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ON GENERALIZATIONS OF THE KKM PRINCIPLE ON ABSTRACT CONVEX SPACES

Sehie Park

The National Academy of Sciences, Republic of Korea, and
School of Mathematical Sciences
Seoul National University
Seoul 151–747, Korea
E-mail : shpark@math.snu.ac.kr

Abstract. We introduce a new concept of abstract convex spaces which are adequate to establish the KKM theory. With this new concept, we can generalize and simplify known results in the theory on convex spaces, H-spaces, G-convex spaces, and others. The KKM type maps are used to obtain coincidence theorems and fixed point theorems. We add a number of examples of abstract convex spaces and generalizations of the KKM principle.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptyness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or others of the corresponding equilibrium problem under consideration. One of the remarkable results on the nonempty intersection is the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in 1929 [16], which is concerned with certain types of multimaps called the KKM maps.

The KKM theory, first named by the author, is the study of KKM maps and their applications; see [19]. Nowadays, it would be better to regard it as the study of applications of various equivalent formulations of the KKM principle. At the beginning, the theory was mainly devoted to the study on convex subsets of topological vector spaces. Later, it has been extended to convex spaces by Lassonde [17], and to C-spaces (or H-spaces) by Horvath [8-11], and others. In the last decade, the KKM theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the author; for details, see [22-31] and references therein.

In this paper, we introduce a new concept of abstract convex spaces which are adequate to establish the KKM theory. With this new concept, we can generalize and simplify known results of the theory on convex spaces, H-spaces, G-convex spaces, and others. The KKM type maps are used to obtain coincidence theorems and

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fixed point theorems. Finally, as historical remarks, we add a number of examples of abstract convex spaces and generalizations of the KKM principle in the chronological order.

2. Abstract convex spaces

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definitions. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if, for any $A \in \langle X \cap D\rangle$, we have $\Gamma_A \subset X$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

An abstract convex space with any topology is called *an abstract convex topological space.*

If the reader prefers, abstract convex spaces can be called A-convex spaces. Examples of abstract convex spaces will be given in Section 5.

Definitions. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \ \ A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called *a* $\mathfrak{K}$*-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation $\mathrm{KKM}(E, Z)$ instead of $\mathfrak{KC}(E, Z)$.

Examples. 1. Every abstract convex space in our sense has a map $F \in \mathfrak{K}(E, Z)$ for any nonempty set Z and for any class of KKM maps $G : D \multimap Z$ with respect to F. In fact, for each $x \in E$, choose $F(x) := Z$ or $F(x) := \{z_0\}$ for some $z_0 \in Z$.

2. Further examples are given in Section 5.

We have abstract convex subspaces:

Proposition 1. *For an abstract convex space $(E \supset D; \Gamma)$, let X be a Γ-convex subset of E, and D' a nonempty subset of $X \cap D$. Let $\Gamma' : \langle D' \rangle \multimap X$ be a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \text{ for } A \in \langle D' \rangle.$$

Then $(X \supset D'; \Gamma')$ itself is an abstract convex space called a subspace.

Proposition 2. *Let $(E \supset D; \Gamma)$ be an abstract convex space, $(X \supset D'; \Gamma')$ a subspace, and Z a set. If $F \in \mathfrak{K}(E, Z)$, then $F|_X \in \mathfrak{K}(X, Z)$.*

Proof. Suppose that a map $G' : D' \multimap Z$ satisfies

$$F|_X(\Gamma'_A) \subset G'(A) \ \text{ for all } A \in \langle D' \rangle.$$

Define a map $G : D \multimap Z$ by

$$G(y) := \begin{cases} G'(y) & \text{for } y \in D' \\ Z & \text{otherwise} \end{cases}$$

Then

$$F(\Gamma_A) = F|_X(\Gamma'_A) \subset G'(A) = G(A) \ \text{ for } A \in \langle D' \rangle; \text{ and}$$

$$F(\Gamma_A) \subset Z = G(A) \ \text{ for } A \in \langle D \rangle \setminus \langle D' \rangle.$$

Since $F \in \mathfrak{K}(E, Z)$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property, and hence so does its subfamily $\{G'(y)\}_{y \in D'}$. Therefore, $F|_X \in \mathfrak{K}(X, Z)$.

