REMARKS ON FIXED POINT THEOREMS FOR NEW CLASSES OF MULTIMAPS

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Abstract

Since the author established fixed point theorems for the admissible classes $\mathfrak{A}_c^\kappa$ of multimaps, there have appeared another classes: the better admissible class $\mathfrak{B}$, the class satisfying Kim's condition $(*)$, the KKM-class, the S-KKM class, and others. In this paper, we study fixed point theorems for those classes and obtain new results better than known ones. Moreover, some fixed point theorems for new classes of closed compact multimaps defined on convex subsets of topological vector spaces are shown to be disguised forms of our previous result.

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1. Introduction

Recently, there have appeared a number of fixed point theorems for new classes of multimaps defined on convex subsets of topological vector spaces. Apparently motivated by the author's works, some other authors obtained a number of seemingly new results. In this paper, we will discuss on such results and try to give sharpened results.

We will begin with the failure of the author's paper [20], which was discovered and discussed by Huang and Jeng in a recent work [7]. Now we have to assume the closed-valuedness of the map T in [20, Theorems 1 and 2] and to withdraw [20, Proposition]. Instead, in Sections 2 and 3, we give another approaches to the contents of [7] and obtain sharpened results.

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In Section 2, we define the almost fixed point property of a multimap $T : X \multimap X$, where X is a nonempty convex subset of a Hausdorff topological vector space, and obtain a generalization (Theorem 2.3) of the well-known Himmelberg theorem. Moreover, we give a basic fixed point theorem (Theorem 2.4) on multimaps satisfying the almost fixed point property and the condition $(*)$ due to Kim [8], and some consequences of the basic theorem.

Section 3 deals with the so-called s-KKM class of multimaps having the almost fixed point property. We obtain sharpened versions of results of Huang and Jeng [7], whose main aim was to correct and generalize earlier works of Kim [8] and Park [20], and a generalization (Theorem 3.4) of results in [2,3,7].

In Section 4, we are mainly concerned with generalizations of the Himmelberg fixed point theorem [6] for convex-valued upper semicontinuous maps. We show that the theorem was extended to acyclic maps $\mathbb{V}$, admissible class $\mathfrak{A}_c^\kappa$, and the better admissible class $\mathfrak{B}$ by the author for topological vector spaces more general than locally convex ones. Recently, some authors dealt with fixed point theorems for seemingly new and broad classes of multimaps. We clarify that some of them are disguised forms of the author's earlier result, and obtain sharpened formulations of them.

2. The almost fixed point property

A t.v.s. means a Hausdorff topological vector space and a multimap (simply, a map) $T : X \multimap Y$ is a function from X into $2^Y \setminus \{\emptyset\}$. All terminology is standard. The closure operation is denoted by cl or $\overline{}$.

In a recent work [21], from the KKM principle and its open version, we deduced a general almost fixed point theorem [21, Theorem 3]. The following is a simple consequence of the theorem:

Theorem 2.1. *Let X be a convex subset of a t.v.s. E. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(x)$ is convex for each $x \in X$. If there is a precompact subset K of X such that $T(x) \cap K \neq \emptyset$ for each $x \in X$, then for every convex neighborhood U of the origin 0 of E, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

Ky Fan [5, Theorem 7] obtained Theorem 2.1 for a locally convex t.v.s. E and for a lower semicontinuous multimap $T : X \multimap E$. For a single-valued map $f : X \to X$, Fan noted that Theorem 2.1 might be regarded as a generalization of the Tychonoff fixed point theorem to noncompact (or precompact) convex sets; see Theorem 2.2 below.

Lassonde [9, Théorème 4] obtained Theorem 2.1 for a compact upper semicontinuous map $T : X \multimap X$ having convex values.

Moreover, in [21], we deduced the following from Theorem 2.1:

Theorem 2.2 (Himmelberg [6]). *Let X be a convex subset of a locally convex t.v.s. E and $T : X \multimap X$ a compact upper semicontinuous multimap with closed convex values. Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

Let X be a convex subset of a t.v.s. E. A multimap $T : X \multimap X$ is said to have the *almost fixed point property* if, for every convex neighborhood U of the origin 0 of E, there exists a point $x_U \in X$ such that $x_U \in T(x_U) + U$ or $T(x_U) \cap (x_U + U) \neq \emptyset$.

