NEW VERSIONS OF THE FAN-BROWDER FIXED POINT THEOREM AND EXISTENCE OF ECONOMIC EQUILIBRIA

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We introduce a generalized form of the Fan-Browder fixed point theorem and apply it to game-theoretic and economic equilibrium existence problem under the more generous restrictions. Consequently, we state some of recent results of Urai (2000) in more general and efficient forms.

1. Introduction

In 1961, using his own generalization of the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem, Fan [2] established an elementary but very basic "geometric" lemma for multimaps and gave several applications. In 1968, Browder [1] obtained a fixed point theorem which is the more convenient form of Fan's lemma. With this result alone, Browder carried through a complete treatment of a wide range of coincidence and fixed point theory, minimax theory, variational inequalities, monotone operators, and game theory. Since then, this result is known as the Fan-Browder fixed point theorem, and there have appeared numerous generalizations and new applications. For the literature, see Park [7, 8, 9].

Recently, Urai [12] reexamined fixed point theorems for set-valued maps from a unified viewpoint on local directions of the values of a map on a subset of a topological vector space to itself. Some basic fixed point theorems were generalized by Urai so that they could be applied to game-theoretic and economic equilibrium existence problem under some generous restrictions.

However, in view of the recent development of the KKM theory, we found that some (not all) of Urai's results can be stated in a more general and efficient way. In fact, compact convex subsets of Hausdorff topological vector spaces that appeared in some of Urai's results can be replaced by mere convex spaces with finite open (closed) covers. Moreover, Urai's main tools are the partition of unity argument on such covers, where the Hausdorff compactness is essential, and the Brouwer fixed point theorem.

In the present paper, we introduce a generalized form of the Fan-Browder fixed point theorem, which is the main tool of our work. Using this theorem instead of Urai's tools,

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we show that a number of Urai's results [12] (e.g., Theorem 1 for the case (K*), Theorem 2 for the case (NK*), Theorem 3 for the case (K*), Theorem 19, and their Corollaries) can be stated in more generalized and efficient forms.

2. Preliminaries

A *multimap* (or simply, *a map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of the set Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) = \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$.

For a set D, let $\langle D\rangle$ denote the set of nonempty finite subsets of D.

Let X be a subset of a vector space and D a nonempty subset of X. We call (X,D) a *convex space* if $\operatorname{co} D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D\rangle$; see Lassonde [5] and Park [7]. If $X = D$ is convex, then $X = (X,X)$ becomes a convex space in the sense of Lassonde [4].

The following version of the KKM theorem for convex spaces is known.

Theorem 2.1. *Let (X,D) be a convex space and $F : D \multimap X$ a multimap such that*

(1) *$F(z)$ is open (resp., closed) for each $z \in D$;*
(2) *F is a KKM map (i.e., $\operatorname{co} N \subset F(N)$ for each $N \in \langle D\rangle$).*

Then $\{F(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for any $N \in \langle D\rangle$, $\operatorname{co} N \cap [\bigcap_{z\in N} F(z)] \neq \varnothing$.)

The closed version is due to Fan [2] and the open version is motivated from the works of Kim [3] and Shih and Tan [10], who showed that the original KKM theorem holds for open-valued KKM maps on a simplex. Later, Lassonde [5] showed that the closed and open versions of Theorem 2.1 can be derived from each other. More general versions of Theorem 2.1 were recently known; for example, see Park [8, 9].

From Theorem 2.1, we deduce the following result.

Theorem 2.2. *Let (X,D) be a convex space and $P : X \multimap D$ a multimap. If there exist $z_1,z_2,\dots,z_n \in D$ and nonempty open (resp., closed) subsets $G_i \subset P^-(z_i)$ for each $i = 1,2,\dots,n$ such that $\operatorname{co}\{z_1,z_2,\dots,z_n\} \subset \bigcup_{i=1}^n G_i$, then the map $\operatorname{co} P : X \multimap X$ has a fixed point $x_0 \in X$ (i.e., $x_0 \in \operatorname{co} P(x_0)$).*

Proof. Let $Y := \operatorname{co}\{z_1,z_2,\dots,z_n\}$ and $D' := \{z_1,z_2,\dots,z_n\} \subset D$ and consider the convex space (Y,D'). Define a map $F : D' \multimap Y$ by $F(z_i) := Y\setminus G_i$ for each $z_i \in D'$. Then each $F(z_i)$ is closed (resp., open) in Y, and

$$\bigcap_{i=1}^{n} F(z_i) = Y\setminus \bigcup_{i=1}^{n} G_i = Y\setminus Y = \varnothing. \tag{2.1}$$

