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# The KKM principle implies many fixed point theorems ☆

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**Abstract**

We show that the KKM principle implies two new general fixed point theorems for the Kakutani maps or the Browder maps. Consequently, we give unified transparent proofs of many of well-known results.


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*Keywords:* Kakutani map; Browder map; KKM principle; Fixed point; Maximal element; Zima type

## 1. Introduction

It is well-known that the Brouwer fixed point theorem, the Sperner lemma, the Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM principle), and many results in topology and nonlinear analysis are mutually equivalent; see [19]. Especially, in [14], it was shown that the KKM principle implies the Brouwer theorem.

For topological spaces $X$ and $Y$, a *multimap* or a *map* $T : X \multimap Y$ is a function from $X$ into the power set of $Y$. A map $T : X \multimap Y$ is *upper semicontinuous* (*u.s.c.*) if for each open subset $G$ of $Y$, the set $\{x \in X\colon\ T(x) \subset G\}$ is open in $X$; *lower semicontinuous* (*l.s.c.*) if for each closed subset $F$ of $Y$, the set $\{x \in X\colon\ T(x) \subset F\}$ is closed in $X$; *continuous* if it is u.s.c. and l.s.c.; and *compact* if the range $T(X) = \{y \in Y\colon\ y \in T(x) \text{ for some } x \in X\}$ is contained in a compact subset of $Y$.

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Let $X$ be a subset of a topological vector space. A multimap $T : X \multimap X$ is called a *Kakutani map* if $T$ is u.s.c. and has nonempty closed convex values $T(x)$ for each $x \in X$. There have appeared many fixed point theorems for Kakutani maps. For the literature, see [15,16].

A multimap $T : X \multimap X$ is called a *Browder map* (or a *$\Phi$-map*) if it has nonempty convex values $T(x)$ for $x \in X$ and open fibers $T^{-}(y) = \{x \in X\colon\ y \in T(x)\}$ for $y \in X$. The Browder fixed point theorem [2] has also many generalizations and variations; see [16,17].

In the present paper, we show that most of well-known fixed point theorems for Kakutani maps or Browder maps are simple consequences of the KKM principle. We obtain two new general fixed point theorems for Browder maps (Theorem 2.1) and for Kakutani maps (Theorem 3.1) and, consequently, give unified transparent proofs of many of well-known results.

The following is the celebrated KKM theorem [14]:

**KKM Principle.** *Let $D$ be the set of vertices of a simplex $S$ and $F : D \multimap S$ a multimap with closed* (*respectively open*) *values such that*

$$\operatorname{co} N \subset F(N) \quad \textit{for each } N \subset D.$$

*Then* $\bigcap_{z\in D} F(z) \neq \emptyset$.

The "open-valued" version of the KKM principle is due to Kim [13] and Shih and Tan [23], and is now known to be a simple consequence of the original "closed-valued" version; see [19]. It is also well-known that the following easily follows from the KKM principle; see Fan [3].

**Lemma.** *Let $X$ be a subset of a topological vector space, $D$ a nonempty subset of $X$ such that* $\operatorname{co} D \subset X$, *and $F : D \multimap X$ a KKM map with closed* (*respectively open*) *values. Then* $\{F(z)\}_{z\in D}$ *has the finite intersection property.*

Note that a map $F : D \multimap X$ is called a *KKM map* if

$$\operatorname{co} N \subset F(N) \quad \text{for each } N \in \langle D\rangle,$$

where $\langle D\rangle$ denotes the class of all nonempty finite subsets of $D$.

Recall that a binary relation $R$ in a set $X$ can be regarded as a multimap $\Phi : X \multimap X$ and conversely by the following obvious way:

$$y \in \Phi(x) \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a multimap $\Phi$ if $\Phi(x_0) = \emptyset$; see [25].

