MULTIMAPS HAVING OPENNESS AND THE BIRKHOFF-KELLOGG THEOREMS

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ABSTRACT. Based on a fixed point theorem for the multimap class $\mathfrak{B}$, we generalize or correct results on homeomorphically convex sets in [P4], on openness of multimaps in [P5], and on the Birkhoff-Kellogg type theorems in [P6].

1. Introduction

Our aim in this paper is to generalize or to correct results in our previous works [P4-6], which were based on an incorrectly stated result (that is, Theorem 1 of [P4,5] and Theorem 0 of [P6]). In this theorem, we had to assume the closedness of the multimap.

In [P4], we claimed a fixed point theorem for multimaps in an admissible class defined on homeomorphically convex sets. In [P5], some known results on the openness of a continuous function $A : X \to E$, where X is a convex subset of a locally convex Hausdorff topological vector space E, are extended to multimaps in an admissible class. Moreover, in [P6], we obtained very general Birkhoff-Kellogg type theorems on eigenvectors of compact multimaps in an admissible class, and some applications to fixed point and best approximation problems.

In this paper, we show that all of the results on compact multimaps in any admissible class $\mathfrak{A}_c^\kappa$ in [P4-6] can be extended to the 'better' admissible class $\mathfrak{B}$.

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2. Preliminaries

A *multimap* or *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^-(y)$ if and only if $y \in T(x)$.

Given two maps $T : X \multimap Y$ and $S : Y \multimap Z$, their *composition* $ST : X \multimap Z$ is defined by $(ST)(x) = S(T(x))$ for $x \in X$.

Bd, Int, and $\overline{}$ denote the boundary, interior, and closure, respectively.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x, y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is a compact subset of Y.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is a closed subset of X. Note that compositions of u.s.c. maps are u.s.c.; the image of a compact set under an u.s.c. map with compact values is compact; and every u.s.c. map T with closed values is closed.

Throughout this paper, t.v.s. means Hausdorff topological vector spaces, and co denotes the convex hull. A *polytope* in a t.v.s. is a compact convex subset of a finite dimensional subspace.

For any topological spaces X and Y and given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite compositions of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Examples of $\mathfrak{A}$ can be seen in [P1-6, PK1,2]. We introduce two more classes:

$F \in \mathfrak{A}_c^\sigma(X,Y) \iff$ for any σ-compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X,Y) \iff$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

Note that $\mathbb{K}_c^\sigma$ due to Lassonde and $\mathbb{V}_c^\sigma$ due to Park et al., the Browder maps (codomains are convex sets, nonempty convex values and open fibers), and locally selectionable maps with convex values are examples of $\mathfrak{A}_c^\sigma$. Examples of $\mathfrak{A}_c^\kappa$ are $\mathfrak{A}_c^\sigma$, approximable maps due to Ben-El-Mechaiekh et al., and others.

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Any class $\mathfrak{A}_c^\kappa$ is called *admissible*. For details, see [P1-2, PK1,2].

In our previous works [P1,2], it is shown that if X is a nonempty convex subset of a locally convex t.v.s., then any compact map in $\mathfrak{A}_c^\sigma(X,X)$ has a fixed point, and furthermore if X is compact, then any map in $\mathfrak{A}_c^\kappa(X,X)$ has a fixed point.

In 1997, the author introduced the 'better' admissible class $\mathfrak{B}$ of multimaps as follows:

> $F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that, for each polytope P in X and for any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point,

where X is a subset of a t.v.s. and Y is a topological space.

Subclasses of $\mathfrak{B}$ are any classes $\mathfrak{A}_c^\kappa$, closed maps in the KKM class due to Chang and Yen and, more generally, in the s-KKM class due to Chang et al., and others.

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E. Examples of admissible sets can be seen in [P3-6] and references therein.