Therefore, the family $\{F(z)\}_{z\in D'}$ does not have the finite intersection property, and hence, F is not a KKM map by Theorem 2.1. Thus, there exists an $N \in \langle D'\rangle$ such that $\operatorname{co} N \not\subset F(N) = \bigcup\{Y\setminus G_i \mid z_i \in N\}$. Hence, there exists an $x_0 \in \operatorname{co} N$ such that $x_0 \in G_i \subset P^-(z_i)$

for each $z_i \in N$; that is, $N \subset P(x_0)$. Therefore, $x_0 \in \operatorname{co} N \subset \operatorname{co} P(x_0)$. This completes our proof. □

Note that Theorem 2.2 is actually equivalent to Theorem 2.1.

Proof of Theorem 2.1 using Theorem 2.2. Suppose that there exists $M \in \langle D \rangle$ such that $\bigcap_{z\in M} F(z) = \emptyset$ under the hypothesis of Theorem 2.1. Let $M := \{z_1, z_2, \ldots, z_n\}$ and define $P : X \multimap D$ by $P^-(z) := X \backslash F(z)$ for $z \in D$. Then for each i, $1 \le i \le n$, the set $G_i := P^-(z_i) = X \backslash F(z_i)$ is closed (resp., open). Moreover, $\operatorname{co} M \subset X = X \backslash \bigcap_{z\in M} F(z) = \bigcup_{z\in M}(X \backslash F(z)) = \bigcup_{i=1}^n G_i$. Therefore, by Theorem 2.2, there exists an $x_0 \in X$ such that $x_0 \in \operatorname{co} P(x_0)$. Hence, there exists $N := \{y_1, y_2, \ldots, y_m\} \subset P(x_0)$ such that $x_0 \in \operatorname{co} N$. Since $y_j \in P(x_0)$ for all j, $1 \le j \le m$, we have $x_0 \in P^-(y_j) = X \backslash F(y_j)$ on $x_0 \notin F(y_j)$. So $x_0 \notin F(N)$ and we have $x_0 \in \operatorname{co} N \not\subset F(N)$. Then F can not be a KKM map, a contradiction. □

In our previous work (Sy and Park [11]), Theorem 2.2 is applied to obtain several forms of the Fan-Browder fixed point theorem, other (approximate) fixed point theorems, and so on.

In fact, from Theorem 2.2, we can easily deduce the following Fan-Browder fixed point theorem.

Corollary 2.3 (Browder [1, Theorem 1]). *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E and let ϕ be a nonempty convex-valued multimap on X to X. If for all $y \in X$, $\phi^-(y)$ is open in X, then ϕ has a fixed point.*

Proof. Put $X = D$ and $\operatorname{co} P = P = \phi$. Since $\{\phi^-(y)\}_{y\in X}$ covers the compact set X, there exists $z_1, z_2, \ldots, z_n \in X$ such that $\bigcup_{i=1}^n \phi^-(z_i) = X \supset \operatorname{co}\{z_1, z_2, \ldots, z_n\}$. Therefore, by putting $G_i = \phi^-(z_i) = P^-(z_i)$ in Theorem 2.2, we have the conclusion. □

Remark 2.4. Browder obtained his theorem by adopting the partition of unity argument subordinated to a finite open cover of the Hausdorff compact subset X and applying the Brouwer fixed point theorem. In our method using the KKM theorem, Hausdorffness is removed and the compactness is replaced by a finite open (resp., closed) cover.

From now on, we consider mainly the case $X = D$ for simplicity. The following is a basis of some results of Urai [12].

Theorem 2.5. *Let X be a convex space, $T : X \multimap X$ a map with convex values, and $K_T := \{x \in X \mid x \notin T(x)\}$. If there exist $z_1, z_2, \ldots, z_n \in X$ and nonempty open (resp., closed) subsets $G_i \subset T^-(z_i)$ for each $i = 1, 2, \ldots, n$ such that $K_T \subset \bigcup_{i=1}^n G_i$, then T has a fixed point.*

Proof. Suppose that T has no fixed point, that is, $X = K_T$. Then, by Theorem 2.2, T has a fixed point, a contradiction. □

3. Fixed point theorems of the Urai type

In this section, we derive some of Urai's results from Theorem 2.5.

