FIXED POINTS, ROBERTS SPACES, AND THE COMPACT AR PROBLEM

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ABSTRACT. We discuss current state of research related to some well-known problems in infinite dimensional topology—the Schauder conjecture, the compact AR problem, and the Banach problem of whether every infinite dimensional compact convex subset X of a metrizable t.v.s. is homeomorphic to the Hilbert cube. We give affirmative answers to the second and third problems. Moreover, for every such X, we show that a closed multimap $F \in \mathfrak{B}(X, X)$ has a fixed point, where $\mathfrak{B}$ denotes the class of the better admissible multimaps. Our new results are applied to Roberts spaces. Some related problems are also discussed.

It is well-known that, in a locally convex Hausdorff t.v.s., any compact convex subset has the fixed point property (by Tychonoff) and has extreme points (by Krein-Milman).

For a long period it was not known whether it was true in every t.v.s. that every compact convex subset has the fixed point property (the Schauder conjecture) or an extreme point (see [25]).

In 1977, Roberts [24] constructed a striking example of compact convex sets without any extreme points providing a counter-example to the Krein–Milman theorem for a non-locally convex t.v.s. Nguyen To Nhu et al. [16,17] showed that all *Roberts spaces* – that is, compact convex sets with no extreme points constructed by Roberts' method of needle point spaces – have the fixed point property; and moreover, that the original example constructed by Roberts is an AR.

In 2001, Robert Cauty obtained the affirmative solution to the Schauder conjecture as follows:

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem 1. [3] *Let E be a Hausdorff topological vector space, C a convex subset of E, and f a continuous function from C into C. If $f(C)$ is contained in a compact subset of C, then f has a fixed point.*

Now, we review some known general fixed point theorems. Single-valued functions will be denoted by $f : X \to Y$, for example, and multimaps or set-valued maps by $F : X \multimap Y$ and are called simply maps. In this paper, all t.v.s. are assumed to be Hausdorff, and all metric spaces are separable.

A *polytope* P in a t.v.s. E is a nonempty compact convex subset of E contained in a finite dimensional subspace of E.

Let $\mathcal{V}$ be a fundamental system of neighborhoods of 0 in a t.v.s. E.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in X.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Examples of admissible t.v.s. are l^p, L^p, the Hardy spaces H^p for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, and others. Moreover, any locally convex subset of an F-normable t.v.s. and any compact convex locally convex subset of a t.v.s. is admissible. Note that an example of a nonadmissible nonconvex compact subset of the Hilbert space l^2 is known. For details, see Hadžić [9], Weber [26,27], and references therein.

Let X and Y be topological spaces. An *admissible class* $\mathfrak{A}_c^\kappa(X,Y)$ of maps $T : X \multimap Y$ is one such that, for each compact subset K of X, there exists a map $S \in \mathfrak{A}_c(K,Y)$ satisfying $S(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite compositions of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous and compact-valued; and
(iii) for each polytope P, each $F \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex sets), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with compact acyclic values), the Powers maps $\mathbb{V}_c$, the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, and others. Further, the Fan-Browder maps (domains are Hausdorff, codomains are convex sets, nonempty convex values and open fibers), $\mathbb{K}_c^+$ due to Lassonde, $\mathbb{V}_c^+$ due to Park et al., and approximable maps $\mathbb{A}^\kappa$ due to Ben-El-Mechaiekh and Idzik are examples of $\mathfrak{A}_c^\kappa$. For the literature, see [21-23].

For a subset X of a t.v.s. E, we defined the *"better" admissible class* $\mathfrak{B}$ of maps as follows:

$F \in \mathfrak{B}(X, Y) \Longleftrightarrow F : X \multimap Y$ is a map such that for any polytope P in X and any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

Note that $\mathfrak{A}_c^\kappa(X, Y) \subset \mathfrak{B}(X, Y)$ for a subset X of a t.v.s. E, and some examples of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ are known. It is also known that closed maps in the KKM class due to Chang and Yen or, more generally, in the s-KKM class due to Chang et al. belong to $\mathfrak{B}$.

The following fixed point theorems were obtained in 1990's by the author [21,22]:

Theorem 2. *Let E be a t.v.s. and X an admissible compact convex subset of E. Then any map $T \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

Theorem 3. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact closed map $T \in \mathfrak{B}(X, X)$ has a fixed point.*

It is not known whether the admissibility of X can be eliminated in Theorems 2 and 3. If so, then these results would be far-reaching generalizations of the Cauty theorem.