The following is the basis of a number of the author's works:

Theorem 2.1 [P3]. *Let X be an admissible convex subset of a t.v.s. E. Then every compact closed map $T \in \mathfrak{B}(X, X)$ has a fixed point.*

Recall that, in [P4,5, Theorem 1; P6, Theorem 0], we used $T \in \mathfrak{A}_c^\kappa(X, X)$ instead of $T \in \mathfrak{B}(X, X)$ in Theorem 2.1, and we should assume T is closed there.

Now we introduce a generalized version of Theorem 2.1 by switching the admissibility of domain of a compact multimap to the Klee approximability (defined below) of codomain.

Let X be a subset of a t.v.s. E. A compact subset K of X is said to be *Klee approximable in* X if for any neighborhood V of the origin 0 of E, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in X.

Note that a subset X of E is admissible (in the sense of Klee) if and only if every compact subset K of X is Klee approximable in X.

Theorem 2.1 can be generalized as follows:

Theorem 2.2. *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}(X, X)$ a closed multimap. If $\overline{F(X)}$ is a Klee approximable compact subset of X, then F has a fixed point.*

Proof. Let V be a neighborhood of 0. Since $\overline{F(X)}$ is compact and Klee approximable in X, there exist a continuous function $h : \overline{F(X)} \to X$ and a polytope P in X such that $x - h(x) \in V$ for all $x \in \overline{F(X)}$ and $h(\overline{F(X)}) \subset P$. Note that $h : \overline{F(X)} \to P$ and $F|_P : P \multimap \overline{F(X)}$. Since $F \in \mathfrak{B}(X, X)$, $F(P) \subset \overline{F(X)}$ and $h : \overline{F(X)} \to P$, $h(F|_P) : P \multimap P$ has a fixed point $x_V \in hF(x_V)$. Let $x_V = h(y_V)$ for $y_V \in F(x_V) \subset \overline{F(X)}$. We have $y_V - h(y_V) = y_V - x_V \in V$. Since $\overline{F(X)}$ is compact, we may assume that the net y_V converges to some $\hat{x} \in \overline{F(X)} \subset X$. Then the net x_V also converges to $\hat{x}$. Since the graph $\mathrm{Gr}(F)$ of F is closed and $(x_V, y_V) \in \mathrm{Gr}(F)$, we have $(\hat{x}, \hat{x}) \in \mathrm{Gr}(F)$, that is, $\hat{x} \in F(\hat{x})$. This completes our proof.

Note that [P3, Theorem 1] was incorrectly stated and it should be replaced by our new Theorem 2.2.

Recall that, in 2001, Robert Cauty [C] obtained the affirmative resolution of the Schauder conjecture as follows:

Theorem 2.3. [C] *Let E be a t.v.s., C a convex subset of E, and f a continuous function from C into C. If $f(C)$ is contained in a compact subset of C, then f has a fixed point.*

3. Fixed points in homeomorphically convex sets

In this section, we devote to correct the results in [P4].

From Theorem 2.1, for a compact closed map $T \in \mathfrak{A}_c^\kappa(X,X)$, we have the following:

Theorem 3.1. *Let E and F be t.v.s. and X a subset of E which is homeomorphic to an admissible convex subset Δ of F. Then any compact closed map $T \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

Proof. Let $h : \Delta \to X$ be the homeomorphism. Then the composition $h^{-1}Th : \Delta \to \Delta$ belongs to $\mathfrak{A}_c^\kappa(\Delta,\Delta)$. Since T is compact, so is $h^{-1}Th$. Since a compact closed map is u.s.c., $h^{-1}Th$ is u.s.c. and compact-valued, and hence it is closed. Therefore, by Theorem 2.1, there exists a $z_0 \in \Delta$ such that $z_0 \in h^{-1}Th(z_0)$ or equivalently $h(z_0) \in Th(z_0)$. Hence $\hat{x} = h(z_0)$ is a fixed point of T. This completes our proof.

Remark. Theorem 3.1 is a correct form of [P4, Theorem 2]. If $T \in \mathfrak{B}(X,X)$ in Theorem 3.2 and satisfies $Th \in \mathfrak{B}(\Delta, X)$, then the conclusion also holds. Moreover, for a single-valued continuous function $T = f$ or for an approachable map T, the admissibility of Δ can be eliminated in view of the Cauty Theorem 2.3; see [P8].