Remark. Note that Proposition 2 holds for $\mathfrak{KC}$ and $\mathfrak{KO}$ instead of $\mathfrak{K}$ whenever Z is a topological space.

3. The KKM principle in abstract convex spaces

We begin with the following simple observation:

Proposition 3. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F : E \multimap Z$ a multimap. Then $F \in \mathfrak{K}(E, Z)$ if and only if for any multimap $G : D \multimap Z$ satisfying*

(3.1) *$F(\Gamma_N) \subset G(N)$ for any $N \in \langle D \rangle$,*

we have $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.

Proof. For the necessity, from (3.1), for any $N \in \langle D \rangle$, we have $F(\Gamma_N) \subset F(E) \cap G(N) = \bigcup_{y \in N}\{F(E) \cap G(y)\}$. Since F is a $\mathfrak{K}$-map, the family $\{F(E) \cap G(y)\}_{y \in D}$ has the finite intersection property. The sufficiency is clear.

Remark. If Z has any topology and if $F \in \mathfrak{KO}(E, Z)$ [resp., $F \in \mathfrak{KC}(E, Z)$], then we have to assume G is open-valued [resp. closed-valued].

For an abstract convex topological space, from Proposition 3 with $E = Z$ and $F = 1_E$, the following recovers the meaning of $1_E \in \mathfrak{KC}(E, E)$ or $1_E \in \mathfrak{KO}(E, E)$:

Proposition 4. *Let $(E, D; \Gamma)$ be an abstract convex topological space. Then the identity map 1_E belongs to $\mathfrak{KC}(E,E)$ [resp. $1_E \in \mathfrak{KO}(E,E)$] if and only if for any multimap $G : D \multimap E$ satisfying*

(4.1) *G has closed [resp. open] values, and*

(4.2) *G is a KKM map,*

$\{G(y)\}_{y\in D}$ has the finite intersection property.

Under an additional requirement, we have the whole intersection property for the map-values of a KKM map:

Proposition 5. *Let $(E, D; \Gamma)$ be an abstract convex topological space, the identity map $1_E \in \mathfrak{KC}(E,E)$ [resp. $1_E \in \mathfrak{KO}(E,E)$], and $G : D \multimap E$ a multimap satisfying (4.1), (4.2), and*

(4.3) *$\bigcap_{z\in M} \overline{G(z)}$ is compact for some $M \in \langle D\rangle$.*

Then we have

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Proof. Since $1_E \in \mathfrak{KC}(E,E)$ [resp. $1_E \in \mathfrak{KO}(E,E)$], by definition, $\{G(y)\}_{y\in D}$ has the finite intersection property. Now the whole intersection property follows from the compactness in (4.3).

Remarks. 1. You may prefer to adopt "compactly" closed [resp. open] values in (4.1). This is impractical and superfluous. In fact, replacing the topology of E by its compactly generated extension, we can eliminate that kind of inadequate terminology; see [24].

2. Some authors call G a transfer closed map when $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$. In this case, the conclusion of Proposition 5 becomes $\bigcap_{y\in D} G(y) \neq \emptyset$.

The following is a basic observation:

Proposition 6. *Let $(E, D; \Gamma)$ be an abstract convex topological space and Z a topological space. If $1_E \in \mathfrak{KC}(E,E)$, then $f \in \mathfrak{KC}(E,Z)$ for any continuous function $f : E \to Z$. This also holds for $\mathfrak{KO}$.*

Proof. Let $G : D \multimap Z$ be a closed-valued map satisfying $f(\Gamma_N) \subset G(N)$ or $\Gamma_N \subset f^{-1}G(N)$ for each $N \in \langle D\rangle$. Since $1_E \in \mathfrak{KC}(E,E)$ and $f^{-1}G : D \multimap E$, $\{f^{-1}G(y)\}_{y\in D}$ has the finite intersection property. Hence, so does $\{G(y)\}_{y\in D}$. Therefore, $f \in \mathfrak{KC}(E,Z)$. Similarly, we can show the case for $\mathfrak{KO}$.