Theorem 3.1. *Let X be a convex space, $\Phi : X \multimap X$ a map with convex values, and $K_\Phi := \{x \in X \mid x \notin \Phi(x)\}$. Suppose that*

(I) *for each $x \in K_\Phi$, there exists an open (resp., a closed) subset $U(x)$ of X containing x and a point $y^x \in X$ such that*

$$z \in U(x) \cap K_\Phi \Longrightarrow y^x \in \Phi(z). \tag{3.1}$$

If K_Φ is covered by finitely many $U(x)$'s, then Φ has a fixed point.

Proof. Suppose that $X = K_\Phi$. Then for any $x \in K_\Phi$, by (I), we have $y^x \in \Phi(z)$ or $z \in \Phi^-(y^x)$ for all $z \in U(x)$, that is, $U(x) \subset \Phi^-(y^x)$. We may assume that $X = K_\Phi = \bigcup_{i=1}^n U(x_i)$ for some $\{x_1, x_2, \ldots, x_n\} \subset K_\Phi$. Note that $U(x_i) \subset \Phi^-(y^{x_i})$ for all $i = 1, 2, \ldots, n$. Put $G_i := U(x_i)$ and $z_i := y^{x_i} \in X$. Then, by Theorem 2.5, Φ has a fixed point, which contradicts $X = K_\Phi$. Hence, $K_\Phi \subsetneq X$ and Φ has a fixed point. □

Corollary 3.2. *Let X be a convex space, $\phi : X \multimap X$ a map with nonempty values, and $K_\phi := \{x \in X \mid x \notin \phi(x)\}$. Suppose that*

(K^*) *there is a map $\Phi : X \multimap X$ with convex values such that for each $x \in K_\phi$, there exist an open (resp., a closed) subset $U(x)$ of X containing x and a point $y^x \in X$ such that*

$$z \in U(x) \cap K_\phi \Longrightarrow z \notin \Phi(z), \quad y^x \in \Phi(z). \tag{3.2}$$

If K_ϕ is covered by finitely many $U(x)$'s, then ϕ has a fixed point.

Proof. Suppose that $X = K_\phi$. Then Φ satisfies the requirement of (I) of Theorem 3.1, and hence, Φ has a fixed point $x_0 \in X$. On the other hand, by (K^*), for each $x \in X = K_\phi$, we should have $x \notin \Phi(x)$. This contradiction leads to $X \not\supset K_\phi$. Therefore, we have the conclusion. □

Remarks 3.3. (1) In case $\Phi = \phi$, Corollary 3.2 reduces to Theorem 3.1.

(2) Urai (see [12, Theorem 1 for the case (K^*)]) obtained Corollary 3.2 under the restriction that

(i) X is a compact convex subset of a Hausdorff topological vector space,
(ii) $U(x)$ is open,
(iii) for each $z \in U(x)$ as in (K^*),

$$z \in K_\phi \Longrightarrow \phi(z) \subset \Phi(z), \quad z \notin \Phi(z),\ y^x \in \Phi(z). \tag{3.3}$$

Corollary 3.4. *Let X be a convex subset of a real Hausdorff topological vector space E, E^* the algebraic dual of E, and $\phi : X \multimap X$ a map with nonempty values. Suppose that*

(K_1^*) *for each $x \in K_\phi := \{x \in X \mid x \notin \phi(x)\}$, there exists a vector $p^x \in E^*$ such that $\phi(x) - x \subset \{v \in E \mid \langle p^x, v\rangle > 0\}$, and, for each $x \in K_\phi$, there exist a point $y^x \in X$ and an open (resp., a closed) subset $U(x)$ containing x such that*

$$z \in U(x) \cap K_\phi \Longrightarrow \langle p^z, y^x - z\rangle > 0. \tag{3.4}$$

If X is covered by finitely many $U(x)$'s for $x \in K_\phi$, then ϕ has a fixed point.

