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Comments on collectively fixed points in generalized convex spaces

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Abstract

Based on a Fan–Browder type fixed point theorem due to the author, we deduce new general collectively fixed point theorems for a family of Browder type multimaps defined on a product of generalized convex spaces.

Keywords: Collectively fixed point; G-Convex spaces; Compactly generated extension; Coercivity condition

1. Introduction

There have appeared so-called collectively fixed point theorems for a family of multimaps defined on a product of (generalized) convex spaces. In certain cases, such multimaps are assumed to be the Browder maps (having nonempty convex values and open fibers) or multimaps in a more general class. As a typical example, we obtained such a result for a family of compact G-convex spaces in [1], and applied it to the von Neumann type intersection theorems due to Fan and Ma, the Fan–Ma type analytic alternative, the Nash–Ma type equilibrium theorem and its consequences; see [2]. For other applications, see the references of [3].

Such collectively fixed point theorems reduce to the Fan–Browder type fixed point theorem whenever the family consists of a single space. Usually, the proofs of such theorems were given by the partition of

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unity argument and the Brouwer or Tychonoff fixed point theorem, as was shown in Browder's classical proof of the Fan–Browder theorem.

However, in recent works [4–6] in the KKM theory, the author was able to obtain generalizations of the Fan–Browder theorem replacing the Hausdorff compactness of the domain by a finite open [resp. closed] cover refining the fibers. Consequently, we can eliminate the partition of unity argument and the Tychonoff theorem.

In the present paper, using our own version of the Fan–Browder theorem, we obtain new general results on collectively fixed points in the frame of G-convex spaces. At the end of this paper, we give some comments on a recent work [3], which is a rich source of particular forms of our new results.

2. Preliminaries

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that, for each $A = \{a_0, a_1, \ldots, a_n\} \in \langle D \rangle$, there exists a subset $\Gamma(A) = \Gamma_A$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$, where $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n an n-simplex with vertices $v_0, v_1, \ldots, v_n$, and $\Delta_J = \mathrm{co}\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\}$ the face of Δ_n corresponding to J. In case $D = A$ is finite, we may assume $\Gamma_J = \phi_A(\Delta_J)$ for all $J \subset A$.

For the case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X; \Gamma) = (X, X; \Gamma)$.

There are a large number of examples of G-convex spaces; see [1,7,8]. Typical examples are any convex subset of a topological vector space, convex spaces in the sense of Lassonde, C-spaces (or H-spaces) due to Horvath, and many others.

For a topological space X and a G-convex space $(Y, D; \Gamma)$, a multimap $T : X \multimap Y$ is called a *Φ-map* provided that there exists a multimap $S : X \multimap D$ satisfying

(a) for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$; and
(b) $X = \bigcup\{\mathrm{Int}\, S^-(y) : y \in D\}$.

Recall that $S^-(y) := \{x \in X : y \in S(x)\}$ is a fiber of $S : X \multimap D$, and that if S is single-valued, then S^- is usually denoted by S^{-1}; that is $S^{-1}(y) := \{x \in X : y = S(x)\}$.

We need the following fixed point theorem due to the author [4–6]:

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space, D a finite set, and $S : X \multimap D$, $T : X \multimap X$ maps such that*

(1.1) *for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$;*
(1.2) *$S^-(z)$ is open [resp. closed] for each $z \in D$; and*
(1.3) $X = \bigcup\{S^-(z) : z \in D\}$.

Then T has a fixed point $x_* \in X$; that is, $x_* \in T(x_*)$.

Recall that Theorem 1 encompasses a large number of generalization of the Fan–Browder fixed point theorem.

Let $\{X_i\}_{i \in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X := \prod_{j \in I} X_j, \qquad X^i := \prod_{j \in I \setminus \{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \backslash \{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its ith coordinate is x_i, and for $j \neq i$ the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where $x^i = \pi^i(x)$ denotes the projection of x in X^i, and $x_i = \pi_i(x)$ the projection of x in X_i.