However, for Kakutani maps, Dobrowolski [6] obtained recently variants of Theorem 2 or 3 as follows:

Theorem 4. [6] *Let E be a t.v.s. and X a compact convex subset of E. Then any map $T \in \mathbb{K}(X, X)$ has a fixed point.*

Theorem 5. [6] *Let E be a metrizable t.v.s. and X an convex subset of E. Then any compact closed map $T \in \mathbb{K}(X, X)$ has a fixed point.*

More early, in 1996, Nguyen To Nhu [15] defined the notion of *weakly admissible* compact convex subsets of a metrizable t.v.s. and showed that such subsets have the fixed point property.

Arandelović [1] extended the notion of weak admissibility to arbitrary t.v.s. and gave a non-metrizable extension of Nhu's result as follows:

Let E be a t.v.s. and $X \subset E$ a nonempty closed convex subset of E. We say that X is *weakly admissible* if for every $V \in \mathcal{V}$ there exist closed convex subsets $X_1, X_2, \cdots, X_n$ of X with $X = \mathrm{co}\,(\bigcup_{i=1}^n X_i)$ and continuous functions $f_i : X_i \to X \cap L, i = 1, 2, \cdots, n$, where L is a finite dimensional subspace of E, such that $\sum_{i=1}^n (f_i(x_i) - x_i) \in V$ for every $x_i \in X_i$ and $i = 1, 2, \cdots, n$.

Recall the following two theorems:

Theorem 6. [15,1] *Let X be a weakly admissible, compact convex subset of a t.v.s. Then X has the fixed point property.*

The Cauty theorem [3] implies that the weak admissibility of X in Theorem 6 is redundant.

In 2001, Theorem 6 was generalized as follows:

Theorem 7. [20] *Let X be a weakly admissible compact convex subset of a t.v.s. Then every Kakutani map $F : X \multimap X$ has a fixed point.*

Note that, in view of Theorem 4, Theorem 7 holds without assuming the weak admissibility of the domain X.

We adopt the following in [12]: An infinite dimensional compact convex subset K of a t.v.s. E is said to have the *simplicial approximation property* if for every $V \in \mathcal{V}$ there exists a finite dimensional compact convex set K_0 in K such that if S is any finite dimensional simplex in K then there exists a continuous function $\psi : S \to K_0$ with $\psi(x) - x \in V$ for all $x \in S$.

The following is known by Kalton, Peck, and Roberts [12, Theorem 9.8]:

Lemma 1. [12] *If K is an infinite dimensional compact convex set in an F-space, then the following are equivalent*:

(i) *K has the simplicial approximation property.*
(ii) *If $\epsilon > 0$, there exists a simplex S in K and a continuous function $r : K \to S$ such that $\|r(x) - x\| < \epsilon$ for every $x \in K$.*

In 2003, Dobrowolski [5] obtained the following:

Lemma 2. [5] *Every compact convex subset K of a metrizable t.v.s. E has the simplicial approximation property.*

Note that Lemma 2 is the affirmative answer to Problem 9.5 in [12] and was given by Nhu [15] when K is weakly admissible.

Actually, Lemma 2 is the main result of [5] with a proof of almost 25 pages and it was assumed that E is separable and has an F-norm $\|\cdot\|$. In fact, it is known that any metrizable t.v.s. can be equipped an F-norm $\|\cdot\|$ such that $\|\lambda x\| \leq \| x\|$ for $x \in X$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$. Note that K can be embedded in the completion $\hat{E}$ of E and $\hat{E}$ is an F-space. Since K has the simplicial approximation property, it is admissible by Lemma 1 (ii).

Therefore, we have the following equivalent form of Lemma 2.

Lemma 3. *Every compact convex subset K of a metrizable t.v.s. E is admissible.*

Combining Lemma 3 with Theorems 2 and 3, we have

Theorem 8. *Let K be a compact convex subset of a metrizable t.v.s. Then every map $T \in \mathfrak{A}_c^\kappa(K,K)$ or every closed map $T \in \mathfrak{B}(K,K)$ has a fixed point.*

Recall that Dobrowolski [5] gave a proof of a particular case of Theorem 8 (and the Cauty theorem) for $f = T \in \mathbb{C}(K,K)$. As we have seen in Theorem 4, Dobrowolski [6] showed that Theorem 8 for $T \in \mathbb{K}(K,K)$ holds for a t.v.s. without assuming metrizability.

Comparing Theorem 8 with Theorems 2–5, we raise the following problem:

Does Theorem 8 hold for any t.v.s. other than metrizable one?