Usually, a KKM theorem is a claim $1_E \in \mathfrak{KC}(E,E)$ for an abstract convex topological space $(E, D; \Gamma)$. For particular forms of the KKM theorem for convex spaces, H-spaces, or G-convex spaces and their applications, there are a large number of works. See Section 5 and the references at the end.

4. Coincidence and fixed point theorems

In the KKM theory, there exist some basic results from which we can deduce several equivalent formulations that can be used to applications. In this section, we introduce some of such basic results.

For abstract convex spaces, we have the following coincidence theorem:

Theorem 1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, $S : D \multimap Z$, $T : E \multimap Z$ maps, and $F \in \mathfrak{K}(E, Z)$. Suppose that*

(1.1) *for each $z \in F(E)$, $M \in \langle S^-(z)\rangle$ implies $\Gamma_M \subset T^-(z)$; and*
(1.2) *$F(E) \subset S(N)$ for some $N \in \langle D\rangle$.*

Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Proof. For each $y \in D$, define $R(y) := F(E) \setminus S(y)$. Then $\bigcap_{y\in N} R(y) = F(E) \setminus \bigcup_{y\in N} S(y) = \emptyset$ by (1.2), that is, the values of the map $R : D \multimap Z$ does not have the finite intersection property. Since $F \in \mathfrak{K}(E, Z)$, $F(\Gamma_M) \not\subset R(M)$ for some $M \in \langle D\rangle$. Hence, there exist $\bar{x} \in \Gamma_M$ and $\bar{z} \in F(\bar{x}) \subset F(E)$ such that $\bar{z} \notin R(M)$. Then, $\bar{z} \in S(y)$ or $y \in S^-(\bar{z})$ for all $y \in M$. This implies $\bar{x} \in \Gamma_M \subset T^-(\bar{z})$ by (1.1). Therefore, $\bar{z} \in F(\bar{x}) \cap T(\bar{x})$.

Remark. If Z has any topology and S has open [resp. closed] values, then R has relatively closed [resp. open] values in $F(E)$. Then we can assume $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

From Theorem 1, we have the following prototype of the Fan-Browder fixed point theorem [3]:

Theorem 2. *Let $(E, D; \Gamma)$ be an abstract convex topological space, and $G : E \multimap D$, $H : E \multimap E$ maps satisfying*

(2.1) *for each $x \in E$, $M \in \langle G(x)\rangle$ implies $\Gamma_M \subset H(x)$; and*
(2.2) *$E = G^-(N)$ for some $N \in \langle D\rangle$.*
(2.3) *G^- has open [resp. closed] values.*

If the identity map $1_E \in \mathfrak{KC}(E, E)$ [resp. $1_E \in \mathfrak{KO}(E, Z)$], then H has a fixed point $\bar{x} \in E$, that is, $\bar{x} \in F(\bar{x})$.

Proof. In Theorem 1, let $E = Z$, $S := G^-$, $T := H^-$, and $F := 1_E$.

Remark. Theorem 2 is originated from [3] and one of the most useful results in the KKM theory.

From Theorem 2, we deduce some new forms of the Fan-Browder type fixed point theorems:

Theorem 3. *Let $(E, D; \Gamma)$ be an abstract convex topological space and $S : E \multimap D$, $T : E \multimap E$ maps such that*

(3.1) *for each $x \in E$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$; and*
(3.2) *there exist $D' := \{z_1, z_2, \cdots, z_n\} \in \langle D\rangle$ and open [resp. closed] subsets $\{G_i\}_{i=1}^n$ of X such that*

$$E = \bigcup_{i=1}^{n} G_i \quad \text{and} \quad G_i \subset S^-(z_i) \quad \text{for each } i.$$

If $1_E \in \mathfrak{KC}(E,E)$ [*resp.* $1_X \in \mathfrak{KO}(E,E)$], *then* T *has a fixed point* $x_* \in E$.