The following generalizes the Himmelberg theorem:

Theorem 2.3. *Let X be a convex subset of a locally convex t.v.s. E. Then any closed compact multimap $T : X \multimap X$ having the almost fixed property has a fixed point.*

Proof. Since E is locally convex and T has the almost fixed point property, for each neighborhood U of 0, there exists $x_U, y_U \in X$ such that $y_U \in T(x_U)$ and $y_U \in x_U + U$. Since $T(X)$ is relatively compact, we can choose a subnet of $\{y_U\}$ with a cluster point $x_0 \in \overline{T(X)}$. Since E is Hausdorff, the corresponding subnet of $\{x_U\}$ also converges to x_0. Since the graph of T is closed in $X \times \overline{T(X)}$, we have $x_0 \in T(x_0)$. This completes our proof.

Remark. We give an example of Theorem 2.3 for which Theorem 2.2 does not work: Let $X = [0, \infty) \subset \mathbb{R}$ and $T(x) := \{0\}$ for $x \in [0, 1)$, $T(x) := \{0, 1\}$ for $x = 1$, and $T(x) := \{1\}$ for $x \in (1, \infty)$. Then T is closed and has almost fixed point property, but $T(1)$ is not convex.

In order to obtain another fixed point theorems for multimaps having the almost fixed point property, we need the following:

Lemma 2.4. *Let X be a convex subset of a t.v.s. E, $\mathcal{V}$ a local base of open neighborhoods of 0 in E, and $T : X \multimap X$ a multimap. Then the following are equivalent:*

- $(*)$ *If $y \in X$ satisfies $y \notin T(y)+U$ for some $U \in \mathcal{V}$, then $y \notin \operatorname{cl}\{x \in X : x \in T(x)+\operatorname{co} V\}$ for some $V \in \mathcal{V}$.*
- $(**)$ $\bigcap_{U\in\mathcal{V}}\{x \in X : x \in T(x) + U\} = \bigcap_{U\in\mathcal{V}} \operatorname{cl}\{x \in X : x \in T(x) + \operatorname{co} U\}$.

Proof. $(*) \Longrightarrow (**)$. For any $U \in \mathcal{V}$, let

$$F_U := \{x \in X : x \in T(x) + \operatorname{co} U\}, \quad F'_U := \{x \in X : x \in T(x) + U\}.$$

Then $(**)$ becomes the following:

$$\bigcap_{U\in\mathcal{V}} F'_U = \bigcap_{U\in\mathcal{V}} \operatorname{cl} F_U.$$

It suffices to show the left hand side includes the right hand one. Suppose that $y \in \bigcap_{U\in\mathcal{V}} \operatorname{cl} F_U$ satisfies $y \notin \bigcap_{U\in\mathcal{V}} F'_U$; that is, $y \notin T(y) + U$ for some $U \in \mathcal{V}$. Then by condition $(*)$, there exists a $V \in \mathcal{V}$ satisfying $y \notin \operatorname{cl} F_V$. This is a contradiction.

$(**) \Longrightarrow (*)$ Clear.

The conditions $(*)$ and $(**)$ are due to Kim [8], and multimaps satisfying $(*)$ are studied in [7,8,20].

For the multimap class satisfyng $(*)$ or $(**)$ we have the following fixed point theorem:

Theorem 2.5. *Let X be a compact convex subset of a t.v.s. E and $T : X \multimap X$ a multimap such that*

(i) *T has the almost fixed point property;*
(ii) *T has closed values; and*
(iii) *T satisfies condition $(**)$.*

Then T has a fixed point.

Proof. We use the notations in Lemma 2.4 and its proof. For any $U \in \mathcal{V}$, by (i), there is an $x_U \in X$ such that $x_U \in T(x_U) + \operatorname{co} U$. Then $F_U \neq \emptyset$ and $F'_U \subset F_U$ for each $U \in \mathcal{V}$. It is clear that $\{F_U : U \in \mathcal{V}\}$ has the finite intersection property. Since each $\operatorname{cl} F_U$ is a closed subset of the compact space X, we have $\bigcap_{U \in \mathcal{V}} \operatorname{cl} F_U \neq \emptyset$. Therefore, by (iii), there exists an $\widehat{x} \in X$ such that

$$\widehat{x} \in \bigcap_{U \in \mathcal{V}} F'_U = \bigcap_{U \in \mathcal{V}} \operatorname{cl} F_U \neq \emptyset,$$

and hence, we have

$$\widehat{x} \in \bigcap_{U \in \mathcal{V}} (T(\widehat{x}) + U) = \operatorname{cl} T(\widehat{x}) = T(\widehat{x})$$

by (ii). This completes our proof.