As an application of the Cauty Theorem 2.3, we obtained in [P7] the following new Fan-Browder type fixed point theorem for compact maps:

Theorem 3.2. *Let E be a t.v.s. and X a convex subset of E. Let $S, T : X \multimap X$ be compact maps such that*

(1) *for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$; and*
(2) $\{\operatorname{Int} S^{-1}(y)\}_{y \in X}$ *covers X.*

Then T has a fixed point.

Remark. In view of Theorem 3.2, [P4, Theorem 3] can be adequately restated.

4. Fixed points of multimaps having the Leray-Schauder condition

From Theorem 2.1, we obtain the following theorem for compact closed maps satisfying the Leray-Schauder condition (LS).

Theorem 4.1. *Let E be a t.v.s. and U a convex neighborhood of the origin 0 of E such that $\overline{U}$ is admissible. Then any compact closed map $F \in \mathfrak{B}(\overline{U}, E)$ satisfying*

(LS) $Fx \cap \{\lambda x : \lambda > 1\} = \emptyset$ *for each $x \in \operatorname{Bd} U$*

has a fixed point.

Proof. Just follow the proof of [P5, Theorem 2] noting that $G = rF \in \mathfrak{B}(\overline{U}, \overline{U})$.

Remark. Note that [P5, Theorem 2] was stated for $F \in \mathfrak{A}_c^\kappa(\overline{U}, E)$, which should be closed. In view of the Cauty Theorem 2.3, for a single-valued continuous function $F = f \in \mathbb{C}(\overline{U}, E)$ or, more generally, for an approachable map F, the admissibility of $\overline{U}$ is redundant; see [P7].

From now on, for a locally convex t.v.s. E, let $S(E)$ be the family of all continuous seminorms of E defining the topology of E.

For $p \in S(E), x_0 \in E$, and $r > 0$, we define

$$B_p(x_0, r) := \{x \in E : p(x - x_0) \leq r\},$$

$$S_p(x_0, r) := \{x \in E : p(x - x_0) = r\}.$$

We say that a map $F : E \multimap E$ is *p-compact* if for a certain closed convex balanced neighborhood U of the origin $0 \in E$ and the gauge p of U, that is, $U = \{x \in E : p(x) \leq 1\}$, the set $\overline{F(nU)}$ is compact for $n = 1, 2, \ldots$, where $nU = \{x \in E : p(x) \leq n\}$. Note that for a single-valued (continuous) map $f : E \to E$, p-compactness is usually called completely continuous.

Theorem 4.2. *Let E be a locally convex t.v.s., $p \in S(E)$, $F \in \mathfrak{B}(E,E)$ a closed p-compact map, $x_0 \in E$ and $r > 0$. If*

(LS)′ $(Fx - x_0) \cap \{\lambda(x - x_0) : \lambda > 1\} = \emptyset$ *for each* $x \in S_p(x_0, r)$,

then F has a fixed point in $B_p(x_0, r)$.

Note that the p-ball $B_p(x_0, r)$ is a convex neighborhood of x_0 and that the boundary of $B_p(x_0, r)$ is precisely the p-sphere $S_p(x_0, r)$. Therefore, Theorem 4.2 clearly follows from Theorem 4.1. Theorem 4.2 improves [P5, Theorem 4] for $F \in \mathfrak{A}_c^\kappa(E, E)$, where F should be closed.

5. Openness of multimaps

In [R] Reichbach considered the problem: Let $A : X \to Y$ be a map of a topological space X into a topological space Y. Under what conditions is $A(X)$ open in Y ?

A number of authors investigated this problem; see [P5]. Especially, Reichbach [R] gave a particular solution of this problem in the case of selfmaps $A : E \to E$ of a Banach space E and some examples, and Nguyen [N] extended and improved Reichbach's result to locally convex t.v.s.