3. Main results

From Theorem 1, we deduce the following collectively fixed point theorem which is the main result of this paper:

Theorem 2. *Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be a family of G-convex spaces, $X := \prod_{i \in I} X_i$, and for each $i \in I$, D_i finite, $S_i : X \multimap D_i$ and $T_i : X \multimap X_i$ multimaps such that*

(2.1) *for each $x \in X$, $M \in \langle S_i(x) \rangle$ implies $\Gamma_i(M) \subset T_i(x)$;*
(2.2) *$S_i^-(z_i)$ is open [resp. closed] for each $z_i \in D_i$; and*
(2.3) $X = \bigcup \{S_i^-(z_i) : z_i \in D_i\}$.

Then there exists a point $x \in X$ such that $x \in T(x) := \prod_{i \in I} T_i(x)$; that is, $x_i = \pi_i(x) \in T_i(x)$ for each $i \in I$.

Proof. Choose a point $a = [a^i, a_i] \in X$ (here, we always assume that each X_i is nonempty). For each $i \in I$, define a function $J_i : X_i \to X$ by $x_i \mapsto [a^i, x_i]$ for each $i \in I$. Then each J_i is an embedding (homeomorphism into). For each $i \in I$, define $S_i' := S_i \circ J_i : X_i \multimap D_i$ and $T_i' := T_i \circ J_i : X_i \multimap X_i$. Then for each $i \in I$, we have the following:

(1) For each $x_i \in X_i$, $M \in \langle S_i'(x_i) \rangle$ implies $\Gamma_i(M) \subset T_i'(x)$. In fact, $S_i'(x_i) = (S_i \circ J_i)(x_i) = S_i[a^i, x_i]$ and $M \in \langle S_i'(x_i) \rangle$ imply $\Gamma_i(M) \subset T_i[a^i, x_i]$ by (2.1). Note that $T_i[a^i, x_i] = (T_i \circ J_i)(x_i) = T_i'(x_i)$.

(2) For each $z_i \in D_i$, $(S_i')^-(z_i) = (S_i \circ J_i)^-(x_i) = J_i^{-1}(S_i^-(z_i))$ is open [resp. closed] since so is $S_i^-(z_i)$ by (2.2) and J_i is continuous.

(3) $X_i = \bigcup \{(S_i')^-(z_i) : z_i \in D_i\}$. In fact, for any $x_i \in X_i$, we have $J_i(x_i) = [a^i, x_i] \in X = \bigcup \{S_i^-(z_i) : z_i \in D_i\}$ by (2.3). Hence $J_i(x_i) \in S_i^-(z_i)$ for some $z_i \in D_i$, and $x_i \in (J_i^{-1} \circ S_i^-)(z_i) = (S_i')^-(z_i)$.

Now we apply Theorem 1 for $(X_i, D_i; \Gamma_i)$. Then T_i' has a fixed point $b_i \in X_i$; that is, $b_i \in T_i'(b_i) = (T_i \circ J_i)(b_i) = T_i[a^i, b_i]$. Let $b = [b^i, b_i] \in X$. It should be noted that the above argument holds for any point $a \in X$. Therefore, we may choose $a = b$. Then, we have $b_i \in T_i[b^i, b_i] = T_i(b)$ and hence $b \in T(b) = \prod_{i \in I} T_i(b)$, and $b_i = \pi_i(b) \in T_i(b)$. This completes our proof. □

Remark. When I is a singleton, Theorem 2 reduces to Theorem 1. Hence, Theorems 1 and 2 are equivalent.