Combining Theorems 2.1 and 2.5, we obtain the following:

Theorem 2.6. *Let X be a compact convex subset of a t.v.s. E and $T : X \multimap X$ a lower [resp. upper] semicontinuous multimap having closed convex values. If T satisfies condition $(**)$, then T has a fixed point.*

Note that Theorem 2.6 for lower semicontinuous maps is a correct form of Kim [8, Theorem 1], Park [20, Theorem 2] and that Theorem 2.6 for upper semicontinuous maps is due to Huang and Jeng [7, Corollary 2.6] with different proof.

Now we give more maps satisfying condition $(**)$:

Lemma 2.7. *Let X be a convex subset of a locally convex t.v.s. Then any closed compact multimap $T : X \multimap X$ satisfies condition $(**)$.*

Proof. Let $\mathcal{V}$ be a local base of convex open neighborhoods of 0 in E. for each $U \in \mathcal{V}$, we have a $V \in \mathcal{V}$ satisfying $V \subset \overline{V} \subset U$. Then the set $F_{\overline{V}} := \{x \in X : x \in T(x) + \overline{V}\}$ is closed. In fact, for any $x \in \operatorname{cl} F_{\overline{V}}$, choose a net $\{x_\alpha\} \in F_{\overline{V}}$ such that $x_\alpha \to x$. For each $x_\alpha \in F_{\overline{V}}$, we have $y_\alpha \in T(x_\alpha)$ such that $x_\alpha - y_\alpha \in \overline{V}$. Since $y_\alpha \in T(X)$ and $\overline{T(X)}$ is compact in X, we may assume that $y_\alpha \to y$ for some $y \in \overline{T(X)} \subset X$. As $\overline{V}$ is closed, we have $x - y \in \overline{V}$, and as T is closed, we have $y \in T(x)$ and hence $x \in (y + \overline{V}) \cap X \subset (T(x) + \overline{V}) \cap X$. Therefore, $x \in F_{\overline{V}}$ and $F_{\overline{V}}$ is closed. Now we have

$$\bigcap_{U \in \mathcal{V}} F_U = \bigcap_{V \in \mathcal{V}} F_{\overline{V}} = \bigcap_{V \in \mathcal{V}} \operatorname{cl} F_{\overline{V}} = \bigcap_{U \in \mathcal{V}} \operatorname{cl} F_U$$

and hence condition $(**)$ holds.

If X itself is compact, Lemma 2.7 reduces to Huang and Jeng [7, Proposition 2.3], whose proof is modified here for completeness.

From Lemma 2.7 and Theorem 2.5, we have the following:

Corollary 2.8. *Let X be a compact convex subset of a locally convex t.v.s. Then any closed multimap $T : X \multimap X$ having the almost fixed point property has a fixed point.*

Corollary 2.8 is a simple consequence of Theorem 2.3 and generalizes the well-known Fan–Glicksberg theorem; see [12,19].

3. The generalized KKM property

Let $\langle X \rangle$ be the set of all nonempty finite subsets of a set X.

Let X be a convex subset of a linear space and Y a topological space. In 1996, Chang and Yen [3] defined

$T \in \mathrm{KKM}(X, Y) \iff T : X \multimap Y$ is a map such that the family $\{S(x) : x \in X\}$ has the finite intersection property whenever $S : X \multimap X$ has closed values and $T(\mathrm{co}\, N) \subset S(N)$ for each $N \in \langle X \rangle$.

Moreover, Chang and Yen [4] introduced the class of S-KKM maps and gave a characterization of such maps and an s-KKM theorem. This was extended to S-KKM maps by Lin and Chang [10] with additional results. This is followed by Chang, Huang, Jeng, and Kuo [2].

Let X be a nonempty set, Y a nonempty convex subset of a linear space and Z a topological space. If $S : X \multimap Y$, $T : Y \multimap Z$ and $F : X \multimap Z$ are three multimaps satisfying

$$T(\mathrm{co}\, S(A)) \subset F(A)$$

for any $A \in \langle X \rangle$, then F is called a generalized S-KKM map with respect to T. If the multimap $T : Y \multimap Z$ satisfies that for any generalized S-KKM map F with respect to T the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. The class S-KKM(X, Y, Z) is defined to be the set $\{T : Y \multimap Z \,|\, T$ has the S-KKM property$\}$; see [4,10].