Motivated by recent developments of fixed point theory for multimaps, in [P5], we generalized Nguyen's results in several ways.

From Theorem 4.2, we have the following:

Theorem 5.1. *Let E be a locally convex t.v.s., $p \in S(E)$ and $A : E \multimap E$ a closed map such that, for some $\beta \neq 0$ and $\overline{y} \in E$,*

(i) *$G \in \mathfrak{B}(E,E)$ defined by $G(x) = x - \beta A(x)$ for $x \in E$ is p-compact; and*

(ii) *$F \in \mathfrak{B}(E,E)$ defined by $F(x) = x - \beta(A(x) - \overline{y})$ satisfies* (LS)′.

Then, for each $x_0 \in E$ and $r > 0$, there exists a point $\overline{x} \in B_p(x_0, r)$ such that $\overline{y} \in A(\overline{x})$.

Proof. Note that G and F are closed since so is A. We apply Theorem 4.2 to F. In fact, since G is p-compact, it follows that F is also p-compact. Moreover, F satisfies (LS)$'$. Therefore, there exists an $\overline{x} \in B_p(x_0, r)$ such that $\overline{x} \in F(\overline{x})$. Since $\beta \neq 0$, we have $0 \in \beta(A(\overline{x}) - \overline{y})$ and $\overline{y} \in A(\overline{x})$.

For a map $A \in \mathbb{C}(E, E)$ satisfying $A(S_p(x_0, r)) \subset B_p(x_0, r)$ instead of (LS)$'$, Theorem 5.1 reduces to Nguyen [N, Corollary]. Note that Theorem 5.1 improves [P5, Theorem 4], where A should be closed.

Let X and Y be topological spaces, $A : X \multimap Y$ a map, and $y_0 \in A(X)$. We say that A is *open* at y_0 if there exists a neighborhood V of y_0 in Y such that $V \subset A(X)$.

Theorem 5.2. *Let E be a locally convex t.v.s., $p \in S(E)$, $A : E \multimap E$ a closed map, and $y_0 \in A(E)$. Suppose that there exists an $r_0 > 0$ such that*

(iii) *for each $\overline{y} \in B_p(y_0, r_0)$, there exist an $x_0 \in A^-(\overline{y})$, a $\beta \neq 0$, and an $r > 0$ satisfying* (i) *and* (ii).

Then $B_p(y_0, r_0) \subset A(E)$; that is, A is open at the point $y_0 \in A(E)$.

Proof. From Theorem 5.1, for every $\overline{y} \in B_p(y_0, r_0)$, there exists an $\overline{x} \in B_p(y_0, r_0)$ such that $\overline{y} \in A(\overline{x})$. Therefore, every $\overline{y} \in B_p(y_0, r_0)$ belongs to the range $A(E)$ of A, and hence A is open at the point $y_0 \in A(X)$.

For a map $A \in \mathbb{C}(E, E)$, Theorem 5.2 reduces to [N, Theorem 2]. Note that Theorem 5.2 improves [P5, Theorem 5], where A should be a closed map.

In order to give an illustration of Theorem 5.2, we give the following:

For $G \in \mathfrak{B}(E, E)$, let

$$N_p(x_0, r) = \sup\{p(u - u_0) : u \in G(x),\ u_0 \in G(x_0),\ x \in S_p(x_0, r)\}$$

for $p \in S(E)$, $x_0 \in E$, and $r > 0$. If G is p-compact, then $N_p(x_0, r)$ is bounded.

Theorem 5.3. *Let E be a locally convex t.v.s., $p \in S(E)$, $G \in \mathfrak{B}(E, E)$ a closed p-compact map, and $A : E \multimap E$ given by $A(x) = x - G(x)$ for $x \in E$. If there exist an $x_0 \in E$ and an $r > 0$ such that*

$$(1) \qquad N_p(x_0, r) < r,$$

then A is open at any $y_0 \in Ax_0$.

Proof. Just follow the proof of [P5, Theorem 6].