From Theorem 2, we have the following particular form:

Theorem 3. *Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be a family of compact G-convex spaces, $X := \prod_{i \in I} X_i$, and for each $i \in I$, $T_i : X \multimap X_i$ a Φ-map. Then there exists a point $x \in X$ such that $x \in T(x) := \prod_{i \in I} T_i(x)$; that is, $x_i = \pi_i(x) \in \prod_{i \in I} T_i(x)$ for each $i \in I$.*

Proof. Since T_i is a Φ-map, for each $i \in I$, there exists a multimap $S_i : X \multimap D_i$ such that $X = \bigcup \{\operatorname{Int} S_i^-(z_i) : z_i \in D_i\}$. Since X is compact, there exists a $D_i' \in \langle D_i \rangle$ such that

$X = \bigcup\{\mathrm{Int}\, S_i^-(z_i) : z_i \in D_i'\}$ for each $i \in I$. Then $(X_i, D_i'; \Gamma_i')$, where $\Gamma_i' := \Gamma_i|\langle D_i'\rangle$, is a G-convex space. Now, we can apply Theorem 2 to obtain the conclusion. □

Remarks. 1. If I is a singleton, Theorem 3 reduces to the Fan–Browder fixed point theorem for G-convex spaces due to the author [9,10].

2. In our previous work [1], a particular form of Theorem 3, under the restriction that $X_i = D_i$ and each X_i is Hausdorff, was obtained by using the technique of partition of unity [that is why the Hausdorffness of X was needed] and the Tychonoff fixed point theorem [which can be replaced by a recent resolution of the Schauder conjecture due to Robert Cauty [11] in 2001].

3. In our work [2], the above particular form of Theorem 3 was applied to obtain a generalization of various Ky Fan type intersection theorems for sets with convex sections, a generalized Fan type minimax theorems or an analytic alternative, the Nash–Ma type equilibrium theorem, the Mazur–Schauder maximum theorem, and a von Neumann–Sion type minimax theorem. In all of the results in [2], in virtue of the above particular theorem, we had to assume the Hausdorffness of X_i's, which are redundant now.

For a G-convex space $(X \supset D; \Gamma)$ and a subset Y of X, a G-convex subspace $(Y, Y \cap D; \Gamma')$ is defined by

$$\Gamma_A' := \Gamma_A \cap Y \text{ for each } A \in \langle Y \cap D\rangle.$$

Theorem 2 can be applied to the case when X is not covered by a finite number of fibers as follows:

Theorem 4. *Let $\{(X_i \supset D_i; \Gamma_i)\}_{i\in I}$ be a family of G-convex spaces, $X := \prod_{i\in I} X_i$, and for each $i \in I$, $S_i : X \multimap D_i$ and $T_i : X \multimap X_i$ multimaps satisfying conditions*

(4.1) *for each $x \in X$, $M \in \langle S_i(x)\rangle$ implies $\Gamma_i(M) \subset T_i(x)$;*
(4.2) *$S_i^-(z_i)$ is open [resp. closed] for each $z_i \in D_i$.*

Suppose that for each $i \in I$,

(a) *there exists a nonempty subset K of X such that*

$$K \subset \bigcup_{z_i \in N_i} S_i^-(z_i) \text{ for some } N_i \in \langle D_i\rangle;$$

(b) *if $X \neq K$, then there exists a G-convex subspace L_{N_i} of $(X_i \supset D_i; \Gamma_i)$ containing $N_i \in \langle D_i\rangle$ such that, for $L := \prod_{i\in I} L_{N_i}$, we have*

$$L\backslash K \subset \bigcup_{z_i \in M_i} S_i^-(x_i) \text{ for some } M_i \in \langle L_{N_i} \cap D_i\rangle.$$

Then there exists a point $x \in X$ such that $x \in T(x) := \prod_{i\in I} T_i(x)$.