As shown in [2], when $X = Y$ and S is the identity map 1_X, then S-KKM(X, Y, Z) reduces to the class KKM(X, Z) introduced by Chang and Yen in [3], and moreover, KKM(Y, Z) is contained in S-KKM(X, Y, Z) for any $S : X \multimap Y$ and generally this inclusion is proper.

If S is a single-valued map $s : X \to Y$, then we can consider the class s-KKM(X, Y, Z).

Lemma 3.1. *Let I be a nonempty set, X a convex subset of a t.v.s. E (not necessarily Hausdorff), $s : I \to X$, $T \in s$-KKM(I, X, X) a multimap, and U a convex open subset of E. If*

$$\overline{T(X)} \subset \bigcup_{z \in A} (s(z) + U) \quad \textit{for some} \quad A \in \langle I \rangle, \tag{1}$$

then there exists an $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Proof. Let $F : I \multimap X$ be a map defined by

$$F(z) := \overline{T(X)}\backslash(s(z) + U) \quad \text{for } z \in I.$$

Since $\overline{T(X)} \subset \bigcup_{z\in A}(s(z) + U)$,

$$\bigcap_{z\in A} F(z) = \overline{T(X)}\backslash \bigcup_{z\in A}(s(z) + U) \subset \overline{T(X)}\backslash\overline{T(X)} = \emptyset,$$

and hence $\{F(z)\}_{z\in I}$ does not have the finite intersection property. Since F has closed values and $T \in s\text{-KKM}(I, X, X)$, there exists a $B \in \langle I\rangle$ such that

$$T(\operatorname{co} s(B)) \not\subset F(B).$$

Hence there exists $y_0 \in T(\operatorname{co} s(B)) \subset T(X)$ such that

$$y_0 \notin F(z) = \overline{T(X)}\backslash(s(z) + U) \quad \text{for all } z \in B.$$

Therefore, $y_0 \in s(z) + U$ or $s(z) \in y_0 - U$ for all $z \in B$. Since $s(B) \subset y_0 - U$ and $y_0 - U$ is convex, we have $\operatorname{co} s(B) \subset y_0 - U$. On the other hand, $y_0 \in T(\operatorname{co} s(B))$ implies $y_0 \in T(x_U)$ for some $x_U \in \operatorname{co} s(B) \subset y_0 - U$. Hence, $y_0 \in x_U + U$ and $T(x_U) \cap (x_U + U) \neq \emptyset$. This completes our proof.

Corollary 3.2. *Let $I, X, E, s,$ and T be the same as in Lemma 3.1. If T is compact and $T(X) \cap s(I)$ is dense in $T(X)$, then T has the almost fixed point property.*

Proof. For any convex open neighborhood U of 0 in E, if $T(X) \cap s(I)$ is dense in $T(X)$ and $\overline{T(X)}$ is compact, then condition (1) holds.

Note that if $X = I$ and $\overline{T(X)} \subset s(I)$, then Corollary 3.2 reduces to Chang et al. [2, Lemma 3.1].

From Corollary 3.2 and Theorem 2.5, we immediately have the following:

Theorem 3.3. *Let X be a compact convex subset of a t.v.s., I a nonempty set, $s : I \to X$, and $T \in s$-KKM(I, X, X) a closed-valued map such that $T(X) \cap s(I)$ is dense in $T(X)$. If T satisfies condition $(**)$, then T has a fixed point.*

Note that if $X = I$ and $\overline{T(X)} \subset s(I)$, then Theorem 3.3 reduces to Huang and Jeng [7, Theorem 2.2], and if $X = I$ and $s = 1_X$, then Theorem 3.3 originates from Park [20, Theorem 1]. If $T : X \multimap X$ is upper semicontinuous with convex values, then $T \in \text{KKM}(X, X) \subset s\text{-KKM}(X, X, X)$; see Huang and Jeng [7]. Therefore, Theorem 3.3 generalizes Theorem 2.6 for upper semicontinuous maps.