Note that Theorem 5.3 generalizes [P5, Theorem 6] for $G \in \mathfrak{A}_c^{\kappa}(E, E)$.

6. The Birkhoff-Kellogg type theorems

In 1922, Birkhoff and Kellogg [BK] obtained a result on invariant directions of continuous maps defined on function spaces. Since then there have appeared many generalizations and applications; see [P6].

From now on, we deal with multimaps in the class $\mathfrak{B}$ satisfying the following condition:

($*$) if $F \in \mathfrak{B}(X, E)$, where E is a t.v.s. and $X \subset E$, and if $\lambda > 0$, then $\lambda F \in \mathfrak{B}(X, E)$ where $(\lambda F)(x) := \lambda(F(x)) \subset E$ for $x \in X$.

For a subset X of a vector space E and a multimap $F : X \multimap E$, we say that F has *an eigenvalue* (*a proper value*) if the inclusion

$$\mu x \in Fx$$

has a solution $x_0 \in X$ for some real $\mu \neq 0$, and that F has *an invariant direction* (*a positive eigenvalue*) whenever $\mu > 0$.

From Theorem 2.1, we obtain the following generalized Birkhoff-Kellogg type theorems:

Theorem 6.1. *Let U be a convex neighborhood of 0 in a t.v.s. E such that $\overline{U}$ is admissible, and $F \in \mathfrak{B}(\operatorname{Bd} U, E)$ a compact closed map. Suppose that there is a compact extension $G \in \mathfrak{B}(\overline{U}, E)$ of F such that*

(0) $\lambda G(\overline{U}) \cap \overline{U} = \emptyset$ *for some number λ.*

Then F has at least an eigenvalue.

Proof. Note that $\lambda G \in \mathfrak{B}(\overline{U}, E)$ is compact and has no fixed point. Let $p : E \to \mathbb{R}$ be the Minkowski functional of U. Since $0 \in U$, p is continuous. Define $r : E \to \overline{U}$ by $r(x) = x$ for $x \in \overline{U}$ and $r(x) = p(x)^{-1}x$ for $x \notin \overline{U}$. Then r is a continuous retraction of E onto $\overline{U}$. Let $F' = r(\lambda G) \in \mathfrak{B}(\overline{U}, \overline{U})$. Since λG is closed and compact, so is F'. Therefore, by Theorem 2.1, F' has a fixed point $x_0 \in \overline{U}$; that is, $x_0 \in r(\lambda G)(x_0)$. We have $x_0 = r(y_0)$ for some $y_0 \in (\lambda G)(x_0)$. Note that $y_0 \notin \overline{U}$ by (0). Therefore, $x_0 = r(y_0) = p(y_0)^{-1}y_0 \in \operatorname{Bd} U$ and hence $p(y_0)x_0 = y_0 \in (\lambda G)(x_0) = (\lambda F)(x_0)$. This completes our proof.

Remark. If $\lambda > 0$, then F has an invariant direction. Theorem 6.1 improves [P6, Theorem 1] for $\mathfrak{A}_c^\kappa$, where F and G should be closed.

Theorem 6.2. *Let U, E, and F be the same as in Theorem 6.1. Suppose that*

(1) *there is a retraction $r' : \overline{U} \to \operatorname{Bd} U$ and $Fr' \in \mathfrak{B}(\overline{U}, E)$; and*

(2) $\lambda F(\operatorname{Bd} U) \cap \overline{U} = \emptyset$ *for some number λ.*

Then F has an eigenvalue.

Proof. Let $G := Fr' \in \mathfrak{B}(\overline{U}, E)$. Then G is a compact closed extension of F and (2) implies (0). Applying Theorem 6.1, we have the conclusion.

Note that Theorem 6.2 generalizes [P6, Theorem 2], where F should be closed.

Similarly, in [P6, Theorems 3-9], the compact map F in $\mathfrak{A}_c^\kappa$ should be replaced by a compact closed map F in $\mathfrak{B}$.

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