Proof. Consider the abstract convex space $(E, D'; \Gamma)$ where $\Gamma : \langle D' \rangle \multimap X$ is actually the restriction $\Gamma|_{\langle D' \rangle}$ of the original Γ. Define a map $G : E \multimap D'$ by $G^-(z_i) = G_i$ for each $z_i \in D'$. Note that $G(x) \subset S(x)$ for each $x \in X$. Now Theorem 2 with $H := T$ works.

Remarks. 1. In Theorem 3, let $E_T := \{x \in E : x \notin T(x)\}$. Then condition $E = \bigcup_{i=1}^n G_i$ in (3.2) can be replaced by $E_T = \bigcup_{i=1}^n G_i$ without affecting the conclusion of Theorem 3. In fact, suppose that T has no fixed point, that is, $E = E_T$. Then by Theorem 3, T has a fixed point, a contradiction.

2. For a G-convex space, Theorem 3 reduces to [25, Theorem 8], which has a number of variants as shown in [25].

Motivated by Horvath [10, 11], we define the following:

Definition. For a given abstract convex space $(E, D; \Gamma)$ and a topological space X, a map $H : X \multimap E$ is called a Φ-*map* (or a *Fan-Browder map*) if there exists a map $G : X \multimap D$ such that

(i) for each $x \in X$, $M \in \langle G(x) \rangle$ implies $\Gamma_M \subset H(x)$; and

(ii) $X = \bigcup \{\operatorname{Int} G^-(y) \mid y \in D\}$.

Definitions. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is an abstract convex space with a base $\mathcal{U}$ of a uniform structure of E. A space $(E, D; \Gamma; \mathcal{U})$ is called a Φ-*space* if, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : E \multimap E$ such that $\operatorname{Gr}(H) \subset U$.

For an $U \in \mathcal{U}$, a point $x \in E$ is called a U-*fixed point* of a map $F : E \multimap E$ if $F(x) \cap U[x] \neq \emptyset$. The map F is said to have the *almost fixed point property* whenever it has a U-fixed point for any $U \in \mathcal{U}$.

Now we have the following almost fixed point theorem:

Theorem 4. *Let* $(E; \Gamma; \mathcal{U})$ *be an abstract convex uniform space such that*

(4.1) *for each* $U \in \mathcal{U}$, *there exists* $V \in \mathcal{U}$ *such that for each* $x \in X$, $M \in \langle V[x] \rangle$ *implies* $\Gamma_M \subset U[x]$.

Let $F \in \mathfrak{KC}(E,E)$ [*resp.* $F \in \mathfrak{KO}(E,E)$] *be a map such that*

(4.2) $F(E)$ *is totally bounded.*

Then F *has the almost fixed point property.*

Proof. Put $E = D = Z$ and $S = T$ in Theorem 1. Each $U, V \in \mathcal{U}$ in (4.1) can be assumed to be open [resp. closed] symmetric members of a base of $\mathcal{U}$. Let $S(x) := V[x]$ and $T(x) := U[x]$ for each $x \in E$. Then, by Theorem 1, there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap U[\bar{x}] \neq \emptyset$. This completes our proof.

Corollary 4.1. *Under the hypothesis of Theorem 4, further if* $(E, \mathcal{U})$ *is separated and if* F *is compact and closed, then it has a fixed point.*

Proof. By Theorem 4, for each $U \in \mathcal{U}$, there is an $x_U \in E$ such that $F(x_U) \cap U[x_U] \neq \emptyset$ and so choose $y_U \in F(x_U) \cap U[x_U] \neq \emptyset$. Since $\overline{F(X)}$ is compact and Hausdorff, we may assume x_U and y_U converge to $\hat{x} \in \overline{F(X)}$. Since the graph of F is closed, we have $\hat{x} \in F(\hat{x})$. This completes our proof.

Corollary 4.2. *Let $(E;\Gamma;\mathcal{U})$ be an abstract convex separated uniform space satisfying condition (4.1) and the identity map $1_E \in \mathfrak{KC}(E,E)$ [resp. $1_E \in \mathfrak{KO}(E,E)$]. If $f : E \to E$ is a compact continuous function, then f has a fixed point.*

Proof. By Proposition 6, if $1_E \in \mathfrak{KC}(E,E)$, so is f.