## 2. The Browder type theorems

From lemma, we obtain our main result in this section as follows:

**Theorem 2.1.** *Let $X$ be a subset of a topological vector space, $D$ a nonempty subset of $X$ such that* $\operatorname{co} D \subset X$, *and $S : D \multimap X$, $T : X \multimap X$ multimaps. Suppose that*

(1.1) *$S(z)$ is open (respectively closed) for each $z \in D$;*
(1.2) *$\operatorname{co} S^-(y) \subset T^-(y)$ for each $y \in X$; and*
(1.3) *$X = S(M)$ for some $M \in \langle D \rangle$.*

*Then $T$ has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

**Proof.** Define a map $F : D \multimap X$ by $F(z) := X \setminus S(z)$ for each $z \in D$. Then each $F(z)$ is closed (respectively open) in $X$ by (1.1), and

$$\bigcap_{z \in M} F(z) = X \setminus \bigcup_{z \in M} S(z) = X \setminus X = \emptyset$$

by (1.3). Therefore, the family $\{F(z)\}_{z \in D}$ does not have the finite intersection property, and hence, $F$ is not a KKM map by lemma. Thus, there exists an $N \in \langle D \rangle$ such that $\operatorname{co} N \not\subset F(N) = \bigcup\{X \setminus S(z)\colon z \in N\}$. Hence, there exists an $x_0 \in \operatorname{co} N$ such that $x_0 \in S(z)$ for all $z \in N$; that is, $N \subset S^-(x_0)$. Therefore, $x_0 \in \operatorname{co} N \subset \operatorname{co} S^-(x_0) \subset T^-(x_0)$ by (1.2). This implies $x_0 \in T(x_0)$ and completes our proof.

**Remark.** In Theorem 2.1, condition (1.3) can be replaced by the following, without affecting its conclusion:

(1.3)′ *There exists an $A \in \langle D \rangle$ such that $S^-(y) \cap A \neq \emptyset$ for each $y \in X$.*

In fact, if for each $y \in X$, there exists a $z \in A$ such that $z \in S^-(y)$ or $y \in S(z)$, then we have $X = S(A)$. Hence (1.3)′ implies (1.3).

Moreover, condition (1.3) can be replaced by the following:

(1.3)″ *$\operatorname{co} M \subset S(M)$ for some $M \in \langle D \rangle$.*

In this case, $T$ has a fixed point $x_0 \in \operatorname{co} M$.

From Theorem 2.1, we obtain a number of generalizations of the Browder fixed point theorem as follows:

**Corollary 2.2.** *Let $X$ be a subset of a topological vector space, $D$ a nonempty subset of $X$ such that $\operatorname{co} D \subset X$, $K$ a nonempty subset of $X$, and $G : X \multimap D$, $H : X \multimap X$ multimaps. Suppose that*

(2.1) *for each $x \in X$, $\operatorname{co} G(x) \subset H(x)$;*
(2.2) *for each $z \in D$, $G^-(z)$ is open (respectively closed);*
(2.3) *$K \subset \bigcup\{G^-(z)\colon z \in N\}$ for some $N \in \langle D \rangle$; and*
(2.4) *there exists a convex subset $L_N$ of $X$ containing $N$ such that*

$$L_N \setminus K \subset \bigcup\{G^-(z)\colon z \in M\}$$

*for some $M \in \langle L_N \cap D \rangle$.*

*Then $H$ has a fixed point in $L_N$.*

**Proof.** Let $S(z) := G^-(z)$ for each $z \in D$ and $T(x) := H^-(x)$ for each $x \in X$. Then (1.1) holds. Moreover, for each $y \in X$, we have

$$\operatorname{co} S^-(y) = \operatorname{co}(G^-)^-(y) = \operatorname{co} G(y) \subset H(y) = T^-(y)$$

by (2.1). Hence condition (1.2) holds. Further,

$$\operatorname{co}(M \cup N) \subset L_N \subset (L_N \backslash K) \cup K \subset \bigcup \{G^-(z)\colon z \in M \cup N\} = S(M \cup N),$$

where $M \cup N \in \langle L_N \cap D \rangle \subset \langle D \rangle$. Hence condition (1.3)″ holds. Therefore, by Theorem 2.1, $T = H^-$ has a fixed point $x_0 \in \operatorname{co}(M \cup N) \subset L_N$, that is $x_0 \in T(x_0)$ or $x_0 \in H(x_0)$. This completes our proof.