Proof. Define $\Phi : X \multimap X$ by $\Phi(x) := \{y \in X \mid \langle p^x, y - x\rangle > 0\}$ if $x \in K_\phi$ and $\Phi(x) = \emptyset$ if $x \in X \backslash K_\phi$. Then Φ has convex values. Then, by (K_1^*), for each $x \in K_\phi$, there exist a subset $U(x)$ containing x and a point $y^x \in X$ such that

$$z \in U(x) \cap K_\phi \Longrightarrow \langle p^z, y^x - z\rangle > 0 \Longleftrightarrow y^x \in \Phi(z), \quad z \notin \Phi(z). \tag{3.5}$$

Therefore, condition (K^*) is satisfied. Hence, by Corollary 3.2, ϕ has a fixed point. □

Remark 3.5. In case where X is compact and each $U(x)$ is open, Corollary 3.4 reduces to Urai [12, Corollary 1.1 for the case (K2) = (K_1^*)].

Corollary 3.6. *Let X be a convex space and $\psi : X \multimap X$. Suppose that a map $\psi : X \multimap X$ such that*

$$x \notin \psi(x) \Longrightarrow x \notin \phi(x), \quad \phi(x) \neq \emptyset, \tag{3.6}$$

satisfies condition (K^*) for $K_\psi = \{x \in X \mid x \notin \psi(x)\}$. If K_ψ is covered by finitely many $U(x)$'s for $x \in K_\phi$, then ψ has a fixed point.

Proof. Suppose that ψ does not have a fixed point. Then ψ is nonempty valued and does not have a fixed point. Moreover, $X = K_\psi \subset K_\phi \subset X$ and hence ϕ satisfies condition (K^*) even for K_ϕ. Now by applying Corollary 3.2 to nonempty-valued map ϕ, we have a fixed point of ϕ, a contradiction. □

Remark 3.7. In case where X is compact and each $U(x)$ is open, Corollary 3.6 reduces to Urai [12, Corollary 1.2 for the case (K2) = (K^*)].

Theorem 3.8. *Let I be a set. For each $i \in I$, let X_i be a convex space, $\Phi_i : \prod_{i\in I} X_i \multimap X_i$ a map with convex values, $\Phi = \prod_{i\in I}\Phi_i : X \multimap X$, and $K_\Phi := \{x \in X \mid x \notin \Phi(x)\}$. Suppose that*

(II) *for each $x \in K_\Phi$, there exist at least one $i \in I$, an element $y^x \in X_i$, and an open (resp., a closed) subset $U(x)$ of X containing x such that*

$$z \in U(x) \cap K_\Phi \Longrightarrow y^x \in \Phi_i(z). \tag{3.7}$$

If K_Φ is covered by finitely many $U(x)$'s, then Φ has a fixed point.

Proof. Suppose that $X = K_\Phi$. Then there exist a finite set $\{x_1, x_2, \dots, x_k\} \subset X$, a cover $\{U(x_1), U(x_2), \dots, U(x_k)\}$ of X, and a finite sequence $y_{i_1}^{x_1}, y_{i_2}^{x_2}, \dots, y_{i_k}^{x_k}$ for some $\{i_1, i_2, \dots, i_k\} \subset I$ satisfying condition (II) for maps $\Phi_{i_1}, \Phi_{i_2}, \dots, \Phi_{i_k}$. For each $x \in X$, let $J(x) := \{i_m \mid x \in U(x_m)\} \subset I$ and $N(x) := \{m \mid x \in U(x_m)\} \subset \{1, 2, \dots, m\}$. Let $\Phi : X \multimap X$ be a map defined by

$$\Phi(x) := \prod_{i\in J(x)} \Phi_i(x) \times \prod_{i\in I\backslash J(x)} X_i \tag{3.8}$$

for $x \in X$. For each $x \in X$, define $y(x) := (y_j)_{j\in I} \in X$ by letting

(1) y_j be a $y^{x_m}_{i_m}$ for some $i_m = j$, $m \in N(x)$, for $j \in J(x)$;
(2) y_j be an arbitrary element of $\Phi_j(x)$ for $j \notin J(x)$.

Then, by considering the open (resp., closed) neighborhood $\bigcap_{m\in N(x)} U(x_m)$ of x in X, the map Φ satisfies condition (I) of Theorem 3.1.

In fact, for each $x \in X$, for each $z \in \bigcap_{m\in N(x)} U(x_m)$, and for each $j \in \{i_1, i_2, \ldots, i_k\}$, $y(x) = (y_j)_{j\in I}$ is an element of $\Phi(z)$ since, for each $j \in J(x)$, $y_j \in \Phi_i(x)$ for all $z \in \bigcap_{m\in N(x)} U(x^m)$.