For Φ-spaces, we have the following:

Theorem 5. *Let $(E,D;\Gamma;\mathcal{U})$ be an abstract convex uniform Φ-space and $F \in \mathfrak{KC}(E,E)$ a compact map. Then F has the almost fixed point property.*

Proof. For any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : E \multimap E$ such that $\mathrm{Gr}(H) \subset U$, that is, there exists a map $G : E \multimap D$ satisfying conditions (i) and (ii) with $X = E$. Since $\overline{F(E)}$ is compact, $F(E) \subset \overline{F(E)} \subset \bigcup\{\mathrm{Int}\, G^-(y) \mid y \in N\}$ for some $N \in \langle D \rangle$. Let $S := G^-$ and $T := H^-$. Since $F \in \mathfrak{KC}(E,E)$, by Theorem 1, there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap H^-(\bar{x}) \neq \emptyset$. Since $\mathrm{Gr}(H) \subset U$ and $H^-(\bar{x}) = \{x \in E \mid \bar{x} \in H(z)\}$, we have $H^-(\bar{x}) \subset U^-[\bar{x}]$. Since we may assume U is symmetric, $F(\bar{x}) \cap U[\bar{x}] \neq \emptyset$. This completes our proof.

Corollary 5.1. *Under the hypothesis of Theorem 5, further if $(E,\mathcal{U})$ is separated and if F is closed, then it has a fixed point.*

Corollary 5.2. *Let $(E,D;\Gamma;\mathcal{U})$ be an abstract convex separated uniform Φ-space such that the identity map $1_E \in \mathfrak{KC}(E,E)$. If $f : E \to E$ is a compact continuous function, then f has a fixed point.*

Remarks. 1. Recently, Amini et al. [1, 2] obtained particular forms of Theorems 4, 5 and Corollaries 5.1 and 5.2 for the S-KKM class for closed-valued maps on a classical convexity space; see Section 5.

2. Recall that each of Theorems 1-5 has a large number of particular cases and applications in various types of generalized convex spaces. Therefore they can be used to generalize and simplify the KKM theory.

In the remainder of this section, we are concerned with metric spaces and give different proof of main results in [1]:

Definition. Let (M,d) be a pseudo-metric space. Define a multimap $\Gamma : \langle M \rangle \multimap M$ by

$$\Gamma_A = \Gamma(A) := \bigcap\{B \mid B \ \text{ is a closed ball containing } \ A\}$$

for each $A \in \langle M \rangle$.

Theorem 6. *Let $(M,d;\Gamma)$ be a pseudo-metric space, X a Γ-convex subspace of M, and $F \in \mathfrak{KC}(X,X)$. If $F(X)$ is totally bounded, then F has the almost fixed point property.*

Proof. For each $\varepsilon > 0$, define a multimap $G : X \multimap X$ by $G(x) := \{y \in X \mid d(x,y) \leq \varepsilon\} \neq \emptyset$. Then each $G(x)$ is Γ-convex. Moreover, since $F(X)$ is totally bounded, $F(X) \subset \bigcup\{\operatorname{Int} G^-(y) \mid y \in N\}$ for some $N \in \langle X \rangle$. Therefore, by Theorem 1 or the proof of Theorem 4, there exists an $\bar{x} \in X$ such that $F(\bar{x}) \cap G^-(\bar{x}) \neq \emptyset$. This completes our proof.

For a metric space, Theorem 6 reduces to [1, Theorem 2.1], whose proof is different from ours.

Corollary 6.1. *Let $(M, d; \Gamma)$ be a metric space, X a Γ-convex subspace of M, and $F \in \mathfrak{KC}(X, X)$. If F is closed and compact, then F has a fixed point.*

Corollary 6.2. *Let $(M, d; \Gamma)$ be a metric space, X a Γ-convex subspace of M, and the identity map $1_X \in \mathfrak{KC}(X, X)$. Then any compact continuous function $f : X \to X$ has a fixed point.*

Note that Corollaries 6.1 and 6.2 are due to Amini et al. [1, Theorem 2.2 and Corollary 2.4].

