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J. Math. Anal. Appl. 291 (2004) 154–164
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Comments on some fixed point theorems in hyperconvex metric spaces ☆

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Received 15 May 2003
Submitted by C.E. Chidume

Abstract

In this paper, we show that a number of known fixed point theorems for the Fan–Browder type maps or acyclic maps defined on (subsets of) hyperconvex metric spaces are simple consequences of the previously known theorems for corresponding maps defined on generalized convex spaces.

1. Introduction

The notion of hyperconvex metric spaces (or simply, hyperconvex spaces) was introduced by Aronszajn and Panitchpakdi [1] in 1956. Later, in 1979, independently Sine [2] and Soardi [3] proved that a bounded hyperconvex space has the fixed point property for nonexpansive maps. Since then many interesting works have appeared for hyperconvex spaces. For the literature, see the end of this paper.

Until recently, the study of hyperconvex spaces concentrated on their relationship with nonexpansive maps. However, Khamsi [4] established the Knaster–Kuratowski–Mazurkiewicz theorem (in short, KKM theorem) for hyperconvex spaces and applied it to obtain a Schauder type fixed point theorem. This line of study was followed by Kirk [5], Kirk and Shin [6], and Park [7–9]. In particular, the second author obtained extensions or

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.10.022

equivalent forms of the KKM theorem, a Fan–Browder type fixed point theorem, and other results for hyperconvex spaces in [7,8]. Moreover, Yuan [10], Isac and Yuan [11], Kirk et al. [12], Tarafdar and Yuan [13] established the KKM theorem, its equivalent formulations, fixed point theorems, and their applications for hyperconvex spaces.

However, most of the above-mentioned works are simple consequences of much more general results. In fact, Horvath [14–18] initiated study of the KKM theory and fixed point theory for C-spaces, which are meaningful generalizations of convex spaces or convex subsets of topological vector spaces. Moreover, in [18], he found that hyperconvex spaces are a particular type of C-spaces and gave a useful selection theorem on l.s.c. multimaps related to C-spaces. Recently, this selection theorem was extended by Ben-El-Mechaiekh and Oudadess [19] following some ideas from the celebrated theory on continuous selections due to Michael. On the other hand, the second author [20–30] initiated study of generalized convex spaces or G-convex spaces, which properly include the class of C-spaces and a large number of spaces having particular type of abstract convexity.

In this paper, we show that many of the fixed point theorems for the Fan–Browder type maps or acyclic maps defined on (subsets of) hyperconvex metric spaces are simple consequences of the previously known theorems for corresponding maps defined on generalized convex spaces. Consequently, we obtain generalized and improved versions of results in [6,10–13,31].

2. Preliminaries

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$. Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context.

For topological spaces X and Y, a multimap $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) (respectively, *lower semicontinuous* (l.s.c.)) if for each closed (respectively, open) set $B \subset Y$, $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$ is closed (respectively, open) in X.

A metric space (H, d) is said to be *hyperconvex* if

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leqslant r_\alpha + r_\beta$.

It is known that the space $\mathbb{C}(E)$ of all continuous real functions on a Stonian space E (that is, an extremely disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space $\mathbb{C}(E)$ for some Stonian space E. Therefore, $(\mathbf{R}^n, \|\cdot\|_\infty)$, l^∞, and L^∞ are concrete examples of hyperconvex spaces.

Results of Aronszajn and Panitchpakti [1, Theorem 1′] and Isbell [32, Theorem 1.1] are combined in the following

Theorem 2.1. *A hyperconvex space is complete and* (*freely*) *contractible.*

The concepts of C-spaces, LC-spaces, and LC-metric spaces were introduced and extensively studied by Horvath in a sequence of papers [14–18]:

A C-*space* (X, Γ) is a topological space X with a multimap $\Gamma : \langle X \rangle \multimap X$ from the set $\langle X \rangle$ of all nonempty finite subsets of X into the power set of X such that

(1) for each $A \in \langle X \rangle$, $\Gamma(A) = \Gamma_A$ is n-connected for all $n \geqslant 0$; and
(2) for all $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

A nonempty subset $Y \subset X$ is said to be Γ-*convex* if $A \in \langle Y \rangle$ implies $\Gamma_A \subset Y$.