From Theorem 2.3 and Corollary 3.2, we immediately have the following:

Theorem 3.4. *Let X be a convex subset of a locally convex t.v.s. I a nonempty set, $s : I \to X$, and $T \in s$-KKM(I, X, X) a closed compact map such that $T(X) \cap s(I)$ is dense in $T(X)$. Then T has a fixed point.*

If $X = I$ and $\overline{T(X)} \subset s(I)$, then Theorem 3.4 reduces to Chang et al. [2, Theorem 3.2] and further if $s = 1_X$, then to Chang and Yen [3, Theorem 2], and if X itself is compact, then to Huang and Jeng [7, Corollary 2.4].

4. Fixed points for new classes of maps

Let us recall some history of analytical fixed point theory; for details, see [19].

In 1991, the author presented the following at the Halifax conference:

Theorem 4.1 [12]. *Let X be a convex subset of a locally convex t.v.s. E. Then every compact map $T \in \mathbb{V}(X,X)$ has a fixed point, where*

$T \in \mathbb{V}(X,X) \iff T : X \multimap X$ is an acyclic map; that is, an upper semicontinuous map having compact acyclic values.

Theorem 4.1 was extended to the class $\mathfrak{A}_c$ [13], to $\mathbb{V}_c^\sigma$ [23], to $\mathfrak{A}_c^\sigma$ [14], and so on.

Theorem 4.2 [14]. *Let X be a convex subset of a locally convex t.v.s. E, and $F \in \mathfrak{A}_c^\sigma(X,X)$. If F is compact, then F has a fixed point.*

Theorem 4.3 [15]. *Let X be a compact convex subset of a t.v.s. E on which E^* separates points. Then any $F \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

Note that we supplied a lot of examples of $\mathfrak{A}_c^\kappa(X,X)$ in [13,14,22] and that many authors misunderstood $\mathfrak{A}$ as $\mathcal{U}$.

In 1996, Chang and Yen [3] introduced the KKM class (see Section 3) and obtained the following:

Theorem 4.4 [3]. *Let X be a convex subset of a locally convex t.v.s. E. Then every closed compact map $T \in KKM(X,X)$ has a fixed point.*

It was known that $\mathfrak{A}_c^\kappa(X,X) \subset \mathrm{KKM}(X,X)$ in [14], but, any single example of closed compact maps $T \in \mathrm{KKM}(X,X)$ such that $T \notin \mathfrak{A}_c^\kappa(X,X)$ is not given in [3].

Note that our Theorem 2.3 generalizes Theorem 4.4 properly. In fact, the map $T \in \mathrm{KKM}(X,X)$ in Theorem 4.4 has the almost fixed point property by Corollary 3.2, but not conversely. For example, the map T in Remark after Theorem 2.3 has the almost fixed point property, but does not belong to $\mathrm{KKM}(X,X)$; cf. [3, p.226].

In 1996, the author defined

$F \in \mathfrak{B}(X,X) \iff F : X \multimap X$ is a map such that, for each polytope (that is, the convex hull of a nonempty finite subset) in X and for any continuous map $f : F(P) \to P$, the composition $f(F|_P)$ has a fixed point.

and noticed that $\mathfrak{A}_c^\kappa \subset \mathfrak{B}$ and that, in the class of closed compact maps, two subclasses $\mathfrak{B}$ and KKM coincide. At the WCNA'96, general versions of Theorem 4.4 were presented [16]. One of the simplest results is the following restatement of Theorem 4.4:

Theorem 4.5 [16]. *Let X be a convex subset of a locally convex t.v.s. E. Then every closed compact map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E. Examples of admissible subsets can be seen in [18,19].

In 1997, at the KKM theory conference at Changhua, the author presented the following with its long history:

Theorem 4.6 [18,19]. *Let E be a t.v.s. and X an admissible* (*in the sense of Klee*) *convex subset of E. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

In 1998, this was restated as follows:

Theorem 4.7 [20]. *Let X be an admissible convex subset of a t.v.s. Then any closed compact map $T \in \mathfrak{K}(X, X)$ has a fixed point, where $\mathfrak{K}$ denotes KKM.*

Moreover, the manuscript of the Changhua talk appeared as [19], where we listed more than sixty papers in chronological order, from which we could deduce particular forms of Theorem 4.6. Further examples of maps in the class $\mathfrak{B}$ were given in [17]. In 1999, Lin and Yu [11] published Theorem 4.7 and generalized versions of the present author's other results.