The following simple consequence of Corollary 2.2 subsumes a number of generalizations of the Browder fixed point theorem:

**Corollary 2.3.** *Let $X$ be a subset of a topological vector space, $D$ a nonempty subset of $X$ such that $\operatorname{co} D \subset X$, $K$ a nonempty compact subset of $X$, and $G : X \multimap D$ a multimap. Suppose that*

(3.1) $K \subset \bigcup\{\operatorname{Int}_X G^-(z)\colon z \in D\}$; *and*
(3.2) *for each $N \in \langle D \rangle$, there exists a compact convex subset $L_N$ of $X$ containing $N$ such that*

$$L_N \backslash K \subset \bigcup \{\operatorname{Int}_X G^-(z)\colon z \in L_N \cap D\}.$$

*Then the multimap $\operatorname{co} G : X \multimap X$ has a fixed point.*

Corollary 2.3 reduces to the following Browder fixed point theorem whenever $X = D = \operatorname{co} D = K = L_N$ for each $N \in \langle X \rangle$:

**Corollary 2.4.** *Let $X$ be a compact convex subset of a topological vector space and $T : X \multimap X$ a map such that*

(4.1) *$T(x)$ is nonempty and convex for each $x \in X$; and*
(4.2) *$T^-(y)$ is open for each $y \in X$.*

*Then $T$ has a fixed point.*

The following shows that Theorem 2.1 properly generalizes the Browder theorem:

**Examples.**

(1) Let $X := [0, \infty) \subset \mathbb{R}$ and $T : X \multimap X$ be defined by $T(x) := [0, x]$ for $x \in X$. Then each $T(x)$ is nonempty and convex. Moreover, $T^-(y) = [y, \infty)$ is *closed* for each $y \in X$. (Here, $T$ is an example of a Kakutani map.) Note that $X$ is covered by a finite number of $T^-(y)$'s and hence Theorem 2.1 with $X = D$ and $S = T$ works.

(2) Let $X := [0, 10) \subset \mathbb{R}$ and $T : X \multimap X$ be defined by $T(x) := (x/2, 10)$ for $x \in X$. Then each $T(x)$ is nonempty and convex. Moreover, $T^{-}(y) = [0, 2y)$ if $y < 5$ and $T^{-}(y) = [0, 10)$ if $y \geqslant 5$, and hence $T^{-}(y)$ is *open* for each $y \in X$. (Here, $T$ is an example of a Browder map.) Note that $X$ is covered by a finite number of $T^{-}(y)$'s and hence Theorem 2.1 with $X = D$ and $S = T$ works.

Only one of the simplest forms of Corollary 2.2 for the case $S$ has *closed* values is known; see Kim [13]:

**Corollary 2.5.** *Let $X$ be a convex subset of a topological vector space and $T : X \multimap X$ a map such that*

(5.1) *$T(x)$ is convex for each $x \in X$;*
(5.2) *$T^{-}(y)$ is closed for each $y \in X$; and*
(5.3) *there exists an $A \in \langle X \rangle$ such that $T(x) \cap A \neq \emptyset$ for each $x \in X$.*

*Then $T$ has a fixed point.*

From lemma, we have a result on the existence of maximal elements and fixed points:

**Theorem 2.6.** *Let $X$ be a subset of a topological vector space, $D$ a nonempty subset of $X$ such that* $\operatorname{co} D \subset X$, *and $S : D \multimap X$, $T : X \multimap X$ multimaps. Suppose that*

(6.1) *$S(z)$ is open for each $z \in D$;*
(6.2) *$\operatorname{co} S^{-}(y) \subset T^{-}(y)$ for each $y \in X$; and*
(6.3) *$\overline{T(X)} \subset S(M)$ for some $M \in \langle D \rangle$.*

*Then either*

(a) *$T^{-}$ has a maximal element $x_0 \in X$; or*
(b) *$T$ has a fixed point $x_1 \in X$.*

**Proof.** Let $K := \overline{T(X)}$ and define a map $F : D \multimap X$ by $F(z) := K \setminus S(z)$ for each $z \in D$. Then each $F(z)$ is closed by (6.1), and as in the proof of Theorem 2.1, we have $\bigcap_{z \in M} F(z) = \emptyset$ by (6.3). Hence $F$ is not a KKM map by Lemma. Thus there exists an $N \in \langle D \rangle$ such that $\operatorname{co} N \not\subset F(N) = \bigcup_{z \in N} (K \setminus S(z))$. Hence, there exists an $x_0 \in \operatorname{co} N$ such that $x_0 \notin K \setminus S(z)$ for all $z \in N$.

