \in \mathcal{U}$, there is $x_U \in X$ such that $U[x_U] \cap T(x_U) \neq \emptyset$.*

Proof. Recall that $(X;\Gamma';\mathcal{U}')$ as in Proposition 5.6 is a G-convex space and that the map $T : X \multimap X$ having the strict KKM property belongs to $\mathfrak{KC}(X,X)$ by Proposition 5.4.

We claim that $(X;\Gamma';\mathcal{U}')$ is locally G-convex. In fact, since $(E;\Gamma;\mathcal{U})$ is locally G-convex, for any $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$ and, for each $x \in E$, $N \in \langle V[x]\rangle$ implies $\Gamma_N \subset U[x]$. We may assume $U, V \in \mathcal{U}'$. For any $x \in X$ and for any $A := \{x_1,\ldots,x_n\} \in \langle V[x]\rangle$, there exists $B := \{y_1,\ldots,x_n\} \in \langle X\rangle$ such that $(x_i,y_i) \in V$ for each i and $\Gamma'_A = \Gamma_B \subset X$. Since $(x,x_i) \in V$ and $(x_i,y_i) \in V$, we may assume $(x,y_i) \in V$ for each i, and hence $B \in \langle V[x]\rangle$. Therefore, $\Gamma'_A = \Gamma_B \subset U[x]$ and the claim holds.

Moreover, any locally G-convex space is a Φ-space by Proposition 3.2. This also shows that X is a Φ-space as an abstract convex space. Therefore, the compact map $T \in \mathfrak{KC}(X,X)$ has the almost fixed point property by Theorem 2.3.

Note that, in Theorem 5.7, E is not necessarily Hausdorff.

Theorem 5.8 [4, Theorem 3.4]**.** *Let X be an almost Γ-convex subset of a locally G-convex Hausdorff space $(E;\Gamma;\mathcal{U})$. If $T \in \mathfrak{KC}(X,X)$ is compact and closed, then T has a fixed point.*

Proof. If X is Hausdorff and T is closed in addition to the hypothesis of Theorem 5.7, T has a fixed point by Corollary 2.4.

Corollary 5.9 [4, Corollary 3.5]**.** *Suppose X is an almost Γ-convex subset of a locally G-convex space $(E;\Gamma;\mathcal{U})$, and $T : X \multimap E$ is a compact Kakutani map. If $T(X) \subset X$, then it has a fixed point.*

Proof. This follows from Theorem 5.8 in view of Proposition 5.5 [4, Proposition 2.5].

Remarks. 1. If $X = E$ is compact, then Corollary 5.9 reduces to [12, Theorem 4.1].

2. The authors of [4] assumed a very particular form $(E;\Gamma)$ of our generalized convex spaces with extra conditions of monotonicity of Γ and of $x \in \Gamma(\{x\})$ for any $x \in E$. These are all redundant. Moreover, in the second half of [4], the authors tried to generalize known results in locally convex topological vector spaces. These can be simplified and improved by following our methods in [9,10].

3. In [2, Theorem 4], the author obtained Theorem 5.8 with certain defect. He applied his result to a coincidence theorem, a matching theorem, and a quasi-equilibrium theorem in a routine way. The author of [2] made several incorrect statements; for example, he claimed Theorem 5.8 without assuming $T \in \mathfrak{KC}(X,X)$ in [2, Theorem 3].

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