Proof. Recall that $(L_{N_i}, L_{N_i} \cap D_i; \Gamma_i')$ is a G-convex space for each $i \in I$, where $\Gamma_i'(A_i) := \Gamma_i(A) \cap L_{N_i}$ for each $A_i \in \langle L_{N_i} \cap D_i\rangle$. For $L = \prod_{i\in I} L_{N_i} \subset X$, $(L \supset D'; \Gamma')$ is a G-convex space where $D' := \prod_{i\in I}(L_{N_i} \cap D_i)$ and $\Gamma'(A) := \prod_{i\in I} \Gamma_i'(\pi_i(A))$ for each $A \in \langle D'\rangle$; see [1, Lemma 4]. Since

$$L = (L\backslash K) \cup K \subset \bigcup_{z_i \in M_i \cup N_i} S_i^-(z_i)$$

and $M_i \cup N_i \in \langle L_{N_i} \cap D_i\rangle$ for each $i \in I$, $(L_{N_i}, M_i \cup N_i; \Gamma_i'|\langle M_i \cup N_i\rangle)$ is a G-convex space.

Now, for each $i \in I$, define $S_i' : L \multimap M_i \cup N_i$ and $T_i' : L \multimap L_{N_i}$ by

$$S_i'(x) := S_i(x) \cap (M_i \cup N_i) \text{ and } T_i'(x) := T_i(x) \cap L_{N_i} \quad \text{for } x \in L.$$

We show that S_i' and T_i' satisfy the requirements of Theorem 2 as follows:

(2.1) For each $x \in L$, $M \in \langle S_i'(x) \rangle$ implies $\Gamma_i'(M) \subset T_i'(x)$. In fact, $M \in \langle S_i'(x) \rangle$ implies $M \in \langle S_i(x) \rangle$ and $M \subset M_i \cup N_i \subset L_{N_i} \cap D_i$. Then $\Gamma_i'(M) = \Gamma_i(M) \subset T_i(x)$ by (4.1) and $\Gamma_i'(M) = \Gamma_i(M) \subset L_{N_i}$ since L_{N_i} is a G-convex subspace. Therefore, $\Gamma_i'(M) \subset T_i(x) \cap L_{N_i} = T_i'(x)$.

(2.2) $(S_i')^-(z_i)$ is open [resp. closed] for each $z_i \in M_i \cup N_i$. In fact, $(S_i')^-(z_i) = L \cap S_i^-(z)$ is relatively open [resp. closed] in L.

(2.3) $L = \bigcup\{(S_i')^-(z_i) : z_i \in M_i \cup N_i\}$ for each $i \in I$. In fact,

$$L = L \cap \bigcup_{z_i \in M_i \cup N_i} S_i^-(z_i) = \bigcup_{z_i \in M_i \cup N_i} (L \cap S_i^-(z_i)) = \bigcup_{z_i \in M_i \cup N_i} (S_i')^-(z_i).$$

We apply Theorem 2 to $(L, L_{N_i}, M_i \cup N_i, S_i', T_i')$ instead of (X, X_i, D_i, S_i, T_i). Then there exists a point $x \in L$ such that $x \in T'(x) := \prod_{i \in I} T_i'(x)$; that is, $x_i = \pi_i(x) \in T_i'(x) \subset T_i(x)$ for each $i \in I$. This completes our proof. □

From Theorem 4, we have the following particular form of Theorem 4 and the noncompact version of Theorem 3:

Theorem 5. *Let $\{(X_i \supset D_i; \Gamma_i)\}_{i \in I}$ be a family of G-convex spaces, $X := \prod_{i \in I} X_i$, and for each $i \in I$, $T_i : X \multimap X_i$ a Φ-map with the companion map $S_i : X \multimap D_i$. Suppose that for each $i \in I$,*

(a) *there exists a nonempty compact subset K of X;*

(b) *if $X \neq K$, for each $N_i \in \langle D_i \rangle$, there exists a compact G-convex subspace L_{N_i} of $(X_i \supset D_i; \Gamma_i)$ containing N_i such that, for $L := \prod_{i \in I} L_{N_i}$, we have*

$$L \backslash K \subset \bigcup \{\mathrm{Int}_X S_i^-(z_i) : z_i \in L_{N_i} \cap D_i\}.$$

Then there exists a point $x \in X$ such that $x \in T(x) := \prod_{i \in I} T_i(x)$.