Therefore, Φ has a fixed point by Theorem 3.1, and we have a contradiction. □

Corollary 3.9. *Let I be a set. For each $i \in I$, let X_i be a convex space, $\phi_i : X = \prod_{i\in I} X_i \multimap X_i$ a map with nonempty values, $\phi = \prod_{i\in I} \phi_i : X \multimap X$, and $K_\phi := \{x \in X : x \notin \phi(x)\}$. Suppose that*

(NK*) *for each $i \in I$, there is a map $\Phi_i : X \multimap X_i$ such that for each $x = (x_j)_{j\in I} \in X$, $x_i \notin \phi_i(x) \Rightarrow x_i \notin \Phi_i(x)$; and for each $x \in K_\phi$, there exist at least one $i \in I$, an element $y^x \in X_i$, and an open (resp., a closed) subset $U(x)$ of X containing x such that*

$$z \in U(x) \cap K_\phi \Longrightarrow y^x \in \Phi_i(z). \tag{3.9}$$

If K_ϕ is covered by finitely many $U(x)$'s, then ϕ has a fixed point.

Proof. Suppose that $X = K_\phi$. Then Φ as in Theorem 3.8 satisfies the requirement (II) of Theorem 3.8, and hence, Φ has a fixed point. On the other hand, by (NK*), for each $x \in X = K_\phi$, we should have $x \notin \Phi(x)$. This is a contradiction. □

Remark 3.10. (1) In case $\Phi = \phi$, Corollary 3.9 reduces to Theorem 3.8.

(2) Urai (see [12, Theorem 2 for the case (NK*)]) obtained Corollary 3.9 under more restrictions.

Corollary 3.11. *Let I be a set. For each $i \in I$, let X_i be a convex space and $\psi_i : \prod_{i\in I} X_i \multimap X_i$ a map. Define $\psi = \prod_{i\in I} \psi_i : X \multimap X$. Suppose that for each $i \in I$, a nonempty-valued map $\phi_i : X \multimap X_i$ exists such that for each $x = (x_j)_{j\in I}$,*

$$x_i \notin \psi_i(x) \Longrightarrow x_i \notin \phi_i(x) \tag{3.10}$$

(typically, each ϕ_i may be chosen as a selection of ψ_i when ψ_i is nonempty-valued), and that each ϕ_i satisfy condition (NK) in Corollary 3.9 for $K_\psi = \{x \in X \mid x \notin \psi(x)\}$. If K_{ψ_i} is covered by finitely many $U(x)$'s, then Φ has a fixed point.*

Proof. Suppose that ψ does not have a fixed point. Then $\phi = \prod_{i\in I} \phi_i$ does not have a fixed point either. Hence, we have $X = K_\phi = K_\psi \subset \{x \in K \mid x \notin \prod_{i\in I} \phi_i(x)\} \subset X$ so that each ϕ_i satisfies condition (NK*) in Corollary 3.9 even when we take $K_\phi = \{x \in X \mid x \notin \phi(x)\}$ instead of $K_\psi = \{x \in X \mid x \notin \psi(x)\}$. Since ϕ is nonempty-valued, by Corollary 3.9, ϕ has a fixed point, a contradiction. □

Remark 3.12. In case X is compact and each $U(x)$ is open, Corollary 3.11 reduces to Urai [12, Corollary 2.1 for the case (NK*)].

4. Nash equilibrium existence theorems

In this section, we indicate that theorems in Section 3 can be applied to some economic equilibrium problems as in Urai [12, Sections 3 and 4]. We give generalized forms of only two theorems of Urai [12, Theorems 2 and 4].

Let I be a nonempty set of *players* and, for each $i \in I$, X_i the *strategy set* of the player i, where X_i is merely assumed to be a convex space. The payoff structure for games is given as *preference maps* $P_i : X = \prod_{j\in I} X_j \multimap X_i$, $i \in I$, satisfying for each $x = (x_j)_{j\in I} \in X$, $x_i \notin P_i(x)$ (the irreflexivity) for all $i \in I$. The set $P_i(x)$ may be empty and interpreted as the set of all strategies for player i which is better than x_i if the strategies of other players $(x_j)_{j\in I, j\neq i}$ are fixed.

A strategic form game is denoted by $(X_i, P_i)_{i\in I}$ in which a sequence of strategies $(x_i)_{i\in I} \in X$ is called a *Nash equilibrium* if $P_i((x_i)_{i\in I}) = \varnothing$ for all $i \in I$.