A C-space (X, Γ) is called an LC-*space* (or a *locally* H-*convex space* [33]) if X is a Hausdorff uniform space and there exists a basis $\{V_\lambda\}_{\lambda \in I}$ for the uniform structure such that for each $\lambda \in I$, $\{x \in X \mid E \cap V_\lambda[x] \neq \emptyset\}$ is Γ-convex whenever $E \subset X$ is Γ-convex, where

$$V_\lambda[x] = \{x' \in X \mid (x, x') \in V_\lambda\}.$$

For example, any nonempty convex subset X of a locally convex Hausdorff topological vector space is an LC-space with $\Gamma_A = \operatorname{co} A$, the convex hull of $A \in \langle X \rangle$.

A triple $(X, d; \Gamma)$ is called an LC-*metric space* whenever (X, d) is a metric space and (X, Γ) is a C-space such that open balls are Γ-convex, and any neighborhood $\{x \in X \mid d(x, Y) < r\}$ of a Γ-convex set $Y \subset X$ is also Γ-convex.

Horvath [7, Theorem 9] obtained the following

Theorem 2.2. *Any hyperconvex space H is a complete LC-metric space with*

$$\Gamma(A) = \Gamma_A := \bigcap \{B \mid B \text{ is a closed ball containing } A\}$$

for each $A \in \langle H \rangle$.

Note that Γ_A itself is hyperconvex. From now on, a hyperconvex space $(H, d; \Gamma)$ is simply denoted by H, and $\mathrm{BI}(H)$ denotes the set of nonempty closed ball intersections in H. Elements of $\mathrm{BI}(H)$ are sometimes called *admissible subsets* of H; see [4].

3. Better admissible maps on *G*-convex spaces

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n an n-simplex with vertices $\{v_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \operatorname{co}\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and, in case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X \supset X; \Gamma)$ by $(X; \Gamma)$.

It should be noted that ϕ_A depends on $A \in \langle D\rangle$. Hence, for example, for any A, J in $\langle D\rangle$, if $A \supset J$ as above, $\phi_A|_{\Delta_J} : \Delta_J \to \Gamma(J)$ might be different from $\phi_J : \Delta_k \multimap \Gamma(J)$. In case when the family $\{\phi_A\}_{A\in\langle D\rangle}$ can be so chosen that for any $A \supset J$ as above, we have

$$\phi_A\left(\sum_{j=0}^{k}\lambda_j v_{i_j}\right) = \phi_J\left(\sum_{j=0}^{k}\lambda_j v_j\right) \quad \text{for any } \lambda_j \geqslant 0,\ \sum_{j=0}^{k}\lambda_j = 1;$$

then we can put $\Gamma(A) = \phi_A(\Delta_n)$ for all $A \in \langle D\rangle$.

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be *Γ-convex* if for each $N \in \langle D\rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$; and for a nonempty set $Z \subset X$, its *Γ-convex hull* is defined by

$$\Gamma\text{-}\operatorname{co}(Z) = \bigcap\{A \subset X \mid A \text{ is a } \Gamma\text{-convex subset of } X \text{ containing } Z\}.$$

For a G-convex space $(X; \Gamma)$ and $Z \subset X$, it is easily seen that

$$\Gamma\text{-}\operatorname{co}(Z) := \bigcup\bigl\{\Gamma\text{-}\operatorname{co}(A) \mid A \in \langle Z\rangle\bigr\}.$$

Examples of G-convex spaces can be found in [20–23,27] and references therein.

Theorem 3.1. *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ two maps satisfying*

(1) *for each $z \in D$, $S(z)$ is open (respectively, closed);*
(2) *for each $y \in X$, $M \in \langle S^-(y)\rangle$ implies $\Gamma_M \subset T^-(y)$; and*
(3) *$X = S(N)$ for some $N \in \langle D\rangle$.*

Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

Theorem 3.1 is obtained in [25] and applied to various forms of the Fan–Browder theorem, the Ky Fan intersection theorem, and the Nash equilibrium theorem for G-convex spaces.