For Roberts spaces, we know the following:

Lemma 4. [24] *In every metrizable complete needle point space E a compact convex subset K_R without extreme points can be constructed.*

In [15], Nhu observed that all Roberts spaces are 'small generated' and weakly admissible, and asked:

Is every (*small generated*) *compact convex subset K of a metrizable t.v.s. weakly admissible?*

In virtue of Lemma 3, K itself is admissible and hence, so is K_R in Lemma 4.

From Theorem 8, we obtain

Theorem 9. *For every Roberts space $X = K_R$, a map $T \in \mathfrak{A}_c^\kappa(X,X)$ or a closed map $T \in \mathfrak{B}(X,X)$ has a fixed point.*

Note that Nhu et al. [17] obtained Theorem 9 for $\mathbb{C}$, and Okon [18,19] for $\mathbb{K}$.

Recall that, in 1960, Klee [13,14] showed that any admissible compact convex subset of a metrizable t.v.s. is an AR. Therefore, in virtue of Lemma 3, simply we have the affirmative solution to the long-standing compact AR problem as follows:

Theorem 10. *A compact convex subset K of a metrizable t.v.s. is an AR.*

Corollary. *Any Roberts space is an AR.*

This is given for the original Roberts space by Nhu et al. [16].

Moreover, by the so-called generalized Schauder fixed point theorem [8, p.94] that every compact continuous single-valued selfmap of an AR space has a fixed point, we have a particular form of Theorem 9:

Corollary. [17] *Every Roberts space has the fixed point property.*

Moreover, from Theorem 10 we have the following resolution of the Banach problem:

Theorem 11. *An infinite-dimensional compact convex subset K of a metrizable t.v.s. is homeomorphic to the Hilbert cube Q.*

Proof. Dobrowolski and Toruńczyk [7] showed that K is homeomorphic to Q if it is an AR.

Theorem 11 gives also the affirmative answer to Problem 9.6 of [12]. Moreover, we have

Corollary. *Every Roberts space is homeomorphic to Q.*

We may compare Theorem 11 with the following well-known result:

Theorem 12. (Keller-Klee) *Every metrizable infinite-dimensional compact convex subset K of a locally convex t.v.s. is homeomorphic to Q.*

Kalton, Peck, and Roberts [12, p.218] noted that if a compact convex set is homeomorphic to the Hilbert cube then it has the simplicial approximation property. In other words, Theorem 11 $\Longrightarrow$ Lemma 2. Recall that we followed Lemma 2 $\Longrightarrow$ Lemma 3 $\Longrightarrow$ Theorem 10 $\Longrightarrow$ Theorem 11. Therefore, in certain sense, those four statements are equivalent to each other.

Finally, we consider another general fixed point result related to the notion of convexty totally bounded (simply, c.t.b.) subsets due to Idzik [10], who established some (almost) fixed point theorems for Kakutani maps having relatively compact c.t.b. ranges.

A subset of a t.v.s. E is said to be *convexly totally bounded* if for every $V \in \mathcal{V}$, there exist a finite subset $\{x_i\}_{i=1}^n \subset B$ and a finite family of (open) convex subsets $\{C_i\}_{i=1}^n$ of V such that $B \subset \bigcup_{i=1}^n (x_i + C_i)$. Note that, in the above definition, $\{x_i\}$ can be chosen in E; see Idzik and Park [11].

Idzik [10] gave examples of c.t.b. sets as follows:

(1) Every compact set in a locally convex t.v.s.
(2) Any compact set which is locally convex or of the Zima type.

Further examples are given in [4] as follows:

(3) Every compact convex subset of $E = l^p,\ 0 < p < 1$.
(4) More generally, every compact convex subset of a t.v.s. E on which E^* separates points.

For more examples, see [4,26,27]. Later Idzik asked:

Is a c.t.b. compact convex subset admissible?

Note that in [4], examples of admissible sets which are not c.t.b. were given.

More recently, Arandelović [1] asked:

Is a c.t.b. compact convex subset weakly admissible?

Recall that a subset Y of a t.v.s. E is said to be *almost convex* if for any neighborhood V of the origin 0 in E and for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of Y, there exists a finite subset $\{z_1, z_2, \cdots, z_n\}$ of Y such that $z_i - y_i \in V$ for each $i = 1, 2, \cdots, n$ and $\mathrm{co}\{z_1, z_2, \cdots, z_n\} \subset Y$; see [10].

The following well-known result is due to Idzik [10]:

Theorem 13. [10] *Let E be a Hausdorff t.v.s., X an almost convex subset of E, and $T \in \mathbb{K}(X, X)$. If $\overline{T(X)}$ is a compact c.t.b. subset of X, then T has a fixed point.*

Note that this theorem has been a rich source of a lot of problems and has neither an elementary proof nor a generalization yet.