When I is a singleton, the Nash equilibrium is just a *maximal element* for the relation P_i on X_i.

Theorem 4.1 (maximal element existence). *Let X be a convex space and $P : X \multimap X$ a map such that for all $x \in X$, $x \notin P(x)$. Suppose that a map $\phi : X \multimap X$ satisfies condition (I) for $K_P := \{x \in X \mid P(x) \neq \varnothing\}$ in Theorem 3.1 and that for any $x \in X$,*

$$P(x) \neq \varnothing \Longrightarrow \phi(x) \neq \varnothing, \quad x \notin \phi(x). \tag{4.1}$$

If K_P is covered by finitely many $U(x)$'s, then there is a maximal element $x^* \in X$ with respect to P, that is, $P(x^*) = \varnothing$.

Proof. Assume the contrary, that is, for all $x \in X$, $P(x) \neq \varnothing$. Then $\{x \in X \mid x \notin P(x)\} = X = K_p := \{x \in X \mid P(x) \neq \varnothing\}$. Therefore, P satisfies all the requirements for ψ mentioned in Theorem 3.1 so that P has a fixed point, a contradiction. □

Remark 4.2. In case when X is a compact convex subset of a Hausdorff topological vector space, Theorem 4.1 extends Urai [12, Theorem 3 for the case (K*)]. Moreover, the special case of Theorem 4.1 in which $P = \phi$ satisfies condition (I), gives us a generalization of Yannelis and Prabhakar [13, Corollary 5.1] on the maximal element existence.

As Theorem 3.1 gives the maximal element existence, Theorem 3.8 gives the following Nash equilibrium existence.

Theorem 4.3 (Nash equilibrium existence). *For a strategic form game $(X_i, P_i)_{i\in I}$, the Nash equilibrium exists whenever the following conditions are satisfied:*

(A1) *for each $i \in I$, X is a nonempty convex space;*
(A2) *for each $i \in I$, $P_i : X = \prod_{j\in I} X_j \multimap X_i$, satisfying for all $x = (x_j^j)_{j\in I} \in X$, $x_i \in P_i(x)$;*
(A3) *for each P_i, a nonempty-valued map $\phi_i : X \multimap X_i$ is defined such that for all $x = (x_j)_{j\in I} \in X$,*

$$P_i(x) \neq \varnothing \Longrightarrow x_i \notin \phi_i(x); \tag{4.2}$$

(A4) *for each* $i \in I$, ϕ_i *fulfills condition (II) in* Theorem 3.8 *for* $K = \{x \in X \mid P_i(x) \neq \emptyset$ *for some* $i\}$*;*
($*$) X *is covered by finitely many* $U(x)$*'s.*

Proof. Suppose the contrary, that is, for each $x \in X$, there is at least one $i \in I$ such that $P_i(x) \neq \emptyset$. Then we have $\{x \in X \mid x \notin \prod_{i\in I} P_i(x)\} = X = \{x \in X \mid P_i(x) \neq \emptyset$ for some $i\} = K \subset X$. Hence, P_i, $i \in I$, satisfies all the requirements for ψ_i, $i \in I$, in Corollary 3.11 with respect to condition (II) (instead of (NK*)), so that $P = \prod_{i\in I} P_i$ has a fixed point, which violates condition (A2). □

Remark 4.4. Urai [12, Theorem 4] is a particular form of Theorem 4.3 under the restriction that

(1) each X_i is a compact convex subset of a Hausdorff topological vector space,
(2) $U(x)$ is open,
(3) assume (NK*) instead of condition (II).

Similarly, some of other results in Urai [12, Sections 3 and 4] might be improved by following our method, and we will not repeat.

5. Comments on some other results in Urai [12]

Urai [12, page 109] stated that the Fan-Browder fixed point theorem follows from the case (K*) of [12, Theorem 1] (hence from Corollary 3.2). Similarly, we obtain the following form of Theorem 2.2 (or Corollary 2.3) from Corollary 3.2.

Theorem 5.1. *Let* X *be a convex space and* $\phi : X \multimap X$ *a map with nonempty convex values. If there exists* $\{y_1, y_2, \dots, y_n\} \subset X$ *such that* $\phi^-(y_i)$ *is open (resp., closed) for each i,* $1 \leq i \leq n$*, and* $X = \bigcup_{i=1}^n \phi^-(y_i)$*, then* ϕ *has a fixed point.*

Proof. We will use Corollary 3.2 with $\Phi = \phi$. For each $x \in X$, there exist a subset $U(x) := \phi^-(y_i)$ containing x and a point y^x for some i. Then

$$z \in U(x) \cap K_\phi \Longrightarrow z \notin \phi(z), \qquad z \in U(x) = \phi^-(y^x) \ [\text{or } y^x \in \phi(z)]. \tag{5.1}$$

Hence condition (K*) holds. Hence, by Corollary 3.2, ϕ has a fixed point.