From Theorem 3.1, we deduced the following [26]

Theorem 3.2. *Let $(X \supset D; \Gamma)$ be a G-convex space and $A : X \multimap X$ be a multimap such that $A(x)$ is Γ-convex for each $x \in X$. If there exist $z_1, z_2, \ldots, z_n \in D$ and nonempty open (respectively, closed) subsets $G_i \subset A^-(z_i)$ for $i = 1, 2, \ldots, n$ such that $X = \bigcup_{i=1}^n G_i$, then A has a fixed point.*

Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of multimaps from X into Y as follows [22,24]:

$F \in \mathfrak{B}(X, Y) \Leftrightarrow F : X \multimap Y$ is a multimap such that for any $N \in \langle D\rangle$ with $|N| = n + 1$ and any continuous map $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point.

For any topological space E and a G-convex space $(X, D; \Gamma)$, a map $T : E \multimap X$ is called a *Φ-map* if there exists a map $S : E \multimap D$ such that

(i) for each $y \in E$, $M \in \langle S(y)\rangle$ implies $\Gamma_M \subset T(y)$; and
(ii) $E = \bigcup\{\operatorname{Int} S^-(x) \mid x \in D\}$.

The concept of Φ-maps is originated from Horvath [14] and motivated by the works of Fan and Browder; see [22]. Hence the Φ-maps are usually called the Fan–Browder maps.

It is known that if E is a Hausdorff compact space, then T has a continuous selection $f : E \multimap X$ (that is, $f(x) \in T(x)$ for all $x \in E$) [20]. Therefore, a Φ-map belongs to $\mathfrak{B}$ if its domain is Hausdorff.

For a particular type of G-convex spaces, we can establish fixed point theorems for the class $\mathfrak{B}$ as follows.

A G-convex space $(X, D; \Gamma)$ is called a *Φ-space* if X is a Hausdorff uniform space and for each entourage V there is a Φ-map $T : X \multimap X$ such that $\mathrm{Gr}(T) \subset V$. This concept is originated from Horvath [14], where a number of examples were given.

The following is our main result of [22] whose proof is given here for the completeness.

Theorem 3.3. *Let $(X, D; \Gamma)$ be a Φ-space and $F \in \mathfrak{B}(X, X)$. If F is closed and compact, then F has a fixed point.*

Proof. Let $\mathcal{V} = \{V_\lambda\}_{\lambda\in I}$ be a basis of the Hausdorff uniform structure of X. Let $K = \overline{F(X)}$ be the closure of the range of F. Since $(X, D; \Gamma)$ is a Φ-space, for each $\lambda \in I$, there is a Φ-map $T_\lambda : X \multimap X$ such that $\mathrm{Gr}(T_\lambda) \subset V_\lambda$. Since K is compact, it is known that $T_\lambda|_K$ has a continuous selection $f_\lambda : K \to \Gamma_N$ for some $N \in \langle D\rangle$ such that $f_\lambda = \phi_N \circ p$, where $p : K \to \Delta_n$ is a continuous map; see [20]. Since $F \in \mathfrak{B}(X, K)$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \subset K \xrightarrow{p} \Delta_n$$

has a fixed point $a_\lambda \in \Delta_n$; that is, $a_\lambda \in (p \circ F \circ \phi_N)a_\lambda$. Hence,

$$x_\lambda := \phi_N(a_\lambda) \in (\phi_N \circ p \circ F)x_\lambda = (f_\lambda \circ F)x_\lambda$$

and there exists $y_\lambda \in F(x_\lambda) \subset K$ such that $x_\lambda = f_\lambda(y_\lambda) \in T_\lambda(y_\lambda)$; that is, $(x_\lambda, y_\lambda) \in V_\lambda$. Therefore