Case I. If $x_0 \in X \setminus K \subset X \setminus T(X)$, then $x_0 \notin T(X)$ and $T^{-}(x_0) = \emptyset$.

Case II. If $x_0 \in K$, then $x_0 \in S(z)$ for all $z \in N$; that is, $N \subset S^{-}(x_0)$. Therefore, by (6.2), $x_0 \in \operatorname{co} N \subset \operatorname{co} S^{-}(x_0) \subset T^{-}(x_0)$. This implies $x_0 \in T(x_0)$ and completes our proof.

**Remarks.**

(1) Note that in (6.3), $\overline{T(X)}$ denotes the closure of $T(X)$ with respect to $X$.
(2) Condition (6.3) is implied by any of the following:

(6.3)′ $X$ is compact and $X = S(D)$.
(6.3)″ $T$ is compact (that is, $\overline{T(X)}$ is compact in $X$) and $\overline{T(X)} \subset S(D)$.
(6.3)‴ $\overline{T(X)} \backslash S(M)$ is compact for some $M \in \langle D \rangle$ and $\overline{T(X)} \subset S(D)$.

From Theorem 2.6, we have the following:

**Corollary 2.7.** *Let $X$, $D$, $S$, and $T$ be the same as in Theorem* 2.6. *Suppose that*

(7.1) $X = \bigcup\{\mathrm{Int}_X S(z)\colon z \in D\}$;
(7.2) $\mathrm{co}\, S^-(y) \subset T^-(y)$ *for each* $y \in X$; *and*
(7.3) $\overline{T(X)} \backslash \bigcup\{\mathrm{Int}_X S(z)\colon z \in M\}$ *is compact for some* $M \in \langle D \rangle$.

*Then $T$ has a fixed point $x_0 \in X$.*

**Proof.** Consider the map $\mathrm{Int}_X S : D \multimap X$ instead of $S$ in Theorem 2.6. Then (7.1) and (7.3) $\Rightarrow$ (6.3)‴ $\Rightarrow$ (6.3). Note that

$$\mathrm{co}(\mathrm{Int}_X S)^-(y) \subset \mathrm{co}\, S^-(y) \subset T^-(y) \quad \text{for } y \in X,$$

and hence (7.2) $\Rightarrow$ (6.2). Moreover, by (7.1), for each $x \in X$, we have a $z \in D$ such that $x \in \mathrm{Int}_X S(z) \subset S(z)$, and hence $S^-(x) \neq \emptyset$. Therefore, $T^-(x) \neq \emptyset$ for all $x \in X$ by (7.2) and the conclusion (a) of Theorem 2.6 cannot occur, and hence, we have the conclusion.

Note that Corollary 2.7 generalizes the Browder fixed point theorem (Corollary 2.4), which is also known to be equivalent to the Brouwer fixed point theorem.

## 3. The Kakutani type theorems

From lemma, we deduce the following main result of this section:

**Theorem 3.1.** *Let $X$ be a convex subset of a topological vector space $E$. Let $T : X \multimap X$ be an u.s.c.* (*respectively a l.s.c.*) *multimap with nonempty convex values such that the following holds*:

(Z) *for each neighborhood $U$ of* 0 *in $E$, there exists a neighborhood $V$ of the origin* 0 *in $E$ such that*

$$\mathrm{co}\big(V \cap \big(T(X) - T(X)\big)\big) \subset U.$$

*If $T(X)$ is totally bounded, then for any neighborhood $U$ of* 0 *in $E$, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

**Proof.** Let $U$ be a neighborhood of the origin 0 in $E$ and $V$ a symmetric open neighborhood of 0 satisfying (Z). Then there exists a symmetric open neighborhood $W$ of 0 such that $\overline{W} + \overline{W} \subset V$. Since $K := \overline{T(X)}$ is totally bounded in $X$, there exists a finite subset $D := \{x_1, x_2, \ldots, x_n\} \subset T(X)$ such that $K \subset \bigcup_{i=1}^n (x_i + W)$.