Proof. Since T_i is a Φ-map with the companion map S_i for each $i \in I$, we have $X = \bigcup\{\mathrm{Int}\, S_i^-(z_i) : z_i \in D_i\}$. Since K is a compact subset of X, for each $i \in I$, there exists $N_i \in \langle D_i \rangle$ such that $K \subset \bigcup\{\mathrm{Int}\, S_i^-(z_i) : z_i \in N_i\}$.

If $X = K$, then the conclusion follows from Theorem 3.

Suppose $X \neq K$. Then by (b), for each $i \in I$, there exists a compact G-convex subspace L_{N_i} of $(X_i \supset D_i; \Gamma_i)$ containing N_i such that $L = \prod_{i \in I} L_{N_i}$ is compact. Since

$$L = (L \backslash K) \cup (L \cap K) \subset \bigcup_{z_i \in L_{N_i} \cap D_i} \mathrm{Int}\, S_i^-(z_i) \cup \bigcup_{z_i \in N_i} \mathrm{Int}\, S_i^-(z_i)$$

and L is compact, we have

$$L \backslash K \subset \bigcup_{z_i \in M_i} \mathrm{Int}\, S_i^-(z_i) \text{ for some } M_i \in \langle L_{N_i} \cap D_i \rangle.$$

Therefore, the conclusion follows from Theorem 4. □

Remarks. 1. We showed that Theorem 4 implies Theorem 5. Similarly, slightly modifying the proof of Theorem 4, we can easily show that Theorem 3 implies Theorem 5; that is, the noncompact version is a consequence of the corresponding compact version of the collectively fixed point theorem.

2. For a singleton I, Theorem 5 is a particular form of the main result in [9,12].

4. Comments on known results

Recently, in [3], by applying the partition of unity argument and the Tychonoff fixed point theorem, its authors obtained collectively fixed point theorems for a family of multimaps defined on a product of noncompact G-convex spaces. They claimed that their results improve, unify, and generalize a number of known results in at least eight published works; see the references of [3].

Their main result runs as follows:

Theorem 3.1 ([3]). *Let $(X_i, \Gamma_i)_{i\in I}$ be a family of G-convex spaces where I is an (finite or infinite) index set. Let $X = \prod_{i\in I} X_i$ and for each $i \in I$, let $F_i, G_i : X \to 2^{X_i}$ be two set-valued mappings such that for each $i \in I$, the following conditions hold.*

(i) *For each $x \in X$ and $N_i \in \mathcal{F}(F_1(x))$, $\Gamma_i(N_i) \subset G_i(x)$.*
(ii) *For each nonempty compact subset K of X, $K = \bigcup_{y_i\in X_i}(\text{cint } F_i^{-1}(y_i) \cap K)$.*
(iii) *There exists a nonempty subset X_i^0 of X_i such that for each $N_i \in \mathcal{F}(X_i)$, there is a compact G-convex subset L_{N_i} of X_i containing $(X_i^0 \cup N_i)$, and the set $D_i = \bigcap_{y_i\in X_i^0}(\text{cint } F_i^{-1}(y_i))^c$ is empty or compact in X, where $(\text{cint } F_i^{-1}(y_i))^c$ denotes the complement of $\text{cint } F_i^{-1}(y_i)$ in X.*

Then there exists a point $\hat{x} = (\hat{x}_i)_{i\in I}$ such that $\hat{x} \in G_i(\hat{x})$ for each $i \in I$.

We show that Theorem 3.1 [3] follows from our Theorem 5.

Proof. Recall that $\mathcal{F}(A) = \langle A\rangle$ for a set A.

First, by switching the product topology of X to its compactly generated extension, the cint can be replaced by Int.