$$(x_\lambda, y_\lambda) \in V_\lambda \cap \mathrm{Gr}(F) \subset X \times K.$$

Since K is compact, we may assume that $\{y_\lambda\}_{\lambda\in I}$ converges to some $x_0 \in K$. Since $(x_\lambda, y_\lambda) \in V_\lambda$ for all $\lambda \in I$, $\{x_\lambda\}_{\lambda\in I}$ also converges to $x_0 \in K$. Since F is closed and $(x_\lambda, y_\lambda) \in \mathrm{Gr}(F)$, we should have $(x_0, x_0) \in \mathrm{Gr}(F)$. Therefore, F has a fixed point $x_0 \in K$. □

Particular forms of Theorem 3.3 were known by Horvath [14] and Park and Kim [27]. In [21–24], it was shown that Theorem 3.3 subsumes a large number of fixed point theorems related to approachable maps on G-convex spaces, acyclic maps on locally G-convex spaces, and Kakutani maps on Φ-spaces and on hyperconvex spaces.

4. Fan–Browder maps on hyperconvex spaces

The following is a version of Theorem 3.2 for hyperconvex spaces.

Theorem 4.1. *Let H be a hyperconvex space and $A : H \multimap H$ a multimap. If there exists $y_1, y_2, \ldots, y_n \in H$ and open (respectively, closed) subsets $G_i \subset A^-(y_i)$ for $i = 1, 2, \ldots, n$ such that $H = \bigcup_{i=1}^n G_i$, then there exists a point x_0 such that $x_0 \in \Gamma(A(x_0))$, where $\Gamma(A(x_0))$ is the closed ball intersection containing $A(x_0)$.*

Note that if A has admissible values (that is, values are closed ball intersections) in Theorem 4.1, then A has a fixed point.

For the case H itself is compact, particular forms of Theorem 4.1 appear in [7, Theorem 3], [10, Theorems 3.3–3.6], [11, Theorems 3.2, 3.3, 4.6], [12, Theorem 3.1, Corollary 3.6], and [13, Theorems 3.3–3.6, Corollary 4.8].

The following is a dual form of Theorem 4.1.

Theorem 4.2. *Let H be a hyperconvex space and $B : H \multimap H$ a multimap. If there exist $x_1, x_2, \ldots, x_n \in H$ and open (respectively, closed) subsets $G_i \subset B(x_i)$ for $i = 1, 2, \ldots, n$ such that $H = \bigcup_{i=1}^n G_i$, then there exists a point x_0 such that $x_0 \in \Gamma(B^-(x_0))$.*

Proof. Put $A := B^-$ and $A^- := B$ in Theorem 4.1. □

Note that, if B^- has admissible values, then B has a fixed point.

Particular forms of Theorem 4.2 are given in [10, Theorems 3.1, 3.2], [11, Theorem 3.1], and [13, Theorems 3.1, 3.2].

The following is a Ky Fan type matching theorem.

Theorem 4.3. *Under the hypothesis of Theorem 4.2, there exist a finite subset $\{y_1, y_2, \ldots, y_r\} \subset H$ and $x_0 \in \Gamma(\{y_1, y_2, \ldots, y_r\})$ such that $x_0 \in \bigcap_{j=1}^r B(y_j)$.*

Proof. By Theorem 4.2, there exists a point $x_0 \in \Gamma(B^-(x_0))$. Since $\Gamma(B^-(x_0))$ is a Γ-convex hull, there exists a finite subset $\{y_1, y_2, \ldots, y_r\} \subset B^-(x_0)$ such that $x_0 \in \Gamma(\{y_1, y_2, \ldots, y_r\})$. Hence we have $y_j \in B^-(x_0)$ or $x_0 \in B(y_j)$ for all $j = 1, 2, \ldots, r$. □

Particular forms appear in [10, Theorem 4.4, Corollary 4.5].

The following is already known [9, Theorem 5.5], [21, Theorem 8].