Since then generalizations of KKM classes to S-KKM or s-KKM classes follow. One of the main targets of such works was to try to generalize Theorems 4.4 and 4.7. However, the main results in [2,3,7] are generalized and unified by Theorem 3.4 in this paper.

For the s-KKM class, we need the following:

Lemma 4.8. *Let X be a convex subset of a t.v.s., I a nonempty set, $s : I \to X$ a surjection, and $T \in s$-KKM(I, X, X). If T is closed and compact, then $T \in \mathfrak{B}(X, X)$.*

Proof. Let P be a polytope in X, $T' := T|_P$, and $f : T(P) \to P$ a continuous map. Let $I' := s^{-1}(P) \subset I$ and $s' := s|_{I'} : I' \to P$.

Claim 1. $T \in s'$-KKM(I', X, X).

Let $F' : I' \multimap X$ be closed-valued such that

$$T(\operatorname{co} s'(A)) \subset F'(A) \quad \text{for all } A \in \langle I' \rangle. \tag{2}$$

Define $F : I \multimap X$ by $F(z) := F'(z)$ for $z \in I'$ and $F(z) := X$ for $z \in I \backslash I'$. Then F is closed-valued with respect to the relative topology of X. Now, from (2), we have

$$T(\operatorname{co} s(A)) \subset F(A) \quad \text{for all } A \in \langle I \rangle.$$

Since $T \in s$-KKM(I, X, X), $\{F(z)\}_{z \in I}$ has the finite intersection property, and hence, so does $\{F'(z)\}_{z \in I'}$.

Claim 2. $f \circ T' \in s'$-KKM(I', P, P).

Let $F' : I' \multimap P$ be closed-valued such that

$$(f \circ T')(\operatorname{co} s'(A)) \subset F'(A) \quad \text{for all } A \in \langle I' \rangle.$$

Then we have

$$T(\operatorname{co} s'(A)) = T'(\operatorname{co} s'(A)) \subset (f^{-1} \circ F')(A) \quad \text{for all } A \in \langle I' \rangle.$$

Since $f^{-1} \circ F'$ is closed-valued and $T \in s'$-KKM(I', X, X), $\{(f^{-1} \circ F')(z)\}_{z \in I'}$ has the finite intersection property, and hence, so does $\{F'(z)\}_{z \in I'}$.

Claim 3. $f \circ T' : P \multimap P$ has a fixed point.

Since T is closed and compact, it is upper semicontinuous with closed values, and hence, so is T'. Moreover, $f \circ T'$ is upper semicontinuous and closed-valued, and hence, it is closed and compact. Since $f \circ T' \in s'$-KKM(I', P, P) by Claim 2 and $(f \circ T')(P) \subset P = s(I')$, by Theorem 3.4, $f \circ T'$ has a fixed point.

Therefore $f \circ (T|_P)$ has a fixed point and hence $T \in \mathfrak{B}(X, X)$. This completes our proof.

Remark. If $s : I \to X$ is not a surjection, then Lemma 4.9 does not hold. For example, let $X = [0, 1]$ and $s : X \to X$ be defined by $s(x) := x/2$ for $x \in X$. Let $T : X \multimap X$ be defined by $T(x) := \{1\}$ if $x \in [0, 1/2)$, $T(x) := \{0, 1\}$ if $x = 1/2$, and $T(x) := \{0\}$ if $x \in (1/2, 1]$. Then $T \in s$-KKM(X, X, X) by noting $1 \in T(s(x))$ for any $x \in X$; see [2, p.219]. Note that T is closed and compact, but $T \notin \mathfrak{B}(X, X)$.

From Lemma 4.8 and Theorem 4.6, we have the following:

Theorem 4.9. *Let E be a t.v.s. and X an admissible* (*in the sense of Klee*) *convex subset of E, I a nonempty set, $s : I \to X$ a surjection, and $T \in s$-KKM(I, X, X). If T is closed and compact, then T has a fixed point.*

Note that if I is a nonempty subset of X, then Theorem 4.9 reduces to Chang, Huang, and Jeng [1, Theorem 3.1].

It should be noticed that the main fixed point theorems in [1,3,11], and others are disguised forms of our Theorem 4.6. Most of other results in those papers are also formally generalized (but not practical) or disguised forms of earlier works of the author on the classes $\mathfrak{A}_c^\kappa$ or $\mathfrak{B}$ of multimaps.

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