Urai [12, Theorem 19] obtained an extension of the KKM theorem, which can be shown to be a simple consequence of Theorem 2.1. □

Theorem 5.2. *Let* (X,D) *be a convex space and* $\{C_z\}_{z\in D}$ *a family of subsets of* X*. Suppose that* $\operatorname{co} N \subset \bigcup_{z\in N} C_z$ *for each* $N \in \langle D\rangle$ *(i.e.,* $z \mapsto C_z$ *is a KKM map* $D \multimap X$*) and that*

(KKM1) *for each* $x \in X$*, if* $x \notin C_z$ *for some* $z \in D$*, then there are an open neighborhood* $U(x)$ *of* x *in* X *and* $z' \in D$ *such that* $U(x) \cap C_{z'} = \emptyset$*.*

If $\bigcap_{z\in M} \overline{C_z}$ *is compact for some* $M \in \langle D\rangle$*, then there exists* $x^* \in X$ *such that* $x^* \in X$ *such that* $x^* \in \bigcap_{z\in D} C_z$*.*

Proof. Since $\operatorname{co} N \subset \bigcup_{z\in N} C_z \subset \bigcup_{z\in N} \overline{C_z}$ for each $N \in \langle D\rangle$, by Theorem 2.1, the family $\{\overline{C_z}\}_{z\in D}$ has the finite intersection property. Since $K := \bigcap_{z\in M} \overline{C_z}$ is compact, the family $\{K \cap \overline{C_z}\}_{z\in D}$ has nonempty intersection. Therefore, there exists an $x_* \in X$ such that

$x_* \in \bigcap_{z\in D} \overline{C_z}$. Suppose that $x_* \notin C_z$ for some $z \in D$. Then $u(x_*) \cap C_{z'} = \varnothing$ for some open neighborhood $u(x_*)$ of x_* and some $z' \in D$, by (KKM1). However, $x_* \in \overline{C}_{z'}$ implies $U(x_*) \cap C_{z'} = \varnothing$, a contradiction. Therefore, $x_* \in C_z$ for all $z \in D$. This completes our proof. □

Remark 5.3. Urai [12, Theorem 19] obtained the preceding result under the assumption that X is a nonempty compact convex subset of a Hausdorff topological vector space E. Actually, condition (KKM1) is equivalent to $\bigcap_{z\in D} C_z = \bigcap_{z\in D} \overline{C_z}$. In this case, the map $z \mapsto C_z$ is said to be *transfer closed-valued* by some authors.

Final Remarks. (1) In most of our results, we showed that compact convex subsets of Hausdorff topological vector spaces in some of Urai's results can be replaced by convex spaces with finite covers consisting of open (closed) neighborhoods of points of those spaces. Urai's main tools are the partition of unity argument on such covers and the Brouwer fixed point theorem. This is why he needs Hausdorffness and compactness. However, our method is based on a new Fan-Browder type theorem (Theorem 2.2), which is actually equivalent to the KKM theorem and to the Brouwer theorem.

(2) Moreover, some of Urai's requirements, for examples (K*) and (NK*), are replaced by a little general ones, for examples (I) and (II), respectively, in our results. Note that other results in Urai's paper which are not amended in the present paper might be improved by following our method.

(3) Urai [12, page 90] noted that (in some of his results) "the structure of vector space is superfluous, however, and a certain definition for a continuous combination among finite points on E under the real coefficient field will be sufficient," and so that " the concept of abstract convexity (like Llinares [6]) would be sufficient for all of the argument" in certain case. In fact, Llinares' MC spaces and many other spaces with certain abstract convexities are unified to generalized convex spaces (simply, G-convex spaces) by the present author since 1993. There have appeared a large numbers of papers on G-convex spaces. Actually, the materials in Section 2 were already extended to G-convex spaces; see Park [8, 9].

(4) For further information on the topics in this paper, the readers may consult the references [14, 15, 16]. Our method would be useful to improve a number of other known results.

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