Theorem 4.4. *Let H be a hyperconvex space and $S, T : H \multimap H$ two maps such that*

(1) *for each $x \in H$, $N \in \langle S(x) \rangle$ implies* $\mathrm{BI}(N) \subset T(x)$; *and*
(2) $H = \bigcup \{\mathrm{Int}\, S^-(y) \mid y \in H\}$.

If T is compact, then T has a fixed point.

5. Acyclic maps on Φ-spaces

In this section, we show that the recent fixed point results in [31] and others are simple consequences of Theorem 3.3.

Recall that for a hyperconvex space $H = (H, d; \Gamma)$, a subset $A \in \mathrm{BI}(H)$ is said to be *admissible* [4]. A nonempty subset $A \subset H$ is said to be *sub-admissible* [31] if for each finite subset N of A, we have $\Gamma_N \subset A$.

As a G-convex space $H = (H; \Gamma)$, a sub-admissible subset A is simply Γ-convex, and hence is a G-convex space. Moreover, we have

Lemma 5.1. *Every sub-admissible subset X of a hyperconvex space H is a Φ-space.*

Proof. Note that X is a metric subspace of H. For any $\lambda > 0$, we give a Φ-map $T : X \multimap X$ such that $d(x, y) \leqslant \lambda$ for all $x \in X$ and $y \in T(x)$. Define $T(x) := \{y \in X \mid d(x, y) \leqslant \lambda\}$ for $x \in X$. Then for each $M \in \langle T(x)\rangle$, we have $\Gamma_M \subset T(x)$. In fact, $\Gamma_M = \bigcap\{B \mid B \text{ is a closed ball containing } M\} \subset T(x) = B(x, \lambda)$. Moreover, for each $x \in X$, there exists $y \in X$ such that $d(x, y) < \lambda$ (since we can choose $y := x$). Therefore

$$x \in \{x \in X \mid d(x, y) < \lambda\} \subset \mathrm{Int}_X T^-(y) \subset T^-(y)$$

shows $X = \bigcup\{\mathrm{Int}_X T^-(y) \mid y \in X\}$. This completes our proof. □

From Theorem 3.3 and Lemma 5.1, we have the following

Theorem 5.1. *Let X be a sub-admissible subset of a hyperconvex space H. Then any closed compact map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

For topological spaces X and Y, we adopt the following:

$F \in \mathbb{V}(X, Y) \Leftrightarrow F : X \multimap Y$ is an acyclic map; that is, an upper semicontinuous multimap with compact acyclic values.

It is well known that $\mathbb{V}$ is one of the typical examples of $\mathfrak{B}$; see [22]. Now we show that Theorem 5.1 subsumes all of the results in [31]. First, we have the following [31, Theorem 2.1]

Corollary 5.1. *Let H be a hyperconvex space and X a sub-admissible subset of H. Suppose that F is an upper semicontinuous map with closed acyclic values from X into a compact subset of X. Then F has a fixed point.*

From Theorem 2.1, we have the following

Lemma 5.2. *Every admissible subset X of a hyperconvex space H is hyperconvex and contractible.*

The following lemma is given in [31, Proposition 1.4].

Lemma 5.3. *Every compact sub-admissible subset X of a hyperconvex space H is admissible.*

From Corollary 5.1, we deduce the following [31, Theorem 2.2]

Corollary 5.2. *Let H be a hyperconvex space and X a sub-admissible subset of H. Suppose that F is an upper semicontinuous multimap with closed sub-admissible values from X into a compact subset of X. Then F has a fixed point.*

Proof. By Lemma 5.3, the values of F are admissible subsets of H, and hence by Lemma 5.2, are contractible. Since every contractible subset is acyclic, the conclusion follows from Corollary 5.1. □

In [31], its authors assumed the closedness of X in Corollary 5.2, which is redundant.

We need the following [9, Theorem 3.2]

Lemma 5.4. *Let X be a metric space, $A \subset X$ with $\dim_X A \leqslant 0$, H a hyperconvex space, and $F : X \multimap H$ a lower semicontinuous map with closed values such that $F(x)$ is sub-admissible for $x \notin A$. Then F admits a continuous selection.*

In view of Lemma 5.4, the above fixed point results can be applied to coincidence result.