If $T$ is u.s.c., for each $i$, let

$$F(x_i) := \{x \in X\colon\ T(x) \cap (x_i + \overline{W}) = \emptyset\}.$$

Then, each $F(x_i)$ is open in $X$. Moreover we have

$$\bigcap_{i=1}^{n} F(x_i) = \left\{x \in X\colon\ T(x) \cap \bigcup_{i=1}^{n} (x_i + \overline{W}) = \emptyset\right\} = \emptyset$$

since $T(X) \subset K \subset \bigcup_{i=1}^{n}(x_i + W)$. Now we apply Lemma to $X$ with $D$ defined as above. Since its conclusion does not hold, $F : D \multimap X$ cannot be a KKM map. Therefore, there exist a subset $\{x_{i_1}, \ldots, x_{i_k}\} \subset D$ and an $x_U \in \operatorname{co}\{x_{i_1}, \ldots, x_{i_k}\}$ such that $x_U \notin \bigcup_{j=1}^{k} F(x_{i_j})$. Hence $T(x_U) \cap (x_{i_j} + \overline{W}) \neq \emptyset$ for each $j$; and consequently

$$T(x_U) \cap (x_{i_j} + V) \neq \emptyset \quad \text{or} \quad V \cap (T(x_U) - x_{i_j}) \neq \emptyset.$$

If $T$ is l.s.c., for each $i$, let

$$F(x_i) := \{x \in X\colon\ T(x) \cap (x_i + W) = \emptyset\}.$$

Then, each $F(x_i)$ is closed in $X$. By a similar method, we obtain the same conclusion.

Therefore, there exists a $z_j \in T(x_U) = \operatorname{co} T(x_U)$ such that

$$z_j - x_{i_j} \in V \cap (T(x_U) - x_{i_j}) \subset \operatorname{co}(V \cap (T(X) - T(X))).$$

Since $x_U \in \operatorname{co}\{x_{i_j}\}_{j=1}^{k}$, there exists a $y_U \in \operatorname{co}\{z_j\}_{j=1}^{k} \in T(x_U)$ such that

$$y_U - x_U \in \operatorname{co}(V \cap (T(X) - T(X))) \subset U \text{ and } y_U \in T(x_U).$$

Therefore, we have $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. This completes our proof.

From Theorem 3.1, we have the following fixed point theorem for Kakutani maps:

**Corollary 3.2.** *Let $X$ be a convex subset of a Hausdorff topological vector space. Then any compact Kakutani map $T : X \multimap X$ has a fixed point in $X$ whenever the condition* (Z) *holds.*

**Proof.** By Theorem 3.1, for each neighborhood $U$ of 0, there exist $x_U, y_U \in X$ such that $y_U \in T(x_U)$ and $y_U \in x_U + U$. Since $T(X)$ is relatively compact, we may assume that the net $\{y_U\}$ converges to some $x_0 \in K$. Since $E$ is Hausdorff, the corresponding net $\{x_U\}$ also converges to $x_0$. Because $T$ is u.s.c. with closed values, the graph of $T$ is closed in $X \times T(X)$ and hence we have $x_0 \in T(x_0)$. This completes our proof.

According to Hadžić [7], the set $T(X)$ in Theorem 3.1 is said to be *of the Zima type* if (Z) holds; see [26].

Actually, Hadžić obtained several particular forms of Corollary 3.2 as follows:

**Examples.**

(1) Hadžić [4, Corollary 2], [6, Theorem 8] obtained Corollary 3.2 under the restriction that $X$ is closed. A number of consequences and applications of her result were given in Hadžić [4,6] and Hadžić and Gajić [9]. Moreover, Arandelović [1] gave a simple proof of a particular form of Hadžić's theorem using the KKM–Fan theorem (Lemma).
(2) Hadžić [5, Theorem 2] obtained a particular form of Corollary 3.2 for a compact convex subset $X$ of a metrizable topological vector space $E$.
(3) Hadžić [8, Theorem 3] obtained a particular form of Corollary 3.2 for a subset $X$ of the Zima type in a complete topological vector space $E$, and applied her result to some economic problems.