Second, for each $i \in I$, $T_i := G_i$ is a Φ-map with the companion map $S_i := F_i$ by (i) and (iii); see (3.2) of the proof in [3].

Third, (ii) clearly implies (a), and (iii) implies (b) [In fact, (3.3) of the proof in [3] says that $X = \bigcup\{\text{Int } F_i^{-}(y_i) : y_i \in L_{N_i}\}$].

Therefore, all of the requirements of Theorem 5 with $X_i = D_i$ are satisfied. □

The following comments on [3] would be helpful to the readers working in the KKM theory.

1. In [3], the technique of continuous partition of unity was used without assuming the Hausdorffness of X_i (or of L_N). This can also be found in a number of Ding's earlier works. Moreover, the Tychonoff fixed point theorem can now be replaced by Cauty's recent resolution of the Schauder conjecture; see [11]. Note that our method in this paper is quite different and is based on our version of the Fan–Browder theorem, which is a simple consequence of our KKM theorem; see [4–6,13].

2. For a topological space $(X, \mathcal{T})$, the compactly generated extension (or the k-extension) $\mathcal{T}_k$ of the original topology $\mathcal{T}$ is a new topology of X finer than $\mathcal{T}$ such that $\mathcal{T}_k$ is the collection of all compactly open [resp. compactly closed] subsets of $(X, \mathcal{T})$. Note that the artificial terminology of compact interior, compact closure, etc., are not practical and can be eliminated by switching the original topology of the underlying space to its compactly generated extension. Therefore, Section 2 of [3] is inadequate.

3. In [3], its authors claimed that their Theorem 3.1 [3] generalizes Theorem 3 of [1], which is Theorem 3 with $X_i = D_i$. However, they are actually equivalent as we noted earlier. In many equilibrium problems, non-compact versions of existence statements are simple consequences of corresponding compact versions. Finally, note that the coercivity condition of Theorem 3.1 [3] is far from elegance.

References

[1] S. Park, Continuous selection theorems in generalized convex spaces, Numer. Funct. Anal. Optim. 20 (1999) 567–583.
[2] S. Park, Generalizations of the Nash equilibrium theorem on generalized convex spaces, J. Korean Math. Soc. 38 (2001) 697–709.
[3] X.P. Ding, J.Y. Park, Collectively fixed-point theorems in noncompact G-convex spaces, Appl. Math. Lett. 16 (2003) 137–142.
[4] S. Park, Fixed points of better admissible maps on generalized convex spaces, J. Korean Math. Soc. 37 (2000) 885–899.
[5] S. Park, Remarks on topologies of generalized convex spaces, Nonlinear Funct. Anal. Appl. 5 (2000) 67–79.
[6] S. Park, New topological versions of the Fan–Browder fixed point theorem, Nonlinear Anal. 47 (2001) 595–606.
[7] S. Park, Elements of the KKM theory for generalized convex spaces, Korean J. Comput. Appl. Math. 7 (2000) 1–28.
[8] S. Park, Fixed point theorems in locally G-convex spaces, Nonlinear Anal. 48 (2002) 869–879.
[9] S. Park, H. Kim, Foundations of the KKM theory on generalized convex spaces, J. Math. Anal. Appl. 209 (1997) 551–571.
[10] S. Park, Remarks on a fixed point problem of Ben-El-Mechaiekh, in: Nonlinear Analysis and Convex Analysis, Proc. NACA'98, Niigata, Japan, July 28–31, 1998, World Sci., Singapore, 1999, pp. 79–88.
[11] R. Cauty, Solution du problème de point fixe de Schauder, Fund. Math. 170 (2001) 231–246.
[12] S. Park, H. Kim, Coincidence theorems for admissible multifunctions on generalized convex spaces, J. Math. Anal. Appl. 197 (1996) 173–187.
[13] S. Park, Coincidence, almost fixed point, and minimax theorems on generalized convex spaces, J. Nonlinear Convex Anal. 4 (2003) 151–164.