Corollary 5.3. *Let X be a sub-admissible subset of a hyperconvex space H, Y a nonempty subset of a hyperconvex space H', and $Z \subset X$ be nonempty compact subset. Suppose that $F : X \multimap Y$ is a lower semicontinuous map with closed sub-admissible values and that $T : Y \multimap Z$ is an upper semicontinuous map with closed sub-admissible values. Then there is a point $\bar{x} \in T(\bar{y})$ and $\bar{y} \in F(\bar{x})$.*

Proof. Note that $F : X \multimap Y \subset H$ has a continuous selection $f : X \to Y$ by Lemma 5.4 (with $A = \emptyset$). Then $T \circ f : X \multimap Z$ is a compact upper semicontinuous multimap with closed sub-admissible values. Hence, by Corollary 5.2, we have a fixed point $\bar{x} \in (T \circ f)(\bar{x})$. Let $\bar{y} = f(\bar{x})$. Then $\bar{x} \in T(\bar{y})$ and $\bar{y} \in T(\bar{x})$. □

In [31, Theorem 2.3], its authors assumed the closedness of X, which is redundant. Note that [31, Theorem 2.4] is a similar application of Corollary 5.1 to another coincidence theorem. Moreover, these two coincidence theorems are used to obtain two minimax results, which might be already known in a more general setting in the frame of C-spaces or G-convex spaces.

The following corollary is known [9, Corollary 4.5], [21, Theorem 7].

Corollary 5.4. *Let H be a hyperconvex space and $F : H \multimap H$ a compact map with closed sub-admissible values. If Φ is u.s.c. or l.s.c., then F has a fixed point.*

In order to add another related results, we need the following well-known fact.

Lemma 5.5. *Let $F, G : X \multimap Y$ be two maps for topological spaces X and Y such that $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$. If*

(1) *F is upper semicontinuous at $x_0 \in X$;*
(2) *$F(x_0)$ is compact; and*
(3) *G is closed;*

then the multimap $F \cap G : x \mapsto F(x) \cap G(x)$ is upper semicontinuous at $x_0 \in X$.

The following corollary generalizes [6, Corollary 3.5].

Corollary 5.5. *Let H be a hyperconvex space and X a closed sub-admissible subset. Let $F : X \multimap H$ be a compact upper semicontinuous multimap with closed sub-admissible values such that $F(x) \cap X \neq \emptyset$ for all $x \in X$. Then F has a fixed point.*

Proof. Note that X is closed in H. Let $G : X \multimap H$ be the constant multimap defined by $G(x) := X$ for all $x \in X$. Then G has closed graph $\mathrm{Gr}(G) = X \times X$ in $X \times H$. Then by Lemma 5.5, the map $F \cap G : X \multimap X$ is upper semicontinuous with nonempty closed sub-admissible values $F(x) \cap X$ for $x \in X$. Moreover, the map is also compact. Therefore, by Corollary 5.2, it has a fixed point $x_0 \in (F \cap G)(x_0) = F(x_0) \cap X$; that is, $x_0 \in X$ and $x_0 \in F(x_0)$. This completes our proof. □

Since any admissible set is sub-admissible, we immediately have the following extremely particular case of Corollary 5.5.

Corollary 5.6. *Let H be a hyperconvex space, X a compact admissible subset of H, and $F : X \multimap H$ an upper semicontinuous map with admissible values for which $F(x) \cap X \neq \emptyset$ for all $x \in X$. Then F has a fixed point.*

Note that Kirk and Shin [6, Corollary 3.5] obtained the above result for a continuous map F and bounded H, and they asked whether their result remains true under the assumption that F is upper semicontinuous rather than continuous. Consequently Corollary 5.5 answers this question affirmatively.

The following corollary is due to Yuan [31, Theorem 1.1].

Corollary 5.7. *Let H be a hyperconvex space and X a compact admissible subset of H. Suppose that $F : X \multimap X$ is an upper semicontinuous map having admissible values. Then F has a fixed point.*

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