5. Historical remarks on the KKM principle

In this section, we give examples of abstract convex spaces $(E, D; \Gamma)$ in the chronological order. For some of them, we have $1_E \in \mathfrak{KC}(E, E)$ [or $1_E \in \mathfrak{KO}(E, E)$].

1. If $E = \Delta_n$ is an n-simplex, D is the set of its vertices, $\Gamma = \mathrm{co}$ is the convex hull operation, then the celebrated KKM principle [16] says that $1_E \in \mathfrak{KC}(E, E)$. In this case, note that $1_E \notin \mathfrak{K}(E, E)$. A simple example for $n = 1$ is as follows: Let $\Delta_1 := [0, 1]$, $D := \{0, 1\}$, and $G(0) := [0, \frac{1}{2}]$, $G(1) := (\frac{1}{2}, 1]$. Then G is a KKM map, but $G(0) \cap G(1) = \emptyset$.

2. If D is a nonempty subset of a topological vector space E (not necessarily Hausdorff), Fan's KKM lemma [6] says that $1_E \in \mathfrak{KC}(E, E)$.

3. Let E be a topological vector space with a neighborhood system $\mathcal{V}$ of its origin. A subset X of E is said to be *almost convex* [7] if for any $V \in \mathcal{V}$ and for any finite subset $A := \{x_1, x_2, \cdots, x_n\}$ of X, there exists a subset $B := \{y_1, y_2, \cdots, y_n\}$ of X such that $y_i - x_i \in V$ for each $i = 1, 2, \cdots, n$ and $\mathrm{co}\, B \subset X$. By choosing $\Gamma_A := B$ for each $A \in \langle X \rangle$, $(X; \Gamma)$ becomes a G-convex space and hence an abstract convex space.

4. If X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology, then $(X, D; \Gamma)$ becomes a *convex space* generalizing the one due to Lassonde [17]. Note that any convex subset of a topological vector space. is a convex space, but not conversely. For a convex space (X, co), Lassonde showed that $1_X \in \mathfrak{KC}(X, X)$.

5. In 1989, Kim [15] and Shih and Tan [32] showed that $1_E \in \mathfrak{KO}(E, E)$ when E is an n-simplex. Therefore, in general, we have

$$\mathfrak{K}(E, E) \subsetneq \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E).$$

6. A well-known subclass of G-convex spaces due to Horvath [8-11] can be generalized as follows: A G-convex space $(X, D; \Gamma)$ is called a *C-space* (or an *H-space*) if each Γ_A is ω-connected (that is, n-connected for all $n \geq 0$) and $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D \rangle$. For a C-space (X, Γ), Horvath showed that $1_X \in \mathfrak{KC}(X, X)$. In particular, Khamsi [12] obtained $1_X \in \mathfrak{KC}(X, X)$ for a hyperconvex metric space X.

7. In early 1990's, the author [18] introduced the admissible class $\mathfrak{A}_c^\kappa(X, Y)$ of multimaps $X \multimap Y$ between topological spaces and showed that this class is contained in the class $\mathfrak{KC}(X, Y)$ when X is a convex space and Y is a Hausdorff space [19]. Motivated by this, Chang and Yen [4] defined the KKM class of maps on convex subsets of topological vector spaces, and further, Chang et al. [5] extended the KKM-class to S-KKM class. On the other hand, the author extended the $\mathfrak{A}_c^\kappa$-class to the 'better' admissible $\mathfrak{B}$-class on convex spaces, supplied a large number of examples, and showed that, in the class of compact closed multimaps from convex spaces to Hausdorff spaces, two subclasses $\mathfrak{B}$ and $\mathfrak{KC}$ coincide [20]. Moreover, recently H. Kim [13] showed that two classes KKM and s-KKM of multimaps from a convex space into a topological space are identical whenever s is surjective [this is the only case S-KKM is slightly meaningful].

8. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. It is possible to assume $\Gamma(A) = \phi_A(\Delta_n)$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$. In case $X \supset D$, the G-convex space is denoted by $(X \supset D; \Gamma)$.

For details on G-convex spaces, see [22-30], where basic theory was extensively developed and lots of examples of G-convex spaces were given.

9. For a G-convex space $(X, D; \Gamma)$ and a topological space Z, we defined the classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ of multimaps $F : X \multimap Z$ [25], and showed that $1_X \in \mathfrak{KC}(X, X) \cap \mathfrak{KO}(X, X)$; see [22-27]. Moreover, we noted that if $F : X \to Z$ is a continuous single-valued map or if $F : X \multimap Z$ has a continuous selection, then $F \in \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$. Furthermore, for a Hausdorff space Z, it is shown that $\mathfrak{A}_c^\kappa(X, Z) \subset \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$ by H. Kim and the author [14].

10. Usually, a *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see [33], where the bibliography lists 283 papers. For any subset $X \subset E$, its *$\mathcal{C}$-convex hull* is defined and denoted by $\text{Co}_{\mathcal{C}} X := \bigcap\{Y \in \mathcal{C} \mid X \subset Y\}$. We say that X is *$\mathcal{C}$-convex* if $X = \text{Co}_{\mathcal{C}} X$. Now we can consider the map $\Gamma : \langle E \rangle \multimap E$ given by $\Gamma_A := \text{Co}_{\mathcal{C}} A$. Then $(E, \mathcal{C})$ becomes our abstract convex space $(E; \Gamma)$.

Notice that our abstract convex space $(E \supset D; \Gamma)$ becomes a convexity space $(E, \mathcal{C})$ for the family $\mathcal{C}$ of all Γ-convex subsets of E.

11. For any metric space (M, d), Amini et al. [1] introduced a convexity structure similar to the one for hyperconvex metric space; see [12]. They defined an $\mathcal{NR}$-metric space (M, d) and showed that, for any subadmissible subset X of M, $1_X \in \mathfrak{KC}(X, X)$ holds. Here, subadmissible subsets are simply Γ-convex subsets as in Theorem 6.

Recall that, early in 1997 [29], for a G-convex space $(X, D; \Gamma)$ and a Hausdorff space Y, we showed that an acyclic map $F : X \multimap Y$ or, more generally, a map $F \in \mathfrak{A}_c^\kappa(X, Y)$ belongs to the class $\mathfrak{KC}$. The authors of [1] repeatedly claimed that they obtained this result in 2005. More early in 1994 [19], the result was obtained for convex spaces and this is the origin of the study of the so-called KKM-class of multimaps.

12. Imitating the original definition of S-KKM maps of Chang et al. [5], Amini et al. [2] defined the S-KKM class for a classical convexity space $(X, \mathcal{C})$ with a nonempty set Z and a topological space Y as follows: If $S : Z \multimap X$, $F : X \multimap Y$, and $G : Z \multimap Y$ are three multimaps satisfying

$$F(\mathrm{Co}_{\mathcal{C}}(S(A)) \subset G(A) \quad \text{for each} \quad A \in \langle Z \rangle,$$

then G is called a $\mathcal{C}$-S-KKM map with respect to F. If the map $F : X \multimap Y$ satisfies the requirement that for any $\mathcal{C}$-S-KKM map G with respect to F, the family $\{\overline{G(z)} \mid z \in Z\}$ has the finite intersection property, then F is said to have the S-KKM property with respect to $\mathcal{C}$. Amini et al. defined

$$S\text{-KKM}_{\mathcal{C}}(Z, X, Y) := \{F : X \multimap Y \mid F \text{ has the } S\text{-KKM property with respect to } \mathcal{C}\}.$$

It should be noted that, by putting $\Gamma_A := \mathrm{Co}_{\mathcal{C}}(S(A))$ for each $A \in \langle Z \rangle$, S-KKM$_{\mathcal{C}}(Z, X, Y)$ becomes simply $\mathfrak{KC}(X, Y)$. Therefore, it seems to be the proper time to eliminate the S-KKM class.

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