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References

[1] I. D. Arandelović, "On the weak admissibility on topological vector spaces and its applications in fixed point theory," *Proc. II Mathematical Conf. in Priština* (Priština, Sept. 25-28, 1996), 89–94.

[2] F. E. Browder, "The fixed point theory of multi-valued mappings in topological vector spaces," *Math. Ann.* **177** (1968), 283–301.

[3] R. Cauty, "Solution du problème de point fixe de Schauder," *Fund. Math.* **170** (2001), 231–246.

[4] E. De Pascale, G. Trombetta, and H. Weber, "Convexly totally bounded and strongly totally bounded sets, Solution of a problem of Idzik," *Ann. del. Scuola Normale Superiore di Pisa, Sci. Fis. Mat.* Ser. IV. Vol. **XX** (1993), 341–355.

[5] T. Dobrowolski, "Revisiting Cauty's proof of the Schauder conjecture," *Abstr. Appl. Anal.* **7** (2003), 407–433.

[6] ———, "Fixed-point theorem for convex-valued mappings," *Preprint.*.

[7] T. Dobrowolski and H. Toruńczyk, "On metric linear spaces homeomorphic to l_2 and compact convex sets homeomorphic to Q," *Bull. Acad. Polon. Sci.* (*Serie Sci. Math.*), **27** (1979), 883–887.

[8] J. Dugundji and A. Granas, *Fixed Point Theory*, PWN – Polish Sci. Publ., Warszawa, 1982.

[9] O. Hadžić, *Fixed Point Theory in Topological Vector Spaces*, Univ. of Novi Sad, Novi Sad, 1984, 337pp.

[10] A. Idzik, "Almost fixed point theorems," *Proc. Amer. Math. Soc.* **104** (1988), 779–784.

[11] A. Idzik and S. Park, "Leray-Schauder type theorems and equilibrium existence theorems," *Differential Inclusions and Optimal Control, Lect. Notes in Nonlinear Anal.* **2** (1998), 191–197.

[12] N. J. Kalton, N. T. Peck, and J. W. Roberts, *An F-space Sampler*, Chap. 9, 198–219, London Math. Soc. Lecture Notes 89, Cambridge Univ. Press, 1985.

[13] V. Klee, "Shrinkable neighborhoods in Hausdorff linear spaces," *Math. Ann.* **141** (1960), 281–285.

[14] ———, "Leray-Schauder theory without local convexity," *Math. Ann.* **141** (1960), 286–296.

[15] N. T. Nhu, "The fixed point property for weakly admissble compact convex sets: Searching for a solution to Schauder's conjecture," *Topology Appl.* **68** (1996), 1–12.

[16] N. T. Nhu, J. M. R. Sanjuro, and T. V. An, "The AR-property for Roberts' example of a compact convex set with no extreme points, Part 1: General result," *Proc. Amer. Math. Soc.* **125** (1997), 3075–3087; "Part 2: Application to the example," *ibid.* **125** (1997), 3089–3098.

[17] N. T. Nhu and L. H. Tri, "No Roberts space is a counter-example to Schauder's conjecture," *Topology* **33** (1994), 371–378.

[18] T. Okon, *When graph meets diagonal: an approximative access to fixed point theory*, Ph.D. Thesis, TU Dresden, Germany, 2001.

[19] ———, "The Kakutani fixed point theorems for Roberts spaces," *Topology Appl.* **123** (2002), 461–470.

[20] ———, "Fixed point theory for Roberts spaces," *Nonlinear Anal.* **47** (2001), 5697–5702.

[21] Sehie Park, “A unified fixed point theory of multimaps on topological vector spaces,” *J. Korean Math. Soc.* **35** (1998), 803–829. “Corrections,” *ibid.* **36** (1999), 829–832.

[22] ———, “Ninety years of the Brouwer fixed point theorem,” *Vietnam J. Math.* **27** (1999), 193–232.

[23] ———, “Continuous selection theorems in generalized convex spaces,” *Numer. Funct. Anal. Optimiz.* **20** (1999), 567–583.

[24] J. W. Roberts, A compact convex set without extreme points, *Studia Math.* **60** (1977), 255–266.

[25] W. Rudin, *Functional Analysis*, McGraw-Hill, Inc., 1973.

[26] H. Weber, “Compact convex sets in non-locally convex linear spaces, Schauder–Tychonoff fixed point theorem,” *Topology, Measure, and Fractals* (Warnemünde, 1991), *Math. Res.* **66**, 37–40, Akademie-Verlag, Berlin, 1992.

[27] ———, “Compact convex sets in non-locally-convex linear spaces,” *Note di Mat.* **12** (1992), 271–289.

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