Note that any subset of a locally convex topological vector space is of the Zima type, and not conversely; see [6,7]. Therefore, the following well-known result follows from Corollary 3.2:

**Corollary 3.3** (Himmelberg [10]). *Let $X$ be a convex subset of a locally convex Hausdorff topological vector space. Then any compact Kakutani map $T : X \multimap X$ has a fixed point.*

The single-valued case of Corollary 3.3 is due to Hukuhara [11].

Weber [24] defined that a subset $K$ of a topological vector space $E$ is said to be *strongly convexly totally bounded* (*sctb*) if every neighborhood $U$ of 0 there exist a convex subset $C$ of $U$ and a finite subset $F$ of $E$ such that $K \subset F + C$.

He showed that if $K$ is totally bounded, then $K$ is sctb iff $K$ is of the Zima type. Therefore in Theorem 3.1 and Corollary 3.2, condition (Z) can be replaced by the following:

(W) *$T(X)$ is sctb.*

Moreover, he also showed that if $K$ is compact and convex, then

$$K \text{ is } sctb \iff K \text{ is of the Zima type} \iff K \text{ is locally convex.}$$

From this, we can obtain some particular forms of Corollary 3.2. For example, we have the following:

**Corollary 3.4.** *Let $X$ be a compact, convex and locally convex subset of a Hausdorff topological vector space $E$. Then any Kakutani map $T : X \multimap X$ has a fixed point.*

Note that Rzepecki [22] obtained a general form of Corollary 3.4 for single-valued compact continuous functions where the domain $X$ is not necessarily compact.

Furthermore, it is known that any compact convex subset $K$ of a topological vector space $E$ on which $E^*$ separates points of $E$ is locally convex; see Weber [24]. Therefore, Corollary 3.4 implies the following:

**Corollary 3.5.** *Let $X$ be a compact convex subset of a topological vector space $E$ on which $E^*$ separates points of $E$. Then any Kakutani map $T : X \multimap X$ has a fixed point.*

Table 1

| $E$ | $f:K\to K$ | | $F:K\multimap K$ | |
|---|---|---|---|---|
| I | Brouwer | 1912 | Kakutani | 1941 |
| II | Schauder | 1927, 1930 | Bohnenblust and Karlin | 1950 |
| III | Tychonoff | 1935 | Fan | 1952 |
| | Hukuhara | 1950 | Glicksberg | 1952 |
| | | | Himmelberg | 1972 |
| IV | Fan | 1964 | Granas and Liu | 1986 |
| | | | Park | 1988 |
| V | Zima | 1977 | Hadžić | 1981, 1982, 1987 |
| | Rzepecki | 1979 | | |
| | Hadžić | 1982 | | |

Note that Corollaries 3.3–3.5 are all consequences of the well-known Idzik fixed point theorem [12], and so is an equivalent form of Corollary 3.2 whenever $\overline{T(X)}$ is of the Zima type. Some other fixed point theorems which can be derived from lemma were given in [18–21].

It is well-known that the Brouwer fixed point theorem is equivalent to the KKM principle and, since all results in this paper imply the Brouwer theorem and are deduced from lemma, they are equivalent to the Brouwer theorem.

Finally, the major particular forms of Corollary 3.2 for Kakutani maps $F:K\multimap K$, where $K$ is a nonempty compact convex subset, can be adequately summarized by the above diagram.

In the diagram, $f$ denotes a (single-valued) continuous function and $F$ a Kakutani map. $K$ in the class I is a compact convex subset of Euclidean spaces, II normed vector spaces, III locally convex Hausdorff topological vector spaces, and IV topological vector spaces $E$ on which $E^*$ separates points. Moreover, $K$ in the class V is a convex subset of Hausdorff topological vector spaces such that $f(K)$ and $F(K)$ are relatively compact subsets of the Zima type. Further, theorems due to Schauder, Hukuhara, Himmelberg, Rzepecki, and some of Hadžić in the diagram are stated for compact maps without assuming compactness of domains. For the literature, see [15–17] and other references in the